Ochiai, T.
Osaka J. Math.
11 (1974), 483-501

SOME REMARK ON THE DEFECT RELATION OF HOLOMORPHIC CURVES

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(Received December 24, 1973)

1. Introduction. Since R. Nevanlinna established the value distribution theory for meromorphic functions in 1925 ([7]), many extensive works have been done for its generalization in one way or another. One of the far reaching generalization was given by H. Cartan ([3]), H. and J. Weyl ([9]) and L.V. Ahlfors ([1]), which is well-known as a classical theory on the defect relation of holomorphic curves. When we formulate their theory in the relatively new language of the holomorphic line bundles as was done originally by S.S. Chern, we strongly suspect that still further development should be possible. However no substantial progress has been made yet beyond their result. Therefore the author believes that it is of some use to give certain result in this direction though it is rather direct from the classical theory. Thus the purpose of this paper is to explain it in somewhat self-contained manner.

Let $F:\boldsymbol{C}\to\boldsymbol{P}^n$ be a holomorphic mapping where $\boldsymbol{C}$ is the complex line and $\boldsymbol{P}^n$ is the n-dimensional complex projective space. We assume F to be *non-degenerate* in the sense that the image $F(\boldsymbol{C})$ does not belong to a hyperplane. Then for each hyperplane Φ we can define a *defect* $\delta_F(\Phi)$, having the properties; 1) $0\leqq\delta_F(\Phi)\leqq 1$; 2) $\delta_F(\Phi)=1$ if $F(\boldsymbol{C})\cap\Phi$ is empty. Roughly speaking $\delta_F(\Phi)$ measures how often $F(\boldsymbol{C})$ intersects with Φ. Then for a set of hyperplanes $\Phi_j(1\leqq j\leqq q)$ in general position, we have

$$\sum_{j=1}^{q}\delta_F(\Phi_j)\leqq n+1 .$$

The above is a very brief outline of the classical theory. Now we remark that the set of all hyperplanes is the complete linear system of divisors given by the hyperplane bundle over $\boldsymbol{P}^n$. Then we are ready to consider the following situation. Let M be a connected compact complex manifold and $\boldsymbol{L}$ be a holomorphic line bundle over M. Let V be a linear subspace of the space $\Gamma(M, \boldsymbol{L})$ of all holomorphic cross-sections of $\boldsymbol{L}$. Each non-zero element ϕ in V defines

1) This work was partially supported by the Sakkokai Foundation, Tokyo, Japan.

a divisor $[\phi]$. Thus V gives us a linear system of divisors $[V]$. Suppose we are given a holomorphic mapping $f: \boldsymbol{C} \to M$ which is *non-degenerate* in the sense that $f(\boldsymbol{C})$ does not belong to a divisor in $[V]$. Now we shall show that for a divisor $[\phi] \in [V]$ we can define a *defect* $\delta_f([\phi])$ having the property; 1) $0 \leqq \delta_f([\phi]) \leqq 1$; 2) $\delta_f([\phi]) = 1$ if $f(\boldsymbol{C}) \cap [\phi]$ is empty. Our main results are Theorem 7.1 and Theorem 7.2 in the section 7;

(1) There exists a positive number e $(0 \leqq e \leqq 1)$ such that $\delta_f([\phi]) \geqq e$ for $[\phi] \in [V]$.

(2) We have $\delta_f([\phi]) = e$ for almost all $[\phi]$ with respect to some canonical positive measure on $[V]$.

(3) For a set of divisors $[\phi_j] \in [V]$ $(1 \leqq j \leqq q)$ in general position, we have

$$\sum_{j=1}^{q} (\delta_f([\phi_j]) - e) \leqq (1-e) \dim V,$$

under some condition on f (*e.g.*, if f is transcendental and of finite type).

In the sections up to six we gather some materials which seem to be more or less known and give a proof to some of them in such a way that it fits to our purpose. More precisely in the section 2 we fix some notations. In the sections 3 and 4 we make an analytic preparation and give a proof to what is essentially the same as the classical Jensen formula. The section 5 is devoted to prove the so-called first fundamental theorem. We remark here that a more generalized version of this theorem is given in [5] and [8]. In the section 6 we recall the classical theory for the case of the projective space. In the section 7 we shall give a proof to our theorems. The final section 8 is for some remark.

Finally the author should mention that he has been strongly influenced by [10] and that he has implicitly made use of some idea from there.

2. Hermitian line bundles. Let M be a connected compact complex manifold. Let $p: \boldsymbol{L} \to M$ be a holomorphic line bundle over M. We denote by $\Gamma(M, \boldsymbol{L})$ the complex vector space of all holomorphic cross-sections of $p: \boldsymbol{L} \to M$. We know that $\Gamma(M, \boldsymbol{L})$ is of finite dimension. For an element ϕ in $\Gamma(M, \boldsymbol{L})$ we denote by $[\phi]$ the divisor of M defined by ϕ. We define *supp* (ϕ), called the *support* of ϕ, by

$$supp\ (\phi) = \{z \in M \mid \phi(z) = 0\}.$$

By a *hermitian fibre metric* h on $\boldsymbol{L}$ we mean a $\boldsymbol{C}^\infty$-mapping $h: \boldsymbol{L} \to \boldsymbol{R}$ such that its restriction onto each fibre $p^{-1}(z)$ is a positive definite hermitian quadratic form. We call a pair $(\boldsymbol{L}, h)$ *a hermitian line bundle* over M, and for a Z in $\boldsymbol{L}$ we simply put $\|Z\| = h(Z)^{1/2}$.

Let $\boldsymbol{L} - 0$ denote the bundle space $\boldsymbol{L}$ minus its zero section. We define a closed 2-form Ω on $\boldsymbol{L} - 0$ by

(2.1) $$\Omega = -\frac{1}{4\pi} dd^c \log h ,$$

where $d^c = i(\bar{\partial} - \partial)$. It is easy to see that there exists uniquely a closed 2-form of type (1.1) ω on M such that

(2.2) $$p^*\omega = \Omega \quad \text{on} \quad \boldsymbol{L} - 0 .$$

The 2-form ω is called the *Chern form* of the hermitian line bundle $(\boldsymbol{L}, h)$ ([2]). Let ϕ be a non-zero element of $\Gamma(M, \boldsymbol{L})$. Then clearly we have

(2.3) $$\omega = -\frac{1}{2\pi} dd^c \log \|\phi\| \quad \text{on} \quad M - supp\ (\phi) .$$

We denote by $\boldsymbol{L}^*$ the dual bundle of $\boldsymbol{L}$. Then $\boldsymbol{L}^*$ has the hermitian fibre metric h^* naturally induced from h. We call $(\boldsymbol{L}^*, h^*)$ the *dual hermitian line bundle* of $(\boldsymbol{L}, h)$. The Chern form ω^* of $(\boldsymbol{L}^*, h^*)$ is given by

