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SOME RELATIONS IN THE GENERALIZED KÄHLERIAN SPACES OF THE SECOND KIND

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Abstract

Starting from the definition of generalized Riemannian space (GR_N) [1], in which a non-symmetric basic tensor g_{ij} is introduced, in the present paper a generalized Kählerian space $G\underset{2}{K}_N$ of the second kind is defined, as a GR_N with almost complex structure F_i^h, that is covariantly constant with respect to the second kind of covariant derivative (equation (2.3)).

Several theorems are proved. These theorems are generalizations of the corresponding theorems relating to K_N. The relations between F_i^h and four curvature tensors from GR_N are obtained.

1 Introduction

A generalized Riemannian space GR_N in the sense of Eisenhart's definition [1] is a differentiable N-dimensional manifold, equipped with a non-symmetric basic tensor g_{ij}. Connection coefficients of this space are generalized Cristoffel's symbols of the second kind. Generally, $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

In a generalized Riemannian space one can define four kinds of covariant derivatives [3], [4]. For example, for a tensor a^i_j in GR_N we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{jm} a^i_p, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{mp} a^p_j - \Gamma^p_{mj} a^i_p,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{pm} a^p_j - \Gamma^p_{mj} a^i_p, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{mp} a^p_j - \Gamma^p_{jm} a^i_p.$$

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In the case of the space GR_N we have five independent curvature tensors [5]:

$$\underset{1}{R}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm},$$
$$\underset{2}{R}^i{}_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp},$$
$$\underset{3}{R}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}(\Gamma^i_{pj} - \Gamma^i_{jp}),$$
$$\underset{4}{R}^i{}_{jmn} = \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}(\Gamma^i_{pj} - \Gamma^i_{jp}),$$
$$\underset{5}{R}^i{}_{jmn} = \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{pn} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}).$$

Kählerian spaces and their mappings were investigated by many authors, for example K. Yano [14], [15], M. Prvanović [12], T. Otsuki [11], N. S. Sinyukov [13], J. Mikeš [2] and many others.

In [6], [7], [8] we defined a generalized Kählerian space GK_N as a generalized N-dimensional Riemannian space with a (non-symmetric) metric tensor

$$g_{ij} = g_{\underline{ij}} + g_{\underset{\vee}{ij}},$$

where $g_{\underline{ij}}$ is symmetric part, and $g_{\underset{\vee}{ij}}$ anty-symmetric one of the metric tensor.

The lowering and the raising of indices one defines by the tensors $g_{\underline{ij}}$ and $g^{\underline{ij}}$ respectively, where $g^{\underline{ij}}$ is defined by the equation

$$g_{\underline{ij}}g^{\underline{jk}} = \delta^k_i, \ i,k = 1,\dots,m.$$

and δ^k_i is Kronecker symbol. Therefore, since the matrix $(g^{\underline{ij}})$ is inverse for $(g_{\underline{ij}})$ it is necessary to be

$$g = det(g_{\underline{ij}}) \neq 0.$$

There exist an almost complex structure F^i_j such that

$$F^h_p(x)F^p_i(x) = -\delta^h_i\,, \tag{1.1}$$
$$g_{\underline{pq}}F^p_iF^q_j = g_{\underline{ij}}, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \tag{1.2}$$
$$F^h_{i\underset{\theta}{|}j} = 0, \quad (\theta = 1,2), \tag{1.3}$$

where $\underset{\theta}{|}$ denotes the covariant derivative of the kind θ with respect to the metric tensor g_{ij}.

In [9] we defined a generalized Kählerian space of the first kind $\underset{1}{GK}_N$ if there exists an almost complex structure $F^i_j(x)$, such that

$$F^h_p(x)F^p_i(x) = -\delta^h_i\,, \tag{1.4}$$
$$g_{\underline{pq}}F^p_iF^q_j = g_{\underline{ij}}, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \tag{1.5}$$
$$F^h_{i\underset{1}{|}j} = 0, \tag{1.6}$$

where $\underset{1}{|}$ denotes the covariant derivative of the first kind with respect to the metric tensor g_{ij}.

