Some Related Problems from Network

Flows, Game Theory and Integer Programming

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Abstract

We consider several important problems for which no polynomially time bounded algorithm is known. These problems are shown to be related in that a polynomial algorithm for one implies a polynomial algorithm for the others.

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1. Introduction

In [2] Cook introduced the concept of P-reducibility. He showed that there exists a polynomial time bounded algorithm to determine if a formula from the Propositional Calculus, in conjunctive normal form, was satisfiable (i.e. was true for some truth assignment to its variables) if and only if the class of languages accepted by polynomial time bounded Turing machines is the same as that acceptable by deterministic Turing machines in polynomial time. The latter problem is a well known "Open Problem" in the theory of computing. Substantial work in identifying other P-equivalent or P-complete problems has been done by Karp [5].

In this paper we show that the following problems are P-complete:

(1) Multicommodity network flows, flows with multipliers, flows with homologous arcs and bundle flows.

(2) Existence of a pure strategy equilibrium point in n-Person Games.

(3) 0-1 Integer programming with only one constraint (Knapsack Problem)

(4) Minimal equivalent digraph

(5) Minimal equivalent Boolean Form

We shall also show that Quadratic Programming and
Linear Programming with one nonlinear constraint are P-hard,
a condition not as strong as P-completeness.

In Section 2 we give a precise formulation of the notions
of P-reducible, P-complete and P-hard. Section 3 contains a
brief description of some of the known members of the class
PC (Polynomially Complete problems) which will be used in
showing other problems equivalent. Section 4 includes detailed
proofs and descriptions of the new problems that have been
found to be in PC or PH (polynomially hard).

2. Definitions

Definition 1 [Knuth]

a) A computational method is a quadruple (Q,I,Ω,f)
satisfying:

   (i) Q contains the subsets I and Ω

   (ii) f is a function from Q into itself.

         f leaves Ω pointwise fixed,

         i.e. $f(q) = q$ $\forall q \in \Omega$.

Informally Q, I, Ω, f represent the states of the computa-
tion, the input set, output set, and the computational rule.

b) Each input $x \in I$ defines a computational sequence
$x_0, x_1, \ldots$ as follows

         $x_0 = x$ and $x_{k+1} = f(x_k)$ for $k \geq 0$

A computational sequence is said to terminate in k
steps if k is the least integer for which $x_k \in \Omega$.

(Note that by the definition of a computational sequence, if $x_k \in \Omega$ then $x_{k+1} \in \Omega$ as $x_{k+1} = f(x_k) = x_k$).

An <u>algorithm</u> is a computational method which terminates in finitely many steps for all inputs $x \in I$.

If the computational rule f is single valued then the computational method is <u>deterministic</u> otherwise it is <u>nondeterministic</u>.

We restrict ourselves to computational methods that are effective (<u>i.e.</u> all the operations to be performed are sufficiently basic that they can be done in a finite amount of time).

Assuming that at each step of the computational sequence only one basic operation (add, multiply, compare, etc.) is performed, we may define the time complexity of the sequence to be the least k for which $x_k \in \Omega$. In the case of a non-terminating computation this is not defined.

The time complexity of an algorithm[1] will in general be some function of its inputs, say $f(\underline{N})$ where $\underline{N}$ is a vector that characterizes the inputs (for example in graph algorithms $\underline{N}$ may be the number of vertices and edges).

If $f(\underline{N})$ is a polynomial in $\underline{N}$ then the algorithm is polynomially time bounded. [In some cases it may be meaningful for $\underline{N}$ to also characterize the outputs, for <u>e.g.</u>: if the number of outputs is an exponential function of the inputs we should not expect to find an algorithm that computes them in

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[1]Unless otherwise specified 'algorithm' will mean 'deterministic algorithm'.

P(N) ;   P a polynomial and N a characterization of the
inputs].

   As our computational model we shall use Turing Machines.
(For a standard treatment of this model see Hopcroft and
Ullman [3].) It should be noted that our results are valid,
independent of the computational model chosen (e.g. random
access machines).

   Definitions 2 through 5 are from Karp [5].

Definition 2   P is the class of languages recognizable by

     by one-tape Turing machines which operate in polynomial

     time.

Definition 3 π is the class of functions from Σ^* into

     Σ^* defined by one-tape Turing machines which operate

     in polynomial time.

Definition 4   Let L and M be languages. Then L α M

     (L is P-reducible to M) if there is a function $f \epsilon \pi$

     such that $f(x) \epsilon M$ = $x \epsilon L$.

   Informally, a problem S is P-reducible to a problem
T if a polynomial algorithm for T implies a polynomial
for S i.e. given a polynomial algorithm for T we can con-
struct a polynomial algorithm for S . In further discussion
we shall use this informal notion of P-reducible. However, a
suitable modification of each of the problems considered will
easily be seen to be P-reducible in the sense of Definition 4.

<u>Definition 5</u>   NP   is the class of languages recognizable in
    polynomial time by nondeterministic Turing Machines.

<u>Open Problem</u>   is NP = P . We may rephrase this as "Is there
    a deterministic polynomial algorithm for all languages
    in NP". Call this problem P1 .