(2.4) $$\omega^* = -\omega .$$

Let us consider some example. We denote by $\boldsymbol{C}^{n+1}$ the $(n+1)$-dimensional complex euclidean space, *i.e.*, $\boldsymbol{C}^{n+1} = \{(z^0, z^1, \cdots, z^n) \mid z^j \in \boldsymbol{C}\}$. If $Z = (z^0, \cdots, z^n)$ and $W = (w^0, \cdots, w^n)$, then the canonical inner product on $\boldsymbol{C}^{n+1}$ is

$$\langle Z, W \rangle = z^0 \bar{w}^0 + \cdots + z^n \bar{w}^n ,$$

and the canonical symmetric bilinear form on $\boldsymbol{C}^{n+1}$ is

$$(Z, W) = z^0 w^0 + \cdots + z^n w^n .$$

We denote by $\boldsymbol{P}^n$ the quotient $(\boldsymbol{C}^{n+1} - \{0\})/\boldsymbol{C}^*$ by the multiplicative group $\boldsymbol{C}^*$ of non-zero complex numbers acting on $\boldsymbol{C}^{n+1} - \{0\}$. Clearly $\boldsymbol{C}^{n+1} - \{0\}$ is a holomorphic principal bundle over $\boldsymbol{P}^n$. Let $p_0 : \boldsymbol{L}_0 \to \boldsymbol{P}^n$ be its associated line bundle over $\boldsymbol{P}^n$. We remark that $\boldsymbol{L}_0 - 0$ can be naturally identified with $\boldsymbol{C}^{n+1} - \{0\}$. Define a mapping $h_0 : \boldsymbol{L}_0 \to \boldsymbol{R}$ by

$$h_0(Z) = \begin{cases} \langle Z, Z \rangle & \text{for} \quad Z \in \boldsymbol{L}_0 - 0 = \boldsymbol{C}^{n+1} - \{0\} , \\ 0 & \text{on the zero-section.} \end{cases}$$

Then h_0 is a hermitian fibre metric on $\boldsymbol{L}_0$. $\boldsymbol{P}^n$ is called the n-dimensional *complex projective space* and the hermitian line bundle $(\boldsymbol{L}_0, h_0)$ is called the *tautological line bundle* of $\boldsymbol{P}^n$. The dual hermitian line bundle $(\boldsymbol{L}_0^*, h_0^*)$ is called the *hyperplane bundle* of $\boldsymbol{P}^n$. We shall see that the vector space $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$ is naturally isomorphic to $\boldsymbol{C}^{n+1}$. In fact let Φ be any element of $\boldsymbol{C}^{n+1}$. Through the identification $L_0 - 0$ with $C^{n+1} - \{0\}$, Φ define a mapping $\phi : \boldsymbol{L}_0 - 0 \to \boldsymbol{C}$ by the formula,

$$\phi(Z) = (\Phi, Z)\,.$$

We now extend the definition of ϕ to entire $\boldsymbol{L}_0$ by putting zero on the zero-section. It is trivial to see that $\phi: \boldsymbol{L}_0 \to \boldsymbol{C}$ is holomorphic and linear on each fibre. Thus ϕ defines an element of $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$. Through the correspondence $\Phi \mapsto \phi$, $\boldsymbol{C}^{n+1}$ is identified with $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$. Let Z be any point in $\boldsymbol{L}_0 - 0 = \boldsymbol{C}^{n+1} - \{0\}$. Then we have

$$\|\phi(p_0(Z))\|^2 = \frac{|(\Phi, Z)|^2}{\langle Z, Z\rangle} \tag{2.5}$$

and

$$supp(\phi) = p_0(\{Z \in \boldsymbol{C}^{n+1} - \{0\} \mid (\Phi, Z) = 0\})\,. \tag{2.6}$$

We have *supp* $(\phi) = [\phi]$, and call *supp* (ϕ) a *hyperplane* in $\boldsymbol{P}^n$. Let ω_0 be the Chern form of $(\boldsymbol{P}^n, \boldsymbol{L}_0)$. Then from (2.1) and (2.2) we have

$$p_0^*\omega_0 = -\frac{1}{4\pi} dd^c \log \langle Z, Z\rangle \quad \text{on} \quad \boldsymbol{C}^{n+1} - \{0\}\,. \tag{2.7}$$

Let ω_0^* be the Chern form of $(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$. From (2.4) and (2.7) we have

$$p_0^*\omega_0^* = \frac{1}{4\pi} dd^c \log \langle Z, Z\rangle \quad \text{on} \quad \boldsymbol{C}^{n+1} - \{0\}\,. \tag{2.8}$$

It is well-known that $(\omega_0^*)^n = \omega_0^* \wedge \cdots \wedge \omega_0^*$ (n-times exterior product) is a volume element on $\boldsymbol{P}^n$ and we have

$$\int_{\boldsymbol{P}^n} (\omega_0^*)^n = 1\,. \tag{2.9}$$

3. Analytic preliminaries. For a real number r we put $D(r) = \{\zeta \in \boldsymbol{C} \mid |\zeta| < e^r\}$ and $B(r) = \{\zeta \in \boldsymbol{C} \mid |\zeta| = e^r\}$. Let $g(\zeta)$ be an entire function different from zero-constant. Then it is well-known that for any real number r, the function $\theta \mapsto \log|g(e^{r+i\theta})|$ is Lebesgue measurable and integrable for $0 \leqq 2\theta \leqq \pi$. Thus we put

$$m(g; r) = \frac{1}{2\pi}\int_0^{2\pi} \log|g(e^{r+i\theta})|\,d\theta\,.$$

Then, as is well-known, we have

(3.1) $m(g; r)$ *is a continuous and increasing function of* r.

Lemma 3.1. *Let* $\{g_j(\zeta)\}_{j=1,2,\dots}$ *be a family of entire functions different from zero-constant which converges uniformly on any compact subset to an entire function* $g(\zeta)$ *different from zero-constant. Then for any fixed* r, $\{m(g_j; r)\}_{j=1,2,\dots}$ *converges to* $m(g; r)$.

Proof. In the case $g(\zeta)$ has no zero on $B(r)$, Lemma 3.1 is easy to see. Since the zeros of $g(\zeta)$ is discrete, there exists $\varepsilon>0$ such that $g(\zeta)$ has no zero on $B(t)$ for $0<|t-r|<\varepsilon$. Then for any s $(r-\varepsilon<s<r)$ we have $m(g_j; s)\leqq m(g_j; r)$ from (3.1) and $\lim_{j\to\infty} m(g_j; s)=m(g; s)$. Thus we see $m(g; s)\leqq \liminf_{j\to\infty} m(g_j; r)$. On the other hand for any t $(r<t<r+\varepsilon)$ we have $m(g_j; r)\leqq m(g_j; t)$ and $\lim_{j\to\infty} m(g_j; t)=m(g; t)$. Hence $\limsup_{j\to\infty} m(g_j; r)\leqq m(g; t)$. Therefore we have $m(g; s)\leqq \liminf_{j\to\infty} m(g_j; r)\leqq \limsup_{j\to\infty} m(g_j; r)\leqq m(g; t)$ for $r-\varepsilon<s<r<t<r+\varepsilon$. From the continuity property in (3.1) we have $\lim_{j\to\infty} m(g_j; r)=m(g; r)$. q.e.d.