2 Generalized Kählerian spaces of the second kind

A generalized N-dimensional Riemannian space with (non-symmetric) metric tensor g_{ij} is a generalized Kählerian space of the second kind $G\underset{2}{K}_N$ if there exists an almost complex structure $F^i_j(x)$ such that

$$F^h_p(x)F^p_i(x) = -\delta^h_i\,, \tag{2.1}$$

$$g_{\underline{pq}}F^p_iF^q_j = g_{\underline{ij}}, \quad g^{\underline{ij}} = g^{\underline{pq}}F^i_pF^j_q, \tag{2.2}$$

$$F^h_{i\underset{2}{|}j} = 0, \tag{2.3}$$

where $\underset{2}{|}$ denotes the covariant derivative of the first kind with respect to the metric tensor g_{ij}. From (2.2), using (2.1), we get

$$F_{ij} = -F_{ji}, \quad F^{ij} = -F^{ji}, \tag{2.4}$$

where we denote

$$F_{ji} = F^p_j g_{\underline{pi}}, \quad F^{ji} = F^j_p g^{\underline{pi}}. \tag{2.5}$$

From here we prove the following theorems.

Theorem 2.1. *For the almost complex structure F^i_j of a generalized Kählerian space of the second kind the relations*

$$\begin{aligned} F^h_{i\underset{1}{|}j} &= 2(F^p_i\Gamma^h_{\underset{\vee}{pj}} + F^h_p\Gamma^p_{\underset{\vee}{ji}}),\\ F^h_{i\underset{3}{|}j} &= 2F^p_i\Gamma^h_{\underset{\vee}{pj}},\\ F^h_{i\underset{4}{|}j} &= 2F^h_p\Gamma^p_{\underset{\vee}{ji}} \end{aligned} \tag{2.6}$$

are valid, where $\Gamma^h_{\underset{\vee}{ij}}$ is the torsion tensor.

Proof. We get the relations (2.6) by using the condition (2.3) □

Let us denote $\overline{F}^h_{ij} = F^p_i\Gamma^h_{\underset{\vee}{jp}}$ and $\overline{\overline{F}}^h_{ij} = F^h_p\Gamma^p_{\underset{\vee}{ij}}$. Then we have

Theorem 2.2. *For the curvature tensors* $\underset{\theta}{R}$, $(\theta = 1, \cdots, 4)$ *of a generalized Kählerian space of the second kind the relations*

$$
\begin{aligned}
&F^p_i \underset{2}{R}^h{}_{pjk} = F^h_p \underset{2}{R}^p{}_{ijk},\\
&F^p_i \underset{1}{R}^h{}_{pij}k - F^h_p \underset{1}{R}^p{}_{ijk} - 4\Gamma^p_{\underset{\vee}{jk}}(F^q_i \Gamma^h_{\underset{\vee}{qp}} + F^h_q \Gamma^q_{\underset{\vee}{pi}})\\
&\quad = 2(\bar F^h_{ik\underset{1}{|}j} - \bar F^h_{ij\underset{1}{|}k} + \bar{\bar F}^h_{ik\underset{1}{|}j} - \bar{\bar F}^h_{ij\underset{1}{|}k}),\\
&F^p_i \underset{3}{R}^h{}_{pjk} - F^h_p \underset{3}{R}^p{}_{ijk} = -2(\bar F^h_{ij\underset{2}{|}k} + \bar{\bar F}^h_{ij\underset{2}{|}k}),\\
&F^p_i \underset{4}{R}^h{}_{pjk} + F^h_p \underset{3}{R}^p{}_{ijk} = 2(\bar{\bar F}^h_{ik\underset{3}{|}j} - \bar F^h_{ij\underset{4}{|}k})
\end{aligned}
\tag{2.7}
$$

are valid.

Proof. From (2.6) by using the covariant derivative of the first kind we have

$$F^h_{i\underset{1}{|}jk} = -2\bar F^h_{ij\underset{1}{|}k} - 2\bar{\bar F}^h_{ij\underset{1}{|}k}, \tag{2.8}$$

and also

$$F^h_{i\underset{1}{|}kj} = -2\bar F^h_{ik\underset{1}{|}j} - 2\bar{\bar F}^h_{ik\underset{1}{|}j}. \tag{2.9}$$

Now, from (2.8) and (2.9) we obtain

$$F^h_{i\underset{1}{|}jk} - F^h_{i\underset{1}{|}kj} = 2(\bar F^h_{ik\underset{1}{|}j} - \bar F^h_{ij\underset{1}{|}k} + \bar{\bar F}^h_{ik\underset{1}{|}j} - \bar{\bar F}^h_{ij\underset{1}{|}k}). \tag{2.10}$$

Using the Ricci identity [5], we get from (2.10)

$$
\begin{aligned}
&F^p_i \underset{1}{R}^h{}_{pjk} - F^h_p \underset{1}{R}^p{}_{ijk} - 2\Gamma^p_{\underset{\vee}{jk}} F^h_{i\underset{1}{|}p}\\
&\quad = 2(\bar F^h_{ik\underset{1}{|}j} - \bar F^h_{ij\underset{1}{|}k} + \bar{\bar F}^h_{ik\underset{1}{|}j} - \bar{\bar F}^h_{ij\underset{1}{|}k}),
\end{aligned}
\tag{2.11}
$$

and from here the second equality (2.7) is valid.