<u>Definition 6</u>   A problem P2 will be said to be P-complete
    if P2 α P3 and P3 α P2 for some P3 already known
    to be P-complete.

    Clearly P-reducible is a transitive relation and
P-complete is an equivalence relation.

<u>Definition 7</u>   P1 (defined above) is in PC (PC is the
    equivalence class of P-complete problems and it includes
    P1)

<u>Definition 8</u>   A problem P2 is P-hard if P3 α P2 and
    P3 ε PC
    Clearly all members of PC are P-hard. In some cases
    we may only be able to show the relation P-hard rather
    than the stronger P-complete relation.

3.   <u>Some Known Members of PC</u>
    To prove some of the reductions we shall make use of some
known members of PC . A brief description of these members
is given below (A more exhaustive list may be found in
Karp [5]).

    (i)   a)   <u>Satisfiability</u>
            Given a formula from the Propositional
        Calculus, in conjunctive normal form (CNF), is
        there an assignment of truth values for which it

is "True".

b) Satisfiability with exactly 3 literals

per clause.

This is the same as (a) except that each

clause of the formula now has exactly 3 literals.

c) Tautology

Given a formula, from the Propositional

Calculus, in disjunctive normal form (DNF) does

if have the value True for all possible assignments

of truth values.

(ii) Sum of subsets of integers

Given a multiset $S = (s_1, \ldots, s_r)$ of positive

integers and a positive integer M does there exist a

sub-multiset of S that sums to M . (This problem

is called the Knapsack problem in [5]. However, here we

shall denote by 'Knapsack Problem' a similar integer

optimization problem.) Note that a multiset is a set

of elements that may not necessarily be distinct.

(iii) Maximum Independent Set

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$.

A set of vertices is independent if no two members of

the set are adjacent in G . A maximum independent set

is an independent set that has a maximum number of

vertices.

(iv) Directed Hamiltonian Cycle

Given a directed graph G , does it have a cycle

that includes each vertex exactly once.

Theorem 1 The following problems are in PC

      (i)   Satisfiability, Satisfiability with exactly three literals per clause, tautology;

   (ii)   Sum of subsets of integers;

 (iii)   Maximum independent set of a graph;

  (iv)   Directed Hamilton Cycle.

Proof (i) is proved in Cook [2]. The rest are proved in Karp [5]. Karp [5] acuually shows that Satisfiability with at most three literals per clause is P-complete. From this result one may trivially show that Satisfiability with exactly three literals per clause is P-complete.

4. Complete and P-hard Problems

In this section we shall show that some frequently encountered problems in various areas such as Network flows, Game Theory, nonlinear and linear optimization are either P-complete or at least P-hard. The reductions will easily be seen to be effective. The polynomial factors involved in the reductions are small (usually a constant or a polynomial of degree 1).

4.1 Integer Network Flows

We define the following network problems:

N(i) Network flows with multipliers.

Let G be a directed graph with vertices s_1, s_2, v_1, ..., v_n and edges (arcs) e_1, e_2, ..., e_m . Let $w^-(v)$ be the set of arcs directed into vertex v and $w^+(v)$ those arcs directed away from v .

G will be said to denote a network with multipliers if:

(a) The source, s_1 , of the network has no incoming arcs, <u>i.e.</u> $w^-(s_1) = \phi$

(b) The sink, s_2 , has no outgoing arcs, <u>i.e.</u> $w^+(s_2) = \phi$

(c) To every vertex v_i (excluding the source and sink) there corresponds an integer $h_i > 0$, called its multiplier.

(d) To each edge, e_i , there corresponds an interval $[a_i, b_i]$.

Conditions (a) - (d) are said to define a transportation network.

We are required to find a flow vector $\Phi = (\phi_1, \phi_2, \ldots, \phi_m)$ such that:

1) $a_i \leq \phi_i \leq b_i$

2) $h(v) \sum\limits_{i \in w^-(v)} \phi_i = \sum\limits_{i \in w^+(v)} \phi_i$ for all $v \in V(G)$
$$v \neq s_1,$$
$$v \neq s_2$$

3) $\sum\limits_{i \in w^-(s_2)} \phi_i$ is maximized

(Note: ϕ_i integer)

In what follows we assume $a_i = 0$.

N(ii) Multicommodity Network Flows

The transportation network is as above, but now $h(v) = 1$ for all v in $V(G)$. We have, however, several different commodities $c_1, c_2, \ldots, c_n$ and some arcs may be labelled <u>i.e.</u> they can carry only certain commodities. Each arc is

assigned a capacity and we wish to know whether a flow
$R = (r_1, r_2, \ldots, r_n)$, where r_i is the quantity of the
i^{th} commodity, is feasible in the network.

N(iii) Integer Flows with Homologous Arcs

The transportation network remains the same. Also
$h(v) = 1$ and there is only one commodity. Certain arcs are
paired and we require that if arcs i, j are paired then
$\phi_i = \phi_j$. We wish to know if a flow of at least F is
feasible in the network.