Let $A(\zeta)$ be a function on $\boldsymbol{C}$ satisfying the condition;

(3.2) *$A(\zeta)$ has an expression of the form*

$$A(\zeta) = |g(\zeta)|a(\zeta)$$

where $g(\zeta)$ is an entire function different from zero-constant and $a(\zeta)$ is positive and C^∞. We define a function μ on $\boldsymbol{C}$ by

$$\mu(\zeta) = \begin{cases} 0 & \text{if } g(\zeta)\neq 0 \\ \text{the order of the zero at } \zeta & \text{if } g(\zeta) = 0\,. \end{cases}$$

$\mu(\zeta)$ is determined by $A(\zeta)$ and independent of the particular choice $g(\zeta)$ in (3.2). We call $\mu(\zeta)$ the *multiplicity function* of $A(\zeta)$. We put $n(A; r)=\sum_{\zeta\in D(r)} \mu(\zeta)$. Then $n(A; r)$ is an increasing and upper semi-continuous function of r. In particular we have

(3.3) *$n(A; r)$ is a Lebesgue measurable function on $\boldsymbol{R}$.*

We define $N(A; r)$ $(r\geqq 0)$ by

$$N(A; r) = \int_0^r n(A; t)dt\,. \tag{3.4}$$

We call $N(A; r)$ the *counting function* of $A(\zeta)$. On the other hand for any fixed r, the function $\theta\mapsto \log|A(e^{r+i\theta})|$ is clearly Lebesgue measurable and integrable. Put

$$m(A; r) = \frac{1}{2\pi}\int_0^{2\pi} \log A(e^{r+i\theta})d\theta\,. \tag{3.5}$$

Then from (3.1) and (3.2) we have

(3.6) *$m(A; r)$ is a continuous function of r.*

We call $m(A; r)$ the *proximity function* of $A(\zeta)$. Since we have $dd^c \log A= dd^c \log a$, we see

(3.7) *$dd^c \log A$ is a smooth 2-form on $\boldsymbol{C}$.*

Hence for $r \geqq 0$ we put

$$T(A; r) = -\frac{1}{2\pi}\int_0^r dt \int_{D(t)} dd^c \log A . \tag{3.8}$$

We call $T(A; r)$ the *characteristic function* of $A(\zeta)$.

Lemma 3.2. *Let the notations be as above. Then for $r>0$ we have*

$$T(A; r) - m(A; 0) = N(A; r) - m(A; r) .$$

Proof. (i) Let $r>0$ be such that $A(\zeta)$ has no zero on $B(r)$. Let $\zeta_1, \cdots, \zeta_l$, be all the distinct zeros of $A(\zeta)$ in $D(r)$. For $\varepsilon>0$ we put $D(j, \varepsilon)=\{\zeta \in \boldsymbol{C} \mid |\zeta-\zeta_j|<\varepsilon\}$ and $B(j, \varepsilon)=\{\zeta \in \boldsymbol{C} \mid |\zeta-\zeta_j|=\varepsilon\}$ $(1 \leqq j \leqq l)$. If we take ε sufficiently small, the Stokes theorem implies

$$\begin{aligned} -\frac{1}{2\pi}\int_{D(r)-\sum_{j=1}^{l} D(j, \varepsilon)} dd^c \log A = & -\frac{1}{2\pi}\int_{B(r)} d^c \log A \\ & + \sum_{j=1}^{l} \frac{1}{2\pi}\int_{B(j, \varepsilon)} d^c \log A . \end{aligned} \tag{3.9}$$

If $w=u+iv$ is a holomorphic local co-ordinate defined on an open subset U of $\boldsymbol{C}$, we have

$$d^c \eta = \frac{\partial \eta}{\partial u} dv - \frac{\partial \eta}{\partial v} du \tag{3.10}$$

for a C^∞-function η on U. On the other hand near the point ζ_j $(1 \leqq j \leqq l)$, $A(\zeta)$ has the form

$$A(\zeta) = |\zeta-\zeta_j|^{\mu(j)} a_j(\zeta)$$

where $\mu(j)=\mu(\zeta_j)$ and $a_j(\zeta)$ is positive and C^∞. Thus we have

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi}\int_{B(j, \varepsilon)} d^c \log A = \lim_{\varepsilon \to 0} \frac{\mu(\zeta_j)}{2\pi}\int_{B(j, \varepsilon)} d^c \log |\zeta-\zeta_j| .$$

Putting $\zeta-\zeta_j=e^{u+iv}$, (3.10) implies

$$\frac{1}{2\pi}\int_{B(j, \varepsilon)} d^c \log |\zeta-\zeta_j| = 1 .$$

Hence we have

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi}\int_{B(j, \varepsilon)} d^c \log A = \mu(\zeta_j) \qquad (1 \leqq j \leqq l) . \tag{3.11}$$

Putting $\zeta=e^{t+i\theta}$, (3.10) together with the assumption on r implies

$$(3.12)\qquad \frac{1}{2\pi}\int_{B(r)} d^c \log A = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial t}\log A\right)(e^{r+i\theta})d\theta = \frac{1}{2\pi}\left[\frac{\partial}{\partial t}\int_0^{2\pi}\log A\ (e^{t+i\theta})d\theta\right]_{t=r}.$$

Therefore we have from (3.9), (3.11) and (3.12)

$$-\frac{1}{2\pi}\int_{B(r)} dd^c \log A = \lim_{\varepsilon\to 0}\left\{-\frac{1}{2\pi}\int_{D(r)-\sum_{j=1}^{l}D(j,\varepsilon)} dd^c \log A\right\} = -\left[\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\log A(e^{t+i\theta})d\theta\right]_{t=r} + \sum_{j=1}^{l}\mu(\zeta_j).$$

Thus we have

$$(3.13)\qquad -\frac{1}{2\pi}\int_{D(r)} dd^c \log A = -\left[\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi}\log A(e^{t+i\theta})d\theta\right]_{t=r} + n(A;r).$$

(ii) Let $0<s<r$ be such that there is no zero of A in $\{\zeta\in\boldsymbol{C}\,|\,e^s\leqq|\zeta|\leqq e^r\}$. Then integrating (3.13) we have

$$(3.14)\qquad -\frac{1}{2\pi}\int_s^r dt\int_{D(t)} dd^c \log A = -m(A;r)+m(A;s)+\int_s^r n(A;t)dt.$$

(iii) From (3.6) it is easy to see that (3.14) implies

$$-\frac{1}{2\pi}\int_s^r dt\int_{D(t)} dd^c \log A = -m(A;r)+m(A;s)+\int_s^r n(A;t)dt.$$

for arbitrary $0\leqq s<r$. In particular we have

$$T(A;r) = -m(A;r)+m(A;0)+N(A;r). \qquad \text{q.e.d.}$$

4. The order and counting functions. Let M be a connected compact complex manifold. Let $(\boldsymbol{L}, h)$ be a hermitian line bundle over M. Let f: $\boldsymbol{C}\to M$ be a holomorphic mapping.

