PRACE IPI PAN • ICS PAS REPORTS

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Some remarks about rough sets

456

January 1982

WARSZAWA

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ISSN 0138 - 0648

Nakład 700 egz. Ark. wyd. 0,25; ark. druk. 0,75. Papier offset. kl. III, 70 g, 70 x 100. Oddano do druku w styczniu 1982 r. W. D. N. Zam. nr 23/0/81

Abstract . Содержание . Streszczenie

We discuss in this paper approximative (rough) operations on sets and "exact" operations on rough sets.

Замечания о прибдиженных множествах

В статьи рассматривается прибдиженные операции на множествах, а также "точные" операции на прибдиженных множествах.

Uwagi o zbiorach przybliżonych

W pracy rozpatrywane są przybliżone operacje na zbiorach oraz "dokładne" operacje na zbiorach przybliżonych.

1. INTRODUCTION

In many branches of computer sciences such as pattern recognition learning algorithms, automatic classification, inductive inference there is a need for approximative methods. Main mathematical tool to deal with such problems is fuzzy set theory. We propose here an alternative approach which is based on a indiscernibility relation, which "glue" together all objects which we are unable to distinguish by means of available means observations (measurements) or expression (language). We introduce the notion of the rough set [6] as a basis of our considerations. Some preliminary results concerning "rough" approach are given in [3], [4], [7], [8], [10] and some applications are mentioned in [2], [5], [1].

2. PRELIMINARY NOTIONS

Let X be a set and $R \subset X \times X$ an equivalence relation on X. An ordered pair $A = \langle X, R \rangle$ will be called an approximation space, and R will be called an indiscernibility relation in A. Equivalence classes of the relation R are called elementary sets in A. Every union of elementary sets in A will be referred to as a composed set in A.

Let Y be a subset of X. By a lower approximation of Y in A, denoted as $\underline{A}Y$, we shall mean the greatest composed set in A contained in Y; an upper approximation of Y in A, denoted $\overline{A}Y$, we shall mean the smallest composed set in A containing Y.

The following properties of approximations are valid for every approximation space $A = \langle X, R \rangle$ and every subset $Y, Z \subset X$.

1. $\underline{A}Y \subset Y \subset \overline{A}Y$,
2. $\underline{A}1 = \overline{A}1 = 1$,
3. $\underline{A}0 = \overline{A}0 = 0$,
4. $\underline{A}\overline{A}Y = \overline{A}\overline{A}Y = \overline{A}Y$,
5. $\overline{A}\underline{A}Y = \underline{A}\underline{A}Y = \underline{A}Y$,
6. $\overline{A}(Y \cup Z) = \overline{A}Y \cup \overline{A}Z$,
7. $\underline{A}(Y \cap Z) = \underline{A}Y \cap \underline{A}Z$,
8. $\overline{A}Y = -\underline{A}(-Y)$,
9. $\underline{A}Y = -\overline{A}(-Y)$,

where 0,1 are denoting the empty set and the set X respectively.

3. ROUGH EQUALITY OF SETS AND ROUGH "OPERATION" ON SETS

We shall say that:

i) Sets $Y, Z \subset X$ are roughly bottom equal in $A = \langle X, R \rangle$, in symbols $Y \underset{A}{\eqsim} Z$, iff $\underline{A}Y = \underline{A}Z$.

ii) Sets $Y, Z \subset X$ are roughly top equal in $A = \langle X, R \rangle$, in symbols $Y \underset{A}{\simeq} Z$, iff $\overline{A}Y = \overline{A}Z$.

iii) Sets $Y, Z \subset X$ are roughly equal in $A = \langle X, R \rangle$, in symbols $Y \underset{A}{\approx} Z$, iff $Y \underset{A}{\eqsim} Z$ and $Y \underset{A}{\simeq} Z$.