The first equality we get directly from the Ricci identity obtained by virtue of the second kind of covariant derivative and using (2.3).

By the same procedure like in the previous two cases, it is easy to prove the third and the fourth equation. □

If we denote with (;) covariant derivative wrt the symmetric connection, then the next theorem follows.

Theorem 2.3. *For the Ricci tensor* R_{ij}*, given by* $g_{\underline{ij}}$ *the relation*

$$R_{hk} = F^p_h F^q_k R_{pq} - g^{\underline{pq}} F^s_h (\mathcal{D}_{s.pqk} + \mathcal{D}_{k.pqs}) \tag{2.12}$$

is valid, where

$$\begin{aligned} \mathcal{D}^h{}_{ijk} &= F^p_{i;k} \Gamma^h_{\underset{\vee}{jp}} - F^p_{i;j} \Gamma^h_{\underset{\vee}{kp}} + F^p_i (\Gamma^h_{\underset{\vee}{jp};k} - \Gamma^h_{\underset{\vee}{kp};j}) \\ &\quad + F^h_{p;k} \Gamma^p_{\underset{\vee}{ij}} - F^h_{p;j} \Gamma^p_{\underset{\vee}{ik}} + F^h_p (\Gamma^p_{\underset{\vee}{ij};k} - \Gamma^p_{\underset{\vee}{ik};j}), \end{aligned} \tag{2.13}$$

and $R_{hk} = R^p_{hkp}$, $\mathcal{D}_{h.ijk} = g_{\underline{ph}} \mathcal{D}^p{}_{ijk}$.

Proof. From (2.3) and (2.6) we get

$$F^h_{i;j} = -(F^p_i \Gamma^h_{\underset{\vee}{jp}} + F^h_p \Gamma^p_{\underset{\vee}{ij}}). \tag{2.14}$$

The integrability conditions of the equation (2.14) are given by

$$F^h_{i;jk} - F^h_{i;kj} = -\mathcal{D}^h{}_{ijk}. \tag{2.15}$$

Using the Ricci identity, from (2.15) we obtain

$$F^h_p R^p{}_{ijk} - F^p_i R^h{}_{pjk} = -\mathcal{D}^h{}_{ijk}. \tag{2.16}$$

Here $R^h{}_{ijk}$ is the curvature tensor with respect to the symmetric affine connection $\Gamma^h_{\underline{ij}}$. Composition with F^i_r in (2.16) gives

$$F^h_p F^q_i R^p{}_{qjk} + R^h{}_{ijk} = -F^p_i \mathcal{D}^h{}_{pjk}. \tag{2.17}$$

Now, from (2.17) by composition with $g_{\underline{hr}}$ we get

$$F_{ph} F^q_i R^p{}_{qjk} + R_{hijk} = -F^p_i \mathcal{D}_{h.pjk}. \tag{2.18}$$

From here we obtain

$$-F^h_p F^q_i R_{pqjk} + R_{hijk} = -F^p_i \mathcal{D}_{h.pjk}. \tag{2.19}$$

From (2.19) by composition with F^i_r we have

$$F^p_h R_{pijk} - F^p_i R_{phjk} = \mathcal{D}_{h.ijk}. \tag{2.20}$$

Using composition with $g^{\underline{ij}}$ in (2.20) we obtain

$$F^p_h R_{pk} - F^p_q R_{ph.k}{}^{q} = g^{\underline{pq}} \mathcal{D}_{h.pqk}. \tag{2.21}$$

The symmetrization in (2.21) with respect to h, k gives the relation (2.12). □

Theorem 2.4. *The Ricci tensors* $\underset{\theta}{R}_{jm}$ $(\theta = 1, \cdots, 5)$ *of the space* $G\underset{2}{K}_N$ *satisfy the relations*

(2.22 a, b, c)
$$\underset{\alpha}{R}_{(pq)}F^p_j F^q_m = \underset{\alpha}{R}_{(jm)} - 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}}F^r_j F^s_m + 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}} + 2g^{\underline{pq}}F^s_h(\mathcal{D}_{s.pqk} + \mathcal{D}_{k.pqs}),\ \ \alpha = 1, 2, 3,$$

(2.22 d)
$$\underset{4}{R}_{(pq)}F^p_j F^q_m = \underset{4}{R}_{(jm)} + 6\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}}F^r_j F^s_m - 6\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}} + 2g^{\underline{pq}}F^s_h(\mathcal{D}_{s.pqk} + \mathcal{D}_{k.pqs}),$$

(2.22 e)
$$\underset{5}{R}_{(pq)}F^p_j F^q_m = \underset{5}{R}_{(jm)} + 2\Gamma^p_{\underset{\vee}{rq}}\Gamma^q_{\underset{\vee}{ps}}F^r_j F^s_m - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}} + 2g^{\underline{pq}}F^s_h(\mathcal{D}_{s.pqk} + \mathcal{D}_{k.pqs}),$$

where (jm) *denotes the symmetrization without division with respect to the indices* j, m.