Lemma 4.1. *Take $\phi\in\Gamma(M, \boldsymbol{L})$ such that $||\phi\circ f||$ is not zero-constant on $\boldsymbol{C}$. Then we have*

(i) *there exists an entire function $g(\zeta)$ such that $\zeta\in\boldsymbol{C}\mapsto g(\zeta)^{-1}\,\phi(f(\zeta))\in\boldsymbol{L}$ is a non-vanishing holomorphic mapping, and in particular*

(ii) *the function $||\phi\circ f(\zeta)||$ satisfies the condition* (3.2).

Proof. From the local triviality of the bundle $\boldsymbol{L}$, we see that near a point $\zeta_0\in\boldsymbol{C}$, $||\phi\circ f||$ has the form

$$(4.1) \qquad ||\phi \circ f(\zeta)|| = |\zeta-\zeta_0|^{\nu} a_0(\zeta)$$

where $a_0(\zeta)$ is positive and C^∞ and ν is a non-negative integer. If ν is positive, ζ_0 is to be a *zero* of $||\phi\circ f||$ with the order ν. From (4.1) the zeros of $||\phi\circ f||$ are discrete in $\boldsymbol{C}$. Then Weierstrass Theorem implies that there exists an entire function $g(\zeta)$ whose zeros are exactly those of $||\phi\circ f||$ with the same order. From (4.1) we see easily that $g(\zeta)^{-1}\phi(f(\zeta))$ is a non-vanishing holomorphic mapping. q.e.d.

Let ϕ be in $\Gamma(M, \boldsymbol{L})$ satisfying the condition;

(4.2) $||\phi \circ f||$ *is not zero-constant on* $\boldsymbol{C}$.

Then from (ii) of Lemma 4.1 we have the multiplicity function $\mu(\zeta)$ of $||\phi\circ f||$ as in the section 3. We call $\mu(\zeta)$ to be the *intersection multiplicity* of $f(\boldsymbol{C})$ and the divisor $[\phi]$ at the point $f(\zeta)$. Also we have the functions $n(||\phi\circ f||; r)$, $N(||\phi\circ f||; r)$, $m(||\phi\circ f||; r)$ and $T(||\phi\circ f||; r)$ for $r>0$ as explained in the section 3.

DEFINITION 4.1. For a $\phi\in\Gamma(M, \boldsymbol{L})$ satisfying (4.2), we put for $r\geqq 0$

$$(4.3) \qquad n_f(\phi; r) = n(||\phi\circ f||; r) = \sum_{\zeta\in D(r)}\mu(\zeta)\,,$$

$$(4.4) \qquad N_f(\phi; r) = N(||\phi\circ f||; r) = \int_0^r n_f(\phi; t)dt\,,$$

$$(4.5) \qquad m_f(\phi; r) = m(||\phi\circ f||; r) = \frac{1}{2\pi}\int_0^{2\pi} \log ||\phi(f(e^{r+i\theta}))||d\theta\,,$$

and

$$(4.6) \qquad T_f(\phi; r) = T(||\phi\circ f||; r) = -\frac{1}{2\pi}\int_0^r dt\int_{D(t)} dd^c \log||\phi\circ f||\,.$$

We call $N_f(\phi; r)$ (resp. $m_f(\phi; r)$) the *counting* (resp. *proximity*) *function* of ϕ.

Let ω be the Chern form of the hermitian line bundle $(\boldsymbol{L}, h)$. For $r>0$ we put

$$(4.7) \qquad T_f(r) = \int_0^r dt\int_{D(t)} f^*\omega\,.$$

DEFINITION 4.2. We call $T_f(r)$ the *order function* of f.

Lemma 4.2. *Let* $\phi\in\Gamma(M, \boldsymbol{L})$ *satisfy* (4.2). *Then we have*

$$T_f(r) = T_f(\phi; r) \qquad \text{for } r>0\,.$$

Proof. From (2.3) we have $f^*\omega=-(1/2\pi)\, dd^c \log ||\phi\circ f||$ on $\boldsymbol{C}$ except the zeros of $||\phi\circ f||$. Then Lemma 4.2 follows from (3.7), (3.8) and (4.6). q.e.d.

Corollary. *If* $\phi \in \Gamma(M, \boldsymbol{L})$ *satisfies* (4.2), *then for* $r>0$ *we have*

$$T_f(r)-m_f(\phi;\, 0) = N_f(\phi;\, r)-m_f(\phi;\, r)\,.$$

Proof. This follows from Lemma 3.2 and Lemma 4.2. q.e.d.

5. The first fundamental theorem. Let $(\boldsymbol{L}, h)$ be a hermitian line bundle over a connected compact complex manifold M. Let V be a fixed linear subspace of $\Gamma(M, \boldsymbol{L})$. We consider V as a normed space by the following norm;

$$norm(\phi) = \sup_{z\in M} \|\phi(z)\| \qquad \text{for } \phi\in V\,. \tag{5.1}$$

Let f: $\boldsymbol{C}\to M$ be a holomorphic mapping. We say f to be *non-degenerate with respect to* V if any $\phi\in V-\{0\}$ satisfies the condition (4.2), or equivalently $f(\boldsymbol{C})$ does not belong to $supp(\phi)$ for $\phi\in V-\{0\}$. In this section we assume that f is non-degenerate with respect to V.

Lemma 5.1. *The mapping* $\phi\in V-\{0\}\mapsto m_f(\phi;r)\in \boldsymbol{R}$ *is continuous for* $r\geqq 0$.

Proof. Let $\{\phi_j\}_{j=1,2,\dots}$ be a sequence in $V-\{0\}$ which converges to $\phi\in V-\{0\}$. From Lemma 4.1 there exists an entire function $g(\zeta)$ such that $\eta(\zeta)=g(\zeta)^{-1}\phi(f(\zeta))$ is a non-vanishing holomorphic mapping. Then from the local triviality of the bundle $\boldsymbol{L}$, it is easy to see that there exist entire functions $g_j(\zeta)$ $(j=1,2,\cdots)$ such that

$$\phi_j\circ f(\zeta) \doteq g_j(\zeta)\eta(\zeta) \qquad \text{for } \zeta\in\boldsymbol{C}\ (j=1,2,\cdots)\,. \tag{5.2}$$

Since $\phi\circ f(\zeta)-\phi_j\circ f(\zeta)=(g(\zeta)-g_j(\zeta))\eta(\zeta)$ for any $\zeta\in\boldsymbol{C}$, on any compact subset X in $\boldsymbol{C}$ we have $norm\,(\phi-\phi_j)\geqq \sup_{\zeta\in X}\|\phi\circ f(\zeta)-\phi_j\circ f(\zeta)\|=\sup_{\zeta\in X}|g(\zeta)-g_j(\zeta)|\,\|\eta(\zeta)\|$. Remarking that $\|\eta(\zeta)\|$ is positive, we see that $\{g_j\}_{j=1,2,\dots}$ converges to g uniformly on X. Thus $\{g_j\}_{j=1,2,\dots}$ converges to g uniformly on any compact subset of $\boldsymbol{C}$. On the other hand from (5.2) we have $m_f(\phi;r)-m_f(\phi_j;r)=m(g;r)-m(g_j;r)$ for $j=1, 2, \cdots$. Then Lemma 5.1 follows from Lemma 3.1. q.e.d.