We can introduce rough set theoretical "operation" on sets in the following way:

$\underset{\overline{A}}{\cup}(Y,Z,U)$ iff $Y \cup Z \underset{A}{\simeq} U$,

$\underset{\underline{A}}{\cup}(Y,Z,U)$ iff $Y \cup Z \underset{A}{\eqsim} U$,

$\underset{A}{\cup}(Y,Z,U)$ iff $Y \cup Z \underset{A}{\approx} U$,

$\underset{\underline{A}}{\cap}(Y,Z,U)$ iff $A \cap Z \underset{A}{\eqsim} U$,

$\underset{\overline{A}}{\cap}(Y,Z,U)$ iff $Y \cap Z \underset{A}{\simeq} U$,

$\underset{A}{\cap}(Y,Z,U)$ iff $Y \cap Z \underset{A}{\approx} U$,

$\underset{\overline{A}}{-}(Y,Z)$ iff $-Y \underset{A}{\simeq} Z$,

$\underset{\underline{A}}{-}(Y,Z)$ iff $-Y \underset{A}{\eqsim} Z$,

$\underset{A}{-}(Y,Z)$ iff $-Y \underset{A}{\approx} Z$.

In fact rough operations $\underset{\overline{A}}{\cup}$, $\underset{\underline{A}}{\cup}$, $\underset{A}{\cup}$, $\underset{\overline{A}}{\cap}$, $\underset{\underline{A}}{\cap}$, $\underset{A}{\cap}$, $\underset{\overline{A}}{-}$, $\underset{\underline{A}}{-}$, $\underset{A}{-}$ are not defined univocally, and consequently we are not allowed to write for example

$$Y \underset{\underline{A}}{\cup} Z = U,$$

$$Y \underset{\underline{A}}{\cap} Z = U,$$

or

$$\underset{\underline{A}}{-} Y = Z,$$

because there are in general many U such that

$$Y \underset{\underline{A}}{\cup} Z = U$$

etc.

4. ROUGH SETS

Let $Y \subset X$ be a set in an approximation space $A = \langle X, R \rangle$, and let us introduce the following definitions:

1) $(\overline{A})Y = \{Z \subset X : Z \underset{A}{\simeq} Y\}$,

2) $(\underline{A})Y = \{Z \subset X : Z \underset{A}{\eqsim} Y\}$,

3) $(A)Y = \{Z \subset X : Z \underset{A}{\approx} Y\}$.

We shall call $(\bar{A})Y$ the upper rough set generated by the set Y in A; $(\underline{A})Y$ - the lower rough set generalized by Y in A and $(A)Y$ - a rough set generated by Y in A.

Thus these three kinds of rough sets are families of families of "ordinary" sets.

The following properties of rough sets are valid:

i) $(A)Y = (\bar{A})Y \cap (\underline{A})Y$,

ii) $\bar{A}Y = \bigcup_{Z \in (\bar{A})Y} Z = \bigcup_{Z \in (A)Y} Z$,

iii) $\underline{A}Y = \bigcap_{Z \in (\underline{A})Y} Z = \bigcap_{Z \in (A)Y} Z$,

for every $Y, Z \subset X$ and every approximation space $A = \langle X, R \rangle$.

Obviously $\approx_A$ $\eqsim_A$ $\simeq_A$ are equivalence relations on the set $\mathcal{P}(X)$, and rough sets (upper and lower) are equivalence classes of the corresponding relations.

Thus with each approximation space $A = \langle X, R \rangle$ we can associate three approximation spaces

$$\underline{A}^* = \langle \mathcal{P}(X), \eqsim_A \rangle ,$$

$$\bar{A}^* = \langle \mathcal{P}(X), \simeq_A \rangle ,$$

$$A^* = \langle \mathcal{P}(X), \approx_A \rangle ;$$

in which $\eqsim_A$, $\simeq_A$, $\approx_A$ are new indiscernibility relations and consequently we are unable to distinguish some subsets of the set $\mathcal{P}(X)$ in the approximation space $A^*(\underline{A}^*, \bar{A}^*)$.