N(iv)   Integer Flows with Bundles

The arcs in the network are divided into sets I_1, $\ldots$,
I_k (the sets may overlap). Each set is called a bundle and
with each bundle is associated a capacity C_i . We wish to
know if a flow $\geq F$ is feasible in the network.
$(\sum_{i \in I_j} \phi_i \leq C_j$, $1 \leq j \leq k)$ and $h(v) = 1$, $\forall v \in V(G)$.

Theorem N: Problems N(i) - N(iv) are in PC .

Proof a) N(i), N(ii), N(iii), N(iv) α Pl .

The Nondeterministic Turing Machine just guesses the flows
in each arc and then verifies conditions (1) and (2). In
addition it does the following:

   i) For N(ii) it verifies that the resultant flow
is $\geq R$.

   ii) For N(iii) the 'homologous conditions' are checked
and $\sum_{i \in w^-(s_2)} \phi_i \geq F$ verified.

   iii) For N(iv) the bundle restrictions are checked and

$$\sum_{i \in w^-(s_2)} \phi_i \geq F \quad \text{verified.}$$

If in $N(i)$ we replace the max $\sum_{i \in w^-(s_2)} \phi_i$ require-

ment to T: $\sum_{i \in w^-(s_2)} \phi_i \geq F$

then from the above it follows that $T \propto Pl$. To see

$N(i) \propto T$ we note that if the length of the input on a

Turing machine's tape is n then the largest number it

can represent is c^n for some constant c which depends

only on the Turing machine. Hence the maximum capacity

of an arc is bounded by c^n and so $\max \sum_{i \in w^-(s_2)} \phi_i \leq k^n$

for some constant k. Now, assume there

is a polynomial $[p(n)]$ algorithm for T then using the

method of bisection we can determine $\max \sum_{i \in w^-(s_2)} \phi_i$ in

at most $\log_2 k^n = n \log_2 k$ applications

of T. This, therefore, gives a polynomial algorithm for

$N(i)$. Therefore, $N(i) \propto T \propto Pl$ and from the transi-

tivity of $\propto$ we conclude $N(i) \propto Pl$. Clearly, this

proof technique can be used to show $N(iii)$ and $N(iv)$

complete when they are changed to maximization problems.

b)   (i) Sum of subsets of integers $\propto N(i)$. We construct

a network flow problem of type $N(i)$ such that

$$\max \sum_{i \in w^-(s_2)} \phi_i = M \quad \text{iff there is a sub-multiset of} \quad S$$

that sums to M. (For notations see Section 3.)
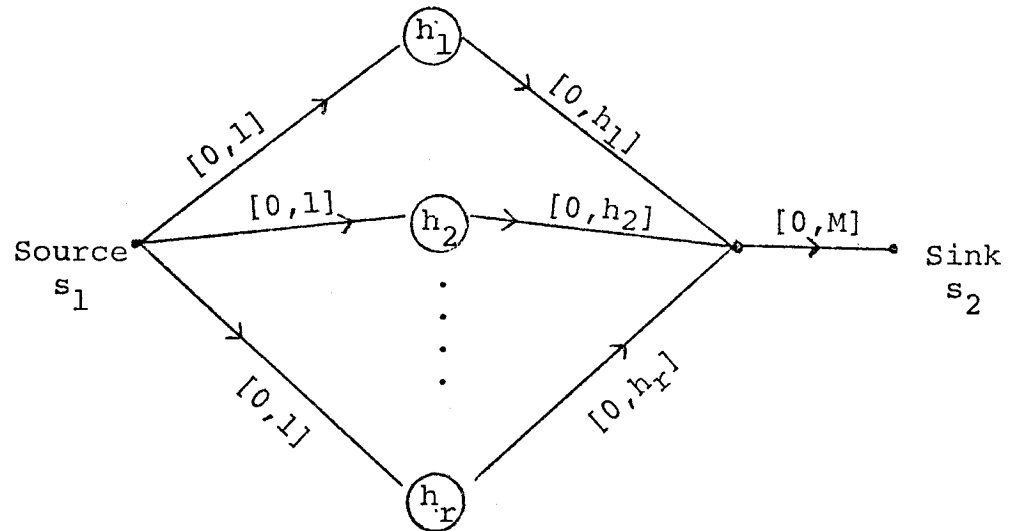
$h_i = s_i \qquad 1 \leq i \leq r$

Clearly max $\sum\limits_{i \in w^-(s_2)} \phi_i = M$ iff some sub-multiset of S sums to M.