Lemma 5.2 (i) *We have* $N_f(\phi;r)=N_f(\lambda\phi;r)$ *for* $\phi\in V-\{0\}$, $r>0$ *and* $\lambda\in\boldsymbol{C}-\{0\}$.
(ii) $N_f(\phi;r)$ *is continuous on* $V-\{0\}$ *for each* $r>0$.

Proof. (i) is trivial. (ii) follows from Corollary to Lemma 4.2 and Lemma 5.1. q.e.d.

Theorem 5.1. (The first fundamental theorem.) *Let* $(\boldsymbol{L}, h)$ *be a hermitian line bundle over a connected compact manifold* M. *Let* V *be a linear subspace of* $\Gamma(M, \boldsymbol{L})$. *Suppose we have a holomorphic mapping* f:$\boldsymbol{C}\to M$ *which is* *non-degenerate with respect to V. Then there exists a positive constant K such that*

$$N_f(\phi;r)<T_f(r)+K \qquad \text{for} \quad r>0 \text{ and } \phi\in V-\{0\}, \tag{5.3}$$

where K is independent of r and ϕ.

Proof. Put $S=\{\phi\in V\,|\,norm\,(\phi)=1\}$. From (i) of Lemma 5.2 it suffices to prove (5.3) for $r>0$ and $\phi\in S$. From Corollary to Lemma 4.2 we have

$$T_f(r)-m_f(\phi;0)=N_f(\phi;r)-m_f(\phi;r).$$

Since ϕ is in S, we have $-m_f(\phi;r)\geqq 0$. On the other hand Lemma 5.1 implies that $-m_f(\phi;0)$ has a finite maximum on the compact set S. Thus we have

$$T_f(r)+\operatorname*{Max}_{\phi\in S}\{-m_f(\phi;0)\}\geqq N_f(\phi;r)$$

for $r>0$ and $\phi\in S$. q.e.d.

6. The case of the projective space. Let the notation be as in the section 2. Let $G:\boldsymbol{C}\to\boldsymbol{C}^{n+1}-\{0\}$ be a holomorphic mapping. Then we define a holomorphic mapping $F:\boldsymbol{C}\to\boldsymbol{P}^n$ by $F=p_0\circ G$. All through this section we assume F to be non-degenerate with respect to $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$. Put $G(\zeta)=(g^0(\zeta),\cdots, g^n(\zeta))$.

Lemma 6.1. *Let ω_0^* be the Chern form of the hyperplane bundle of $\boldsymbol{P}^n$. Then we have*

$$F^*\omega_0^*=\frac{1}{4\pi}dd^c\log\left(\sum_{j=0}^n|g^j|^2\right).$$

Proof. Since we have $F^*\omega_0^*=G^*\circ p_0^*\omega_0^*$, (2.8) implies $F^*\omega_0^*=(1/4\pi)dd^c\log\langle G, G\rangle=(1/4\pi)dd^c\log\left(\sum_{j=0}^n|g^j|^2\right)$. q.e.d.

Lemma 6.2. *Let Φ be an element of $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)=\boldsymbol{C}^{n+1}$. Then we have* $\|\Phi(F(\zeta))\|=|(\Phi, G(\zeta))|/\left(\sum_{j=0}^n|g^j|^2\right)^{1/2}$.

Proof. This follows from (2.5). q.e.d.

Lemma 6.3. *We have $(g^0,\cdots,g^n)\wedge\left(\frac{dg^0}{d\zeta},\cdots,\frac{dg^n}{d\zeta}\right)$ is not zero-constant.*

Proof. Suppose $(g^0,\cdots,g^n)\wedge\left(\frac{dg^0}{d\zeta},\cdots,\frac{dg^n}{d\zeta}\right)=0$ on $\boldsymbol{C}$. Then we have

$$\left(\frac{dg^0}{d\zeta}(\zeta),\cdots,\frac{dg^n}{d\zeta}(\zeta)\right)=c(\zeta)\left(g^0(\zeta),\cdots,g^n(\zeta)\right).$$

It is easy to see $c(\zeta)$ is holomorphic. Thus $g^0(\zeta),\cdots,g^n(\zeta)$ are solutions of the first order linear ordinary differential equation:

$$\frac{dg}{d\zeta} = cg .$$

Hence $g^0, \cdots, g^n$ can not be linearly independent. This is absurd because F is non-degenerate with respect to $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$. q.e.d.

Proposition 6.1. *Let the notation be as above, then we have*

(i) *$T_F(r)$ is an increasing function and* $\lim_{r\to\infty} T_F(r)=\infty$;

and

(ii) $\lim_{r\to\infty} \frac{r}{T_F(r)}=0$ *if and only if F cannot be extended holomorphically to the infinity.*

Proof. Put $F^*\omega_0^*=a(\zeta)d\zeta\wedge d\bar{\zeta}$. From Lemma 6.1 we have

$$F^*\omega_0^* = \frac{1}{2\pi}(\sum_{j=0}^n |g^j|^2)^{-2}\{(\sum_{j=0}^n |g^j|^2)(\sum_{j=0}^n dg^j\wedge d\bar{g}^j) -(\sum_{j=0}^n \bar{g}^j dg^j)\wedge(\sum_{j=0}^n g^j d\bar{g}^j)\} .$$

Thus we have $a(\zeta)\geqq 0$ and $a(\zeta)=0$ holds if and only if

$$(g^0(\zeta), \cdots, g^n(\zeta))\wedge\left(\frac{dg^0}{d\zeta}(\zeta), \cdots, \frac{dg^n}{d\zeta}(\zeta)\right) = 0 .$$

From Lemma 6.3 we see that $a(\zeta)>0$ for a.e. $\zeta\in \boldsymbol{C}$. Hence $\int_{D(r)} F^*\omega_0^*$ is an positive increasing function of r. Then

$$T_F(r) = \int_0^r dt\int_{D(t)} F^*\omega_0^*\geqq\int_0^s dt\int_{D(t)} F^*\omega_0^*+(\int_{D(s)} F^*\omega_0^*)(r-s)$$

for $0<s<r$. From this our assertion (i) follows. To prove (ii) let us assume first F can be extended holomorphically to the infinity. Then $v=\int_C F^*\omega_0^*$ is positive and finite.
Thus we have

$$T_F(r) = \int_0^r dt\int_{D(r)} F^*\omega_0^*\leqq v\int_0^r dt = vr .$$

Hence $r/T_F(r)\geqq 1/v>0$. Therefore $\lim_{r\to\infty} r/T_F(r)=0$ implies that F cannot be extended holomorphically to the infinity. Conversely suppose $\lim_{r\to\infty} r/T_F(r)=0$ does not hold, or more precisely