Thus if $\mathcal{X}$, $\mathcal{Y}$ are some rough (lower, upper) sets in A, then $\mathcal{X} \cap \mathcal{Y}$ is always an empty set and $\mathcal{X} \cup \mathcal{Y}$, $-\mathcal{X}$ are not rough (lower, upper) sets.

In order to deal with approximations of families of sets we introduce the following definition:

a) Every rough set (upper, lower) in A is an elementary family (upper, lower) in $A^*(\bar{A}^*, \underline{A}^*)$.

b) Every union of elementary families (upper, lower) in $A^*(\bar{A}^*, \underline{A}^*)$ is a composed family (upper, lower) in $A^*(\bar{A}^*, \underline{A}^*)$.

Now we are able to introduce lower and upper approximation of a family $\mathcal{X} \subset \mathcal{P}(X)$ in the approximation space $A^*(\bar{A}^*, \underline{A}^*)$ in the same way as in the case of the approximations of subsets in the approximation space $A = \langle X, R \rangle$.

In fact each approximation space $A = \langle X, R \rangle$ generates an infinite sequence of approximation spaces

$$A^0, A^1, A^2, \ldots ,$$

where $A^0 = A$, $A^1 = A^*(\bar{A}^*, \underline{A}^*)$, $A^2 = (A^*)^*$, $((\underline{A}^*)^*, (\bar{A}^*)^*)$, etc., however from practical point of view only approximation spaces A^0 and A^1 seems to have some significance.

5. ROUGH INCLUSION OF SETS

Let $A = \langle X, R \rangle$ be an approximation space. We shall say that

a) Set Y is roughly bottom included in $Z (Y, Z \subset X)$ in the approximation space A, in symbols $Y \underset{A}{\subset} Z$, iff $\underline{A}Y \subset \underline{A}Z$.

b) Set Y is roughly top included in $Z (Y, Z \subset X)$ in the approximation space A, in symbols $Y \overset{A}{\subset} Z$, iff $\bar{A}Y \subset \bar{A}Z$.

iii) Set Y is roughly included in $Z (Y, Z \subset X)$ in the approximation space A, in symbols $Y \textcircled{A} Z$, iff $Y \underset{A}{\subset} Z$ and $Y \overset{A}{\subset} Z$.

Let

$$\wp_{\underline{A}}(Y) = \{Z \subset X : Z \underset{A}{\subset} Y\},$$

$$\wp_{\overline{A}}(Y) = \{Z \subset X : Z \overset{A}{\subset} Y\},$$

$$\wp_{A}(Y) = \{Z \subset X : Z \not\subset Y\}$$

The following property is obvious in view of the definition
of a rough (upper, lower) set:
If $Y \subset X$, then $\wp_A(Y)$ $(\wp_{\overline{A}}(Y), \wp_{\underline{A}}(Y))$ is not a rough set
in A.

6. CONCLUDING REMARKS

The notion of a rough set is defined in this paper as a fa-
mily of roughly equal sets in an approximation space $A = \langle X,R \rangle$.
Rough sets are not closed under set theoretical operations, union,
intersection and oomplement, and the family of rough subset of a
set is not necessarily a rough set.

Rough sets in the approximation space $A = \langle X,R \rangle$ are equi-
valence classes of the relation $\underset{A}{\approx}$ $(\underset{A}{\simeq}, \underset{A}{\eqsim})$ in the approximation
space $A^* = \langle \wp(X), \underset{A}{\approx}$ $(\underset{A}{\eqsim}),$ $\underset{A}{\simeq}) \rangle$ generated by the approxi-
mation space $A = \langle X,R \rangle$. Thus the indiscernibility relation
R, generates new indiscernibility relations $\underset{A}{\approx}, \underset{A}{\eqsim}, \underset{A}{\simeq})$ of
-gher order, which does not allow to distinguish family of sub-
sets of the set X.

This process of generation of indiscernibility relations of
higher orders can be continued to infinity giving a hierarchy of
indiscernibility relations, such that each next indiscernibility
relation in the hierarchy concerns object of higher types.

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