(ii) Tautology α N(ii)

Suppose that the formula P in DNF has n-variables $a_1, a_2, \ldots, a_n$. We shall construct a multicommodity network with n-commodities $c_1, c_2, \ldots, c_n$ such that the flow $R(1, 1, \ldots, 1)$ is feasible iff P is not a tautology. The network of Figure N2 realizes this.
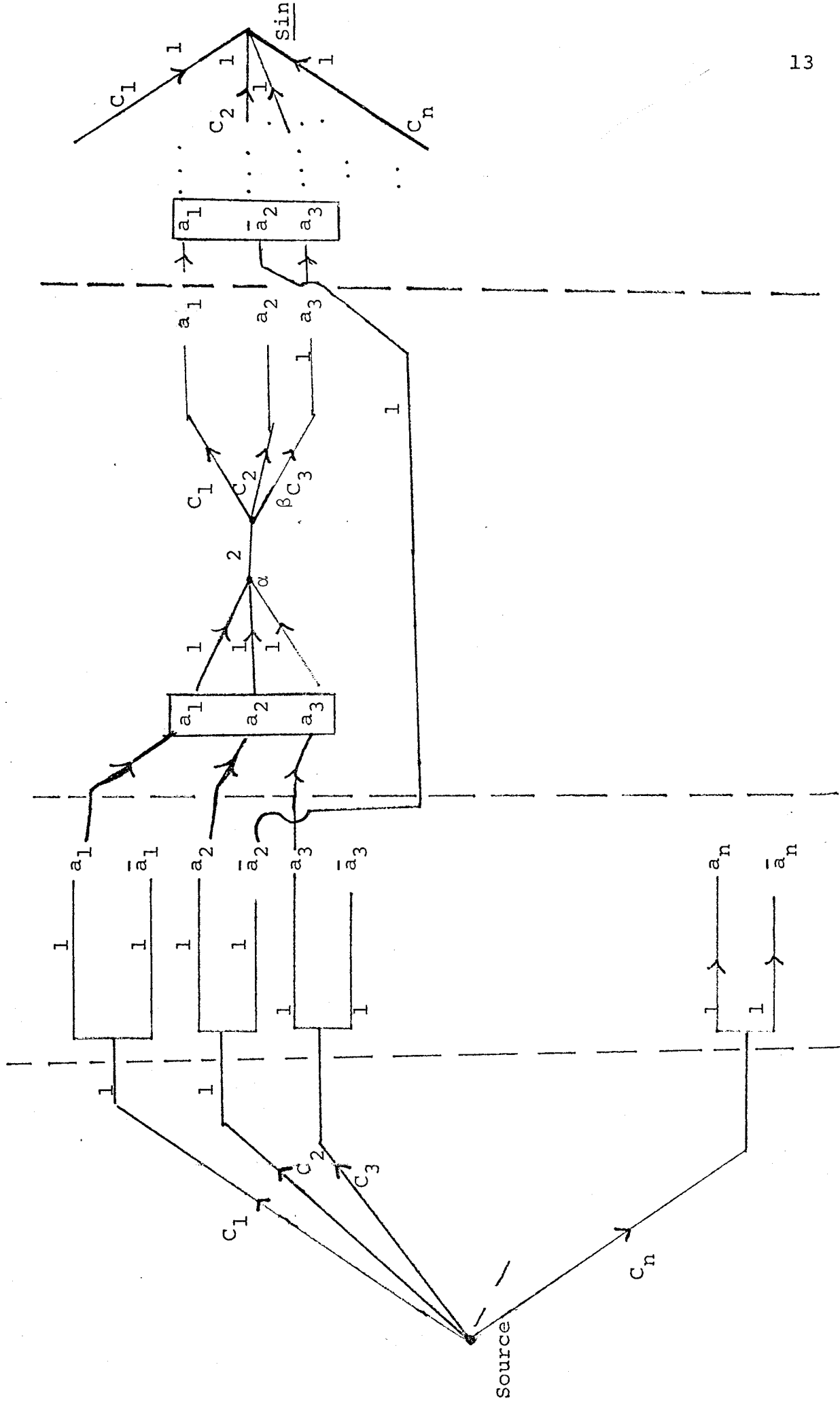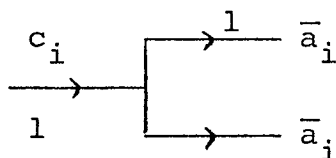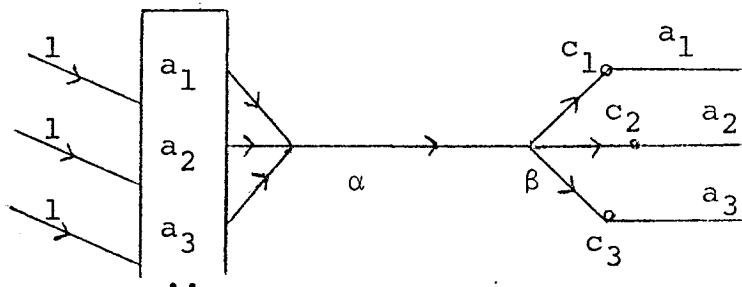
Figure N2   Tautology α Multicommodity Network Flows.

Discussion

[A] This section of the network ensures that there is a flow through only one of the nodes a_i or $\bar{a}_i$. In terms of the formula A , a flow through a_i means a truth assignment of 1 to a_i while a flow through $\bar{a}_i$ means an assignment of 0 to a_i .

[B] For each clause (K_i) in P we have a section of the form

If there are j literals in the clause then arc (α, β) is assigned a capacity of j - 1 . This requires that the truth assignments be such that clause k_i is false (as at least one term in it is false). Node "β" is where the "multicomoodity" property of the network is used. Here the flow through α is correctly separated into its components i.e. we are able to get back the truth values of the variables. The components for each flow are connected in series as in Figure N2.