$$\limsup_{r\to\infty} \frac{r}{T_F(r)} = \beta>0 . \tag{6.1}$$

We claim (6.1) implies that $n_F(\Phi; r)$ is bounded for any $\Phi \in \Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$. In fact suppose otherwise. Then for $c>1/\beta$ there exists $s>0$ such that

$$n_F(\Phi; r) > c \qquad \text{for} \quad r \geqq s.$$

From Theorem 5.1 there exists a positive constant K such that

$$T_F(r)+K > N_F(\Phi; r) = \int_0^r n_F(\Phi; t)\, dt\,.$$

Thus we have

$$T_F(r)+K > \int_0^s n_F(\Phi; t)\, dt + c(r-s)\,.$$

Hence we have

$$\frac{1}{\beta} = \liminf_{r\to\infty} \frac{T_F(r)}{r} \geqq c\,.$$

This is a contradiction. From Lemma 6.2 we see that ag^j+bg^k $(0\leqq j<k\leqq n)$ has only a finite number of zeros for $a, b \in \boldsymbol{C}$. Hence g^j/g^k $(0\leqq j, k\leqq n)$ is a rational function. From this we can conclude that F can be extended holomorphically to the infinity. q.e.d.

Let Φ_j $(1\leqq j\leqq q)$ be elements of $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)=\boldsymbol{C}^{n+1}$. We call $\{\Phi_j\}_{1\leqq j\leqq q}$ to be in general position if $q\geqq n+1$ and $n+1$ of those are linearly independent. We are now ready to recall

Theorem 6.1. (H. Cartan ([3]), L.V. Ahlfors ([1])). *Let the notation be as above. Suppose F is non-degenerate with respect to $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$. Let $\{\Phi_j\}_{1\leqq j\leqq q}$ be a set of elements in $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$ which is in general position. Then we have*

$$(6.2) \qquad (q-n-1)T_F(r) < \textstyle\sum_{j=1}^q N_F(\Phi_j; r) + S(r)$$

where

$$(6.3) \qquad S(r) = O\{\log T_F(r)\} + O\{r\}$$

as $r\to\infty$

(6.4) *through all values if we have* $\limsup_{r\to\infty} \dfrac{\log T_F(r)}{r} < \infty$, *and*

(6.5) *outside a set E if otherwise, where*

$$\int_E e^t\, dt < \infty\,.$$

Proof. For a simple proof see [3]. Since the assertion (6.4) is usually not mentioned in the papers available, we shall make some observation on this in the section 8. q.e.d.

We remark that $\Gamma(\boldsymbol{P}^n, \boldsymbol{L}_0^*)$ has been identified with $\boldsymbol{C}^{n+1}$ in the section 2. From Lemma 5.2 we can consider $N_F(\Phi;r)$ as a continuous function on $\boldsymbol{P}^n$ for $r>0$. With this in mind we recall the following classical result.

Theorem 6.2. (The Crofton's formula.) *We have*

$$T_F(r) = \int_{[\Phi]\in \boldsymbol{P}^n} N_F([\Phi];r)(\omega_0^*)^n .$$

Proof. For the proof see [4].

7. The second fundamental theorem. Let $(\boldsymbol{L}, h)$ be a hermitian line bundle over a connected compact manifold M. We assume dim $\Gamma(M, \boldsymbol{L})\geqq 2$. Let V be a linear subspace of $\Gamma(M, \boldsymbol{L})$ such that $l=\dim V\geqq 2$. Let $f:\boldsymbol{C}\to M$ be a holomorphic mapping which is non-degenerate with respect to V.

Let $\{\phi_1, \cdots, \phi_l\}$ be a fixed basis of V. We may assume that

$$norm\,(\phi_j)\leqq\left(\frac{1}{l}\right)^{1/2} \qquad (1\leqq j\leqq l)\,.$$

From Lemma 4.1 there exists an entire function $g_1(\zeta)$ such that

(7.1) $\zeta\in\boldsymbol{C}\mapsto\eta(\zeta)=g_1(\zeta)^{-1}\phi_1(f(\zeta))\in\boldsymbol{L}$ *is a non-vanishing holomorphic mapping.*

Then there exist non-zero constant entire functions $g_2(\zeta), \cdots, g_l(\phi)$ such that

$$\phi_j(f(\zeta)) = g_j(\zeta)\eta(\zeta) \qquad (2\leqq j\leqq l)\,. \tag{7.2}$$

It is easy to see $\{g_1(\zeta), \cdots, g_l(\zeta)\}$ is determined up to the multiplication of a non-vanishing entire function. From Weierstrass Theorem there exists an entire function $g(\zeta)$ such that

(7.3) $\{g_1/g, \cdots, g_l/g\}$ *has no common zero on* $\boldsymbol{C}$.

We define a holomorphic mapping $G: \boldsymbol{C}\to\boldsymbol{C}^l-\{0\}$ by

$$G(\zeta) = \{g_1(\zeta)/g(\zeta), \cdots, g_l(\zeta)/g(\zeta)\}\,.$$

We put $F(\zeta)=p_0(G(\zeta))$. Thus from f we have a holomorphic mapping $F:\boldsymbol{C}\to\boldsymbol{P}^{l-1}$.

Lemma 7.1. *F is non-degenerate with respect to $\Gamma(\boldsymbol{P}^{l-1}, \boldsymbol{L}_0^*)$ provided f is non-degenerate with respect to V.*

Proof. Suppose F is degenerate. From Lemma 6.2 there exists $(c^1, \cdots, c^l)\in \boldsymbol{C}^l-\{0\}$ such that $\sum_{j=1}^l c^j(g_j/g)=0$. Thus $\sum_{j=1}^l c^j g_j\eta=0$ on $\boldsymbol{C}$. From (7.2) we have $\sum_{j=1}^l c^j\phi_j(f(\zeta))=0$, which means f is degenerate. This is a contradiction. q.e.d.

We define a function $H(\zeta)$ on $\boldsymbol{C}$ by

$$H(\zeta)=\{\textstyle\sum_{j=1}^{l}\|\phi_j(f(\zeta))\|^2\}^{1/2} \qquad \text{for} \quad \zeta\in\boldsymbol{C}\,. \tag{7.4}$$

Then (7.2) implies

$$H(\zeta)=\{\textstyle\sum_{j=1}^{l}|g_j(\zeta)|^2\|\eta(\zeta)\|^2\}^{1/2}=|g(\zeta)|(\sum_{j=1}^{l}|g_j(\zeta)/g(\zeta)|^2)^{1/2}\|\eta(\zeta)\|. \tag{7.5}$$

Hence $H(\zeta)$ satisfies the condition (3.2). (7.5) implies

$$n(H;r)=n(|g|;r)\,. \tag{7.6}$$

We define an identification between V and $\boldsymbol{C}^l$ by $\phi=\sum_{j=1}^{l}c^j\phi_j\mapsto\Phi=(c^1,\cdots,c^l)\in\boldsymbol{C}^l$. We always denote by Φ, Ψ, $\cdots$ the elements in $\boldsymbol{C}^l$ corresponding to ϕ, ψ, $\cdots$ in V.