Proof. (a) We can express the tensor $\underset{1}{R}{}^i{}_{jmn}$ in the form [5]:

$$\underset{1}{R}{}^i{}_{jmn} = R^i{}_{jmn} + \Gamma^i_{\underset{\vee}{jm};n} - \Gamma^i_{\underset{\vee}{jn};m} + \Gamma^p_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{pn}} - \Gamma^p_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{pm}}.$$

By contraction with respect to the indices i, n, and by symmetrization with respect to j, m, we get

$$\underset{1}{R}_{(jm)} = R_{(jm)} - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}. \tag{2.23}$$

From (2.12) and (2.23) we have (2.22a).

(b) The tensor $\underset{2}{R}{}^i{}_{jmn}$ can be expressed in the form [5]:

$$\underset{2}{R}{}^i{}_{jmn} = R^i{}_{jmn} - \Gamma^i_{\underset{\vee}{jm};n} + \Gamma^i_{\underset{\vee}{jn};m} - \Gamma^p_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{pn}} + \Gamma^p_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{pm}}.$$

By contraction with respect to i, n, and then by symmetrization with respect to j, m, we get

$$\underset{2}{R}_{(jm)} = R_{(jm)} - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}},$$

from where, using (2.12), we get the relation (2.22b).

(c) For the tensor $\underset{3}{R}{}^i{}_{jmn}$ we have [5]:

$$\underset{3}{R}{}^i{}_{jmn} = R^i{}_{jmn} + \Gamma^i_{\underset{\vee}{jm};n} + \Gamma^i_{\underset{\vee}{jn};m} - \Gamma^p_{\underset{\vee}{jm}}\Gamma^i_{\underset{\vee}{pn}} + \Gamma^p_{\underset{\vee}{jn}}\Gamma^i_{\underset{\vee}{pm}} - 2\Gamma^p_{\underset{\vee}{mn}}\Gamma^i_{\underset{\vee}{pj}}.$$

Contracting with respect to i, n, and then symmetrizing in relation to j, m, we get

$$\underset{3}{R}_{(jm)} = R_{(jm)} - 2\underset{\vee}{\Gamma}^p_{jq}\underset{\vee}{\Gamma}^q_{pm},$$

from where, using (2.12), we can see that the relation (2.22c) is valid.

(d) The tensor $\underset{4}{R}^i{}_{jmn}$ can be expressed in the form [5]:

$$\underset{4}{R}^i{}_{jmn} = R^i{}_{jmn} + \underset{\vee}{\Gamma}^i_{jm;n} + \underset{\vee}{\Gamma}^i_{jn;m} - \underset{\vee}{\Gamma}^p_{jm}\underset{\vee}{\Gamma}^i_{pn} + \underset{\vee}{\Gamma}^p_{jn}\underset{\vee}{\Gamma}^i_{pm} + 2\underset{\vee}{\Gamma}^p_{mn}\underset{\vee}{\Gamma}^i_{pj}.$$

Contracting with respect to i, n, and symmetrizing with respect to j, m, we get

$$\underset{4}{R}_{(jm)} = R_{(jm)} + 6\underset{\vee}{\Gamma}^p_{jq}\underset{\vee}{\Gamma}^q_{pm}.$$

Using (2.12) we get the relation (2.22d).

(e) The tensor $\underset{5}{R}^i{}_{jmn}$ satisfies the relation [5]:

$$\underset{5}{R}^i{}_{jmn} = R^i{}_{jmn} + \underset{\vee}{\Gamma}^p_{jm}\underset{\vee}{\Gamma}^i_{pn} + \underset{\vee}{\Gamma}^p_{jn}\underset{\vee}{\Gamma}^i_{pm}.$$

Contracting with respect the indices i, n, and than symmetrizing with respect to j, m, we get

$$\underset{5}{R}_{(jm)} = R_{(jm)} + 2\underset{\vee}{\Gamma}^p_{jq}\underset{\vee}{\Gamma}^q_{pm},$$

from where, using (2.12), we get (2.22e). □

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