We now want to know if a flow $R = (1, 1, \ldots, 1)$ is feasible. It is easy to see that such a flow is possible iff there is a truth assignment to $a_1, \ldots, a_n$ for which each clause is false, i.e. iff P is not a Tautology.

(iii) Tautology α N(iii)

The construction is very similar to that for multi-commodity network flows. The network is as in Figure N3. Homologous arcs are marked with the same subscripted Greek letter.

The arcs (α, β) have a capacity that is one less than the number of terms in the clause thereby ensuring that truth assignments that would make the preceding clause True cannot occur. The "homologous conditions" permit the separation of the flow at β into the original "truth assignments".

The maximum capacity of the sink is n. Hence there is a flow $\geq n$ iff there is a consistent assignment of truth values to $a_1, \ldots, a_n$ such that no clause is "True" and hence P is not a Tautology.
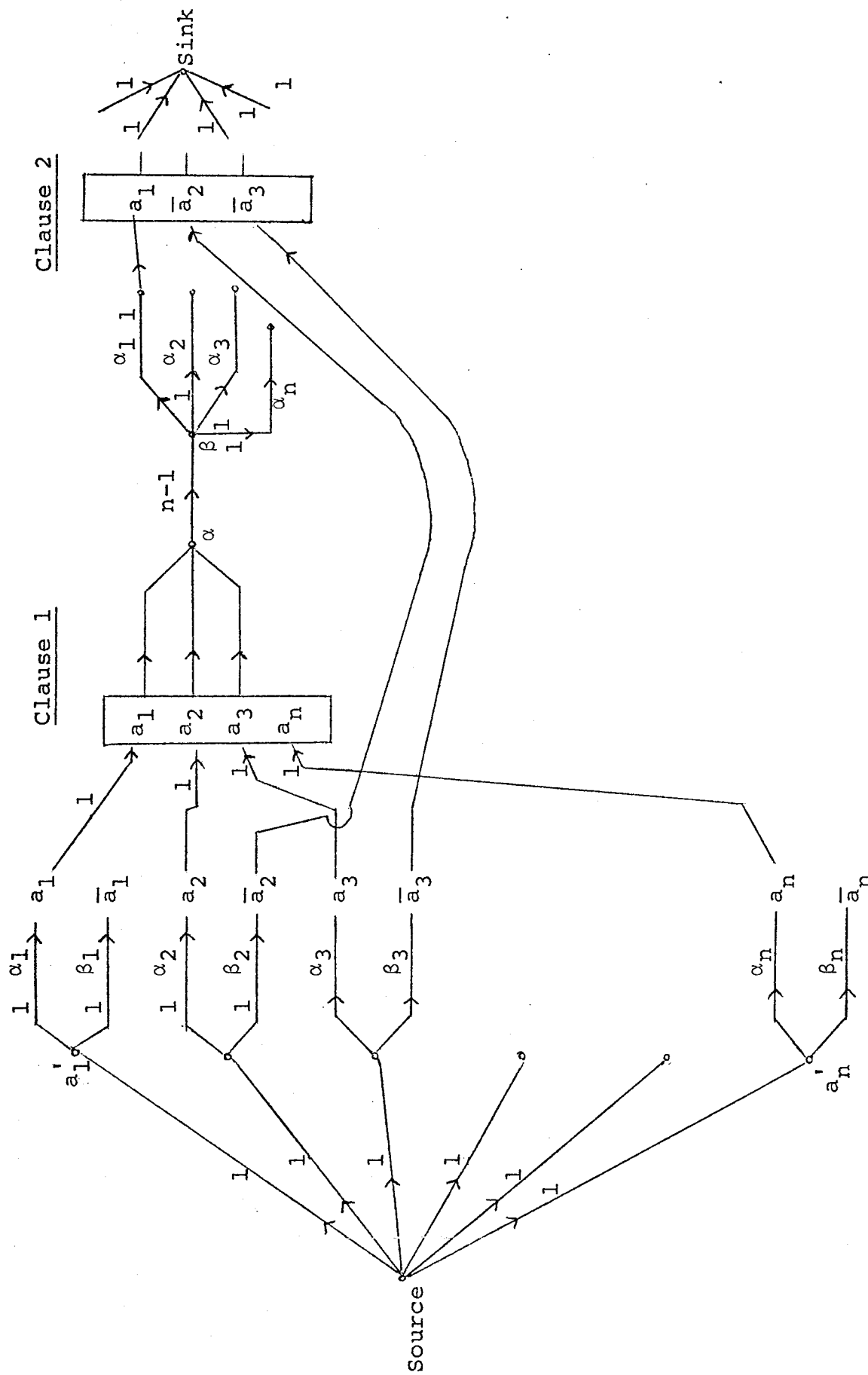
Figure N3: Network with Homologous Arcs.

(iv) Max. Independent Set α N(iv)

Let $G(V,E)$ be an undirected graph for which we want to determine the maximum independent set.