Lemma 7.2. *We have*

$$n_f(\phi;r)=n_F(\Phi;r)+n(H;r)\,.$$

Proof. Put $\phi=\sum_{j=1}^{l}c^j\phi_j$. Thus $\Phi=(c^1,\cdots,c^l)$. Then $\phi\circ f=\sum_{j=1}^{l}c^j\phi_j\circ f=\sum_{j=1}^{l}c^jg_j\eta=g(\sum_{j=1}^{l}c^j(g_j/g))\eta=g(\Phi, F)\eta$. Hence we have

$$\|\phi\circ f\|=|g|\,\|\Phi\circ f\|\{\textstyle\sum_{j=1}^{l}|g_j/g|^2\}^{1/2}\|\eta\|=\|\Phi\circ f\|H\,.$$

q.e.d.

Corollary. *We have*

$$N_f(\phi;r)=N_F(\Phi;r)+N(H;r) \qquad \textit{for} \quad r>0\,.$$

Lemma 7.3. *Let ω be the Chern form of $\boldsymbol{L}$ and ω_0^* be the Chern form of $(\boldsymbol{P}^{l-1}, \boldsymbol{L}_0^*)$. Then we have*

$$-\frac{1}{2\pi}dd^c\log H=f^*\omega-F^*\omega_0^*\,.$$

Proof. From (7.5) we have $H=(\sum_{j=1}^{l}|g_j/g|^2)^{1/2}|g|\,|g_1|^{-1}\|\phi_1\circ f\|$. Hence

$$-\frac{1}{2\pi}dd^c\log H=-\frac{1}{2\pi}dd^c\log\|\phi_1\circ f\|-\frac{1}{4\pi}\log(\textstyle\sum_{j=1}^{l}|g_j/g|^2)\,.$$

From Lemma 6.1 we have

$$-\frac{1}{2\pi}dd^c\log H=f^*\omega-F^*\omega_0^*\,.$$

Proposition 7.1. *We have*

$$T_f(r)-T_F(r)=-m(H;r)+m(H;0)+N(H;r) \qquad \textit{for} \quad r>0\,.$$

Proof. This follows from Lemma 7.3, Lemma 3.2 and (3.8). q.e.d.

Corollary. *There exists a constant K_2 such that*

$$T_f(r) > T_F(r) + K_2 \qquad \text{for} \quad r > 0$$

where K_2 is independent of r.

Proof. Since we have assumed

$$\text{norm}\,(\phi_j) \leqq \left(\frac{1}{l}\right)^{1/2} \qquad (1 \leqq j \leqq l)\,,$$

we have $H(\zeta) \leqq 1$ on $\boldsymbol{C}$. Thus $m(H;r) \leqq 0$. From Proposition 7.1 we have $T_f(r) - T_F(r) \geqq m(H;0) + N(H;r) \geqq m(H;0)$. q.e.d.

DEFINITION **7.1.** For $\phi \in V - \{0\}$ we define $\delta_f(\phi)$ by

$$\delta_f(\phi) = \liminf_{r\to\infty} \left(1 - \frac{N_f(\phi;r)}{T_f(r)}\right).$$

Proposition 7.2. *For any $\phi \in V - \{0\}$ we have*

$$0 \leqq \delta_f(\phi) \leqq 1\,.$$

Proof. The assertion $\delta_f(\phi) \leqq 1$ is trivial. Now from Theorem 5.1 there exists a positive constant K such that

(7.7) $$N_f(\phi;r) < T_f(r) + K \qquad \text{for} \quad r > 0 \text{ and } \phi \in V - \{0\}\,.$$

Thus $\liminf_{r\to\infty} \{1 - (N_f(\phi;r)/T_f(r))\} \geqq \liminf_{r\to\infty} (-K/T_f(r)) = 0$. The last equality follows from (i) of Proposition 6.1 and Corollary to Proposition 7.1. q.e.d.

Lemma 7.4. *Put $e = \liminf_{r\to\infty} \{-m(H;r)/T_f(r)\}$. Then $e \geqq 0$ and $\delta_f(\phi) \geqq e$ for any $\phi \in V - \{0\}$.*

Proof. The assertion $e \geqq 0$ is easy to see. From Proposition 7.1 and Corollary to Lemma 7.2 we have

$$T_f(r) - T_F(r) = -m(H;r) + m(H;0) + N_f(\phi;r) - N_F(\Phi;r)$$

for any $\phi \in V - \{0\}$. From Theorem 5.1 there exists a positive constant K_3 such that

$$N_F(\Phi;r) < T_F(r) + K_3 \qquad \text{for} \quad r > 0 \text{ and } \Phi \in \boldsymbol{C}^l - \{0\}\,.$$

From these formulas we have

$$T_f(r)-N_f(\phi;r)\geqq -K_3-m(H;r)+m(H;0)\,.$$

Hence

$$1-\frac{N_f(\phi;r)}{T_f(r)}\geqq\frac{-m(H;r)}{T_f(r)}+\frac{m(H;0)-K_3}{T_f(r)}$$

Then our assertion follows from (i) of Proposition 6.1 and Corollary to Proposition 7.1. q.e.d.

From (i) of Lemma 5.2 it is easy to see that we have

$$\delta_f(\lambda\phi)=\delta_f(\phi)\qquad\text{for}\quad \phi\in V-\{0\}\text{ and }\lambda\in \boldsymbol{C}^*\,.$$

Thus we can consider δ_f as a function on $\boldsymbol{P}^{l-1}$. (Remark we have identified V with $\boldsymbol{C}^l$ at the beginning of this section.) We put $[\phi]=p_0(\phi)$ for $\phi\in V-\{0\}$.

Theorem 7.1. *Let the notation be as above. We have $\delta_f([\phi])=e$ for almost all $[\phi]\in \boldsymbol{P}^{l-1}$ with respect to the positive measure $(\omega_0^*)^{l-1}$.*

Proof. From Corollary to Lemma 7.2 we have

$$\int_{\boldsymbol{P}^{l-1}}N_f([\phi];r)(\omega_0^*)^{l-1}=\int_{\boldsymbol{P}^{l-1}}N_F([\Phi];r)(\omega_0^*)^{l-1}+N(H;r)$$

for $r>0$. Then Theorem 6.2 implies

$$\int_{\boldsymbol{P}^{l-1}}N_f([\phi];r)(\omega_0^*)^{l-1}=T_F(r)+N(H;r)\qquad\text{for}\quad r>0\,.$$

From Proposition 7.1 we have

$$\int_{\boldsymbol{P}^{l-1}}N_f([\phi];r)(\omega_0^*)^{l-1}=T_f(r)+m(H;r)-m(H;0)\qquad\text{for}\quad r>0\,.$$

Hence (2.9) implies

$$\int_{\boldsymbol{P}^{l-1}}\left(1-\frac{N_f([\phi];r)}{T_f(r)}\right)(\omega_0^*)^{l-1}-\frac{m(H;0)}{T_f(r)}=\frac{-m(H;r)}{T_f(r)}\qquad\text{for}\quad r>0\,.$$

Therefore we see

$$\liminf_{r\to\infty}\int_{\boldsymbol{P}^{l-1}}\left(1-\frac{N_f([\phi];r)}{T_f(r)}\right)(\omega_0^*)^{l-1}\leqq e\,.$$

Since $|1-N_f([\phi];r)/T_f(r)|$ is bounded, Fubini Theorem implies that δ_f is measurable and we have

$$\int_{\boldsymbol{P}^{l-1}}\delta_f([\phi])(\omega_0^*)^{l-1}\leqq e\,.$$

Then Lemma 7.4 implies our assertion. q.e.d.