Construct a network as below:

Let s_1, v_1, ..., v_n, s_2 be the nodes of the network $n = |V|$. From the source node draw an arc of capacity 1 to each of the nodes v_i $1 \leq i \leq n$. From each node v_i draw an arc to the sink node s_2. Mark each such arc v_i. For each node v_i mark the sink arc (v_i, s_2) by v_j if v_j is adjacent to v_i. Define the bundles I_1, I_2, ..., I_n as:

$$I_j = \{e_i \mid arc \ e_i \ is \ marked \ by \ v_j\}$$

Each bundle is assigned a capacity 1. This ensures that if vertex v_i is chosen in the maximum independent set (i.e. there is a non-zero flow through it) then there is no flow through vertices adjacent to v_i (i.e. adjacent vertices are not chosen).

Now there is a flow $\geq F = k$ iff there is an independent set of cardinality $\geq k$. We solve the flow problem for $k = n, n - 1, ..., 1$ and the first k for which we get a feasible flow defines a maximum independent set.

E.g.

G(V,E)   Network
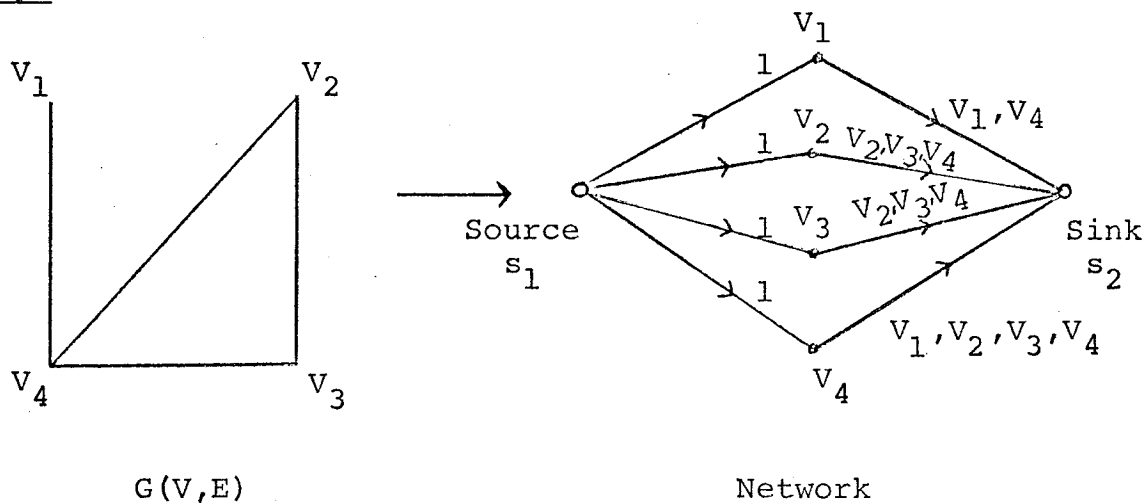
The largest k for which there is a feasible flow is

k = 2 , through vertices V_1 and V_2 . Thus the maxi-

mum independent set of G is of size 2 and one

such set is $\{V_1, V_2\}$. (In this case the solution is

unique.)

4.2   Graph Theory

G1:   Minimal Equivalent Graph of a Digraph

Given a directed graph G(E,V) we wish to remove as

many edges from G as possible, getting a graph G_1 such

that:

(i)   In G there is a path from v_i to v_j

iff there is a path in G_1 from v_i to v_j

(ii) $E(G_1) \subseteq E(G)$ (E(G) is the set of edges of G)

i.e. we want the smallest subset of E(G) such that

the transitive closure of G_1 = Transitive closure of G .

Theorem G1:   G1 is in PC

Proof   (a) G1 α P1

Let   n = # of vertices in   G = |V(G)| then

$|E(G)| \leq 2n(n-1) < 2n^2$.

We can easily construct a NDTM, T, which given G and an

integer k determines if there is a subset of k edges

satisfying (*) .   T can be constructed so as to work in

$O(n^3)$ time.   If NP = DP then there is a deterministic

algorithm that does this in p(n)   time.   We find the smallest

$k \leq n^2$ for which such a subset exists.   After determining

k ,   the k edges can be determined as below:

Define a sequence $\overline{E}$ of maximum length |E(G)| . $\overline{e}_i = 1$ if

edge i is among the k edges 0 otherwise.

Suppose it is already known that $\overline{E} = (i_1, \ldots, i_j)$ is

a correct "partial" choice then we ask if $\overline{E}u(i_{j+1} = 1)$ is.

If Yes then set $\overline{E} = (i, i_2, \ldots, i_j, 1)$

If No then set $\overline{E} = (i, i_2, \ldots, i_j, 0)$

Do this for $j = 0, 1, 2, \ldots, |E| - 1$.

(b) Directed Hamilton Cycle α G1

Note (i) if the directed graph G has a Hamilton cycle then

its transitive closure is the "complete directed graph"

on |V(G)| points. The smallest graph with this transi-

tive closure is the cycle on |V(G)| points. Thus if

there is a Hamilton cycle then this cycle forms the

minimal equivalent graph of G .

(ii) Conversely if the minimal equivalent graph is a cycle

on |V(G)| points then G has a Hamilton cycle.

$\therefore$ G has a Hamilton cycle <=> the minimal equivalent graph of G is a Hamilton cycle.

4.3 n-Person Game Theory

Following Lucas [7] we have:

An n-Person non-cooperative game in normal form consists of a set N of n players denoted 1, 2, ..., n; a finite set N_i = 0, 1, ..., n_i of $(n_i + 1)$ pure strategies for each player $i \in N$; and a payoff function F from $N_1 x ... x N_n$ to R^n .

A _strategy_ n-tuple $(S_1{}^*, ..., S_n{}^*)$ is said to be an equilibrium n-tuple iff for all i, $i \in N$ and $S_i \in N_i$

* $F_i(S_i{}^*, ..., S_n{}^*) \geq F_i(S_i{}^*, ..., S_{i-n}{}^*, S_i, S_{i+1}{}^*, ..., S_n{}^*)$

where F_i is the ith component of F .

i.e. There is no advantage for a player to unilaterally deviate from an equilibrium point.

G_T1: Given a game G = (F, n, $\bar{N}$) does it have an equilibrium point.

Theorem G_T1: GT1 $\in$ PC

Proof (a) GT1 α P1

The nondeterministic Turing Machine just guesses an equilibrium point and verifies that * is satisfied.

(b) Satisfiability (3 literals/clause) α GT1

Let P be the formula in CNF in n-variables. Define an n-person game as below:

Each player has two strategies 0 and 1 . Strategy 0 corresponds to assigning a truth value "False" to the corresponding variable and 1 to a "True" assignment.

Let $P = C_1 \wedge C_2 \wedge \ldots \wedge C_k$

$$C_i = C_{i_1} \vee C_{i_2} \vee C_{i_3}$$

the variables are $x_1, x_2, \ldots, x_n$

Replace each variable in the clause C_i by x_i if $x_i \in C_i$ and by $(1 - x_i)$ if $\bar{x}_i \in C_i$

Replace "$\vee$" by "$+$" , getting C_i' .

E.g. $C_i = x_1 \vee x_2 \vee \bar{x}_3 \Rightarrow C_i' = x_1 + x_2 + (1 - x_3)$
$$= x_1' + x_2' + x_3'$$

In order that C_i' has a $(0,1)$ value
replace $x_1' + x_2' + x_3'$ by

$$f_i(\underline{x}') = x_1'(1-x_2')(1-x_3') + (1-x_1')x_2(1-x_3') + (1-x_1')(1-x_2')x_3'$$
$$+ x_1' x_2' (1-x_3') + x_i(1-x_2')x_3' + (1-x_1')x_2' x_3' + x_1' x_2' x_3'$$

Clearly $f_i(\underline{x}') = 1$ iff $C_i(x)$ is "True".

Define $h_1(\underline{x}') = 2 \prod\limits_{i=1}^{k} f_i(\underline{x}')$

and $F_1(\underline{x}') = \begin{pmatrix} h_1(x') \\ \vdots \\ h_1(x') \end{pmatrix}$

From the above definition of $F_1(\underline{x}')$ it follows that

$$\max\ F_1(\underline{x}') = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix} \quad \text{if} \quad P(\underline{x}) \quad \text{is satisfiable}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{otherwise}$$

Let $G_2(x_1, x_2)$ be a 2-person game with 2-strategies per player and having no equilibrium point.

$$G_2(\underline{x}) = \begin{pmatrix} g_1(\underline{x}) \\ g_2(\underline{x}) \end{pmatrix} \qquad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \le \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

define

$$F_2(\underline{x}) = \begin{pmatrix} g_1(\underline{x}) \\ g_2(\underline{x}) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then $F_2(\underline{x})$ defnnes an n-person game with no equilibrium point.

Set $F(\underline{x}) = F_1(\underline{x}) + F_2(\underline{x}) \left(\begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix} - F_1(\underline{x}) \right)$

Then $F(\underline{X})$ defines an n-person game in which each player has 2-strategies.

For any choice of strategy vector $\underline{x}$ we have either

(i) $F_1(\underline{x}) = 0$, $F(\underline{x}) = 2F_2(\underline{x}) \le \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

By changing the strategies for either x_1 or x_2 we can increase the payoff to x_1 or x_2 respectively as $F_2(\underline{x})$ defines a game with no equilibrium point. If such a change results in $F_1(\underline{x}) = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$ then everyone's payoff increases. In any case such an $\underline{x}$ cannot be an equilibrium point.

(ii) $F_1(x) = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$ Such a point is an equilibrium point

as now $F(x) = F_1(\underline{x}) = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$ and 2 is the maximum payoff

any player can get. So no change from this point, uni-
lateral or otherwise, would be advantageous to any player.
∴ the n-person game defined above has an equilibrium
point iff $P(\underline{x})$ is satisfiable.

As an example for $F_2(x_1, x_2)$ consider:

Strategy	Payoff
(0,0)	[0,1]
(1,0)	[1,0]
(1,1)	[0,1]
(0,1)	[1,0]

$$g_1(\underline{x}) = (2 - x_1 - x_2)(x_1 + x_2)$$
$$g_2(\underline{x}) = (1 - x_1 - x_2)^2$$

Clearly no $\underline{x}$ is a stable (equilibrium) point.