Now take a set of elements ψ_j ($1 \leqq j \leqq q$) in general position from $V = \boldsymbol{C}^l$. Then Theorem 6.1 implies

$$(q-l)\,T_F(r) \leqq \sum_{j=1}^{q} N_F(\psi_j; r) + S(r) \qquad \text{for} \quad r>0\,,$$

where $S(r)$ satisfies the conditions (6.3), (6.4) and (6.5). From Proposition 7.1 and Corollary to Lemma 7.2 we have

(7.8) $(q-l)\,T_f(r) \leqq \sum_{j=1}^{q} N_f(\psi_j; r) - (q-l)\,m(H; r) + S(r)\,.$

(We remark that $m(H;0) \leqq 0$).

DEFINITION 7.1. Let $(\boldsymbol{L}, h)$ be an hermitian line bundle over a connected compact complex manifold M. Let $f: \boldsymbol{C} \to M$ be a holomorphic mapping. Let ω be the Chern form of $(\boldsymbol{L}, h)$. Then we have defined $T_f(r)$ ($r>0$) by

$$T_f(r) = \int_0^r dt \int_{D(t)} f^*\omega\,.$$

We call f to be *transcendental* with respect to $(\boldsymbol{L}, h)$ if we have

$$\lim_{r\to\infty} \frac{r}{T_f(r)} = 0\,.$$

We call f to be *transcendental of finite type* with respect to $(\boldsymbol{L}, h)$ if we have

$$\limsup_{r\to\infty} \frac{\log(|T_f(r)|)}{r} < \infty\,.$$

Theorem 7.2. (The second fundamental theorem.) *Let* $(\boldsymbol{L}, h)$ *be a hermitian line bundle over a connected compact complex manifold* M *such that* $\dim \Gamma(M, \boldsymbol{L}) \geqq 2$. *Let* V *be a linear subspace of* $\Gamma(M, \boldsymbol{L})$ *such that* $l = \dim V \geqq 2$. *Let* $f: \boldsymbol{C} \to M$ *be a holomorphic mapping which is non-degenerate with respect to* V. *Take a set of elements* ψ_j ($1 \leqq j \leqq q$) *of* V *in general position. Then we have*

$$\sum_{j=1}^{q} \{\delta_f([\psi_j]) - e\} \leqq (1-e)\,l\,,$$

if one of the following conditions is satisfied (cf., Definition 7.1):

(i) f *is transcendental of finite type with respect to* $(\boldsymbol{L}, h)$ *or*

(ii) f *is transcendental with respect to* $(\boldsymbol{L}, h)$ *and there exists* $\lim_{r\to\infty} \{-m(H; r)/T_f(r)$, (*which is then equal to e*). *Here e and H have been defined in Lemma* 7.4 *and* (7.4).

Proof. From (7.8) we have

$$\sum_{j=1}^{q} \left\{1 - \frac{N_f(\psi_j; r)}{T_f(r)}\right\} \leqq l + (q-l)\frac{-m(H; r)}{T_f(r)} + \frac{S(r)}{T_f(r)} \tag{7.9}$$

for $r>0$.

If the condition (ii) is satisfied, then (7.9) clearly implies

$$\sum_{j=1}^{q} \delta_f([\psi_j]) \leqq l+(q-l)e+\liminf_{r\to\infty} \frac{S(r)}{T_f(r)} .$$

Then Lemma 7.1, (i) of Proposition 6.1, Corollary to Proposition 7.1, (6.3), (6.4), (6.5) and $\lim_{r\to\infty} \{r/T_f(r)\}=0$ imply

$$\liminf_{r\to\infty} \frac{S(r)}{T_f(r)} = 0$$

Now suppose the condition (i) is satisfied. Then from the same reason as above we can easily see

$$\limsup_{r\to\infty} \frac{\log (T_F(r))}{r} < \infty .$$

Hence from (6.4) we see that $\lim_{r\to\infty} \{S(r)/T_F(r)\}$ exists and is equal to zero. Therefore (7.9) implies

$$\sum_{j=1}^{q} \delta_f([\psi_j]) \leqq l+(q-l) \liminf_{r\to\infty} \left(\frac{-m(H;r)}{T_f(r)}\right) .$$

By the definition $\liminf_{r\to\infty} \{-m(H;r)/T_f(r)\}=e$. Thus Theorem 7.2 has been proved. q.e.d.

8. Some remarks. We first remark that e in Theorem 7.1 and Theorem 7.2 is not necessarily equal to zero. Consider a holomorphic mapping G: $\boldsymbol{C}\to\boldsymbol{C}^3-\{0\}$ defined by

$$G(\zeta) = (\zeta, \zeta^2, e^{g(\zeta)})$$

where $g(\zeta)$ is a transcendental entire function. Put $F(\zeta)=p_0(G(\zeta))$. Let V be a subspace of $\boldsymbol{C}^3=\Gamma(\boldsymbol{P}^2, \boldsymbol{L}_0^*)$ spanned by (1, 0, 0) and (0, 1, 0). Then F is non-degenerate with respect to V and transcendental. It is easy to see

$$e = \liminf_{r\to\infty} \frac{-m(H;r)}{T_f(r)} = 1 - \frac{1}{2\pi} \liminf_{r\to\infty} \frac{\log (e^{2r}+e^{4r})}{r} \frac{r}{T_f(r)} .$$

Since f is transcendental, we have $e=1$.

Finally we would like to comment on the proof of (6.4) in Theorem 6.1. For a meromorphic function $g(\zeta)$ we define $m^+(g; r)$ by

$$m^+(g;r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|g(e^{r+i\theta})|\, d\theta ,$$

where $\log^+$ is defined by

$$\log^+ x = \begin{cases} \log x & \text{if} \quad x \geqq 1 \\ 0 & \text{if} \quad x \leqq 1 . \end{cases}$$

Now let the notation be as in the section 6. What H. Cartan proved in [3] is the following (*c.f.*, p. 14):

$$(8.1) \qquad (q-n-1)T_F(r) < \sum_{j=1}^{q} N_F(\Phi_j; r) + K + K \sum_{\substack{2 \leqq j \leqq q \\ 1 \leqq k \leqq m}} m^+ \left(\frac{\left(\frac{F_j}{F_1}\right)^{(k)}}{\left(\frac{F_j}{F_1}\right)}; r \right),$$

where F_j is as in [3]. Then Theorem 2.2 in [6] and Theorem 3.1 in [6] as well as its proof imply our assertion (6.4) out of (8.1).

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