Set $F_2(\underline{x}) = \begin{pmatrix} g_1(\underline{x})/2 \\ g_2(\underline{x})/2 \end{pmatrix}$

4.4 Optimization

K1: One dimensional (0-1) Knapsack Problem.

The problem is to:

$$\text{maximize:} \qquad \sum_{1}^{n} x_i \, p_i \qquad \qquad \ldots \qquad \text{(i)}$$

$$\text{subject to} \qquad \sum_{1}^{n} x_i \, w_i \leq M$$

$$x_i = 0,1 \qquad 1 \leq i \leq n$$

$$p_i > 0, \quad w_i > 0$$

<u>Theorem Kl</u>: Kl $\in$ PC

<u>Proof</u>   (a) Kl α Pl

Clearly the problem with (i) replaced by

(l') .. $\sum x_i \, p_i \geq Z$ is reducible to Pl . Now if the length

of the input is n , then each $P_i < k^n$ for some k . So

using the method of bisection we can find the optimal Z

in $\log_2 k^n = n \log_2 k$ query steps of (i') for some k ,

$k \leq |\Sigma|$ ($|\Sigma|$ = number of letters in the alphabet for the

NDTM above).

(b)   Sum of subsets of integers α Kl

Let $S = (S_1, \ldots, S_n)$ be the multiset of integers. We

want to find a subset (if one exists) that sums to M . This

may be stated in the form of a Kl problem as below

$$\underline{\text{max}}: \quad \sum x_i \, S_i$$

$$\sum x_i \, S_i \leq M$$

$$x_i = 0,1$$

From this we trivially conclude that the general 0-1

integer programming problem is complete. The 0-1 constraint

may be replaced by the inequalities $x_i \leq 1$ $1 \leq i \leq n$.

[Karp [5] proves a similar result by showing Satisfiability α 0-1 Integer Programming]

PI: Quadratic Programming

Here, the constraints are linear while the optimization function is quadratic.

Theorem PI: PI is P-hard.

Proof Sum of subsets of integers α PI

maximize: $\sum_i x_i (x_i - 1) + \sum_i x_i s_i = f(x)$

subject to: $\sum_i x_i s_i \leq M$ ...   (i)

$$0 \leq x_i \leq 1$$

For $0 < x_i < 1$, $x_i(x_i - 1) < 0$

This, together with (i) implies $f(x) < M$

if for some i, $0 < x_i < 1$

max $f(x) = M$ iff S has a subset that sums to M .

The following variation of this problem may also be shown to be P-hard: linear programming with one nonlinear constraint. Call this problem PI(b) .

Sum of subsets α PI(b)

Just consider the formulation

$$\text{maximize:} \quad \Sigma \ x_i \ s_i$$

$$\text{subject to:} \quad \Sigma_i \ x_i \ s_i \leq M$$

$$\Sigma_i \ x_i \ (x_i - 1) \geq 0$$

$$0 \leq x_i \leq 1$$

4.5 Minimal Equivalent Boolean Form

<u>B1</u>: Given a formula B from the Propositional calculus we wish to find the shortest formula equivalent to it.

<u>Theorem B1</u>: B1 $\in$ PC

<u>Proof</u> (a) B1 α P1

Define B1k to be the problem: Is there a Boolean form of length k equivalent to B . We first show that a polynomial algorithm for P1 implies a polynomial algorithm for B1k . For this we construct a nondeterministic Turing Machine that guesses the Boolean form of length k and then uses the 'Tautology algorithm' to check that it is equivalent to B . If P1 works in $p(n)$ time then the 'Tautology algorithm' works in $p_2(n)$ (as Tautology α P1) time and so the Turing machine constructed above works in $p_2(n)$ time. Hence B1k α P1 . The proof for B1 α B1k is similar to G1(a) . We note that this proof relies heavily on our informal notion of P-reducibility. The proof does not show that B1 is polynomially related to the other problems in PC . If the time complexity of the Tautology problem is $f_1(n)$ and that of P1 $f_2(n)$ then this reduction gives a $f_2(f_1(n))$ algorithm

for B1 . If f_1 (and consequently f_2) is exponential then $f_2(f_1(n))$ is of the form 2^{2^n} . All our other reductions have been of the form $p(n) \cdot f_2(n)$ or $f_2(p(n))$ for some polynomial p .

(b) Tautology α B1

A formula P is a tautology iff its minimal form is '1' .

5 Conclusions

We have extended the class of known P-complete problems
to include some important applications from Network Flows,
Game Theory and Integer Programming. We have also introduced
the concept of P-hard and shown that Quadratic Programming
and Linear Programming with one non linear constraint are P-hard.
The results obtained indicate that many of the problems for
which no polynomial time bounded algorithm is known are related
in terms of time complexity. Indeed the complexity of these
problems may well be exponential. If such is the case then the
next step will be to search for subexponential algorithms and
to investigate the use of heuristics to improve the average
behavior of these algorithms. In [4] we look at the sum of
integers problem and the Knapsack problem. Subexponential
algorithms for these problems are obtained. We also study the
use of heuristics in solving these problems.

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