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' SOME REMARKS ABOUT THE THREE SPACES PROBLEM IN BANACH ALGEBRAS

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Let X be a Banach algebra, not necessarily commutative, with a unit element, and let I be a closed two-sided ideal in X. Thus X/I is also a Banach algebra with unit. Suppose (α) is some property of Banach algebras; we shall say that (α) is a regular property if the fact that I and X/I have property (α) implies that X has property (α). In the present paper we shall show the following properties to be regular:

 (α_1) X is a B*-equivalent algebra;

 (α_2) X is a function algebra.

1. Let X be a Banach algebra with involution and a unit element. X is a B^* -equivalent algebra if it admits a norm (defining the correct topology) which is submultiplicative and for which

$$|x^*x| = |x|^2$$
 for all $x \in X$.

It is well-known [1] that a Banach *-algebra X is B^* -equivalent if and only if there exists a constant C > 0 such that

$$\|x^*x\| \ge C \|x\|^2$$

for all $x \in X$, in the original norm. Thus, in a *B**-equivalent algebra, $x_n^* x_n \to 0$ implies $x_n \to 0$.

Theorem 1.1. Let X be a Banach algebra with a unit element and involution, let I be a closed *-ideal in X. If I and X/I are both B*-equivalent algebras (being in a natural way Banach algebras with involution), then X is a B*-equivalent algebra.

Proof. Suppose on the contrary that X is not B*-equivalent. Then there is a sequence (y_n) in X such that $||y_n|| = 1$ and $y_n^* y_n \to 0$.

Let $\varkappa : X \to X/I$ be the quotient map. One has

$$\varkappa(y_n^* y_n) = \varkappa(y_n^*) \varkappa(y_n) = \varkappa(y_n)^* \varkappa(y_n) \to 0.$$

Hence, since X/I is a B*-equivalent algebra, $\varkappa(y_n) \to 0$. Therefore there is a sequence (i_n) in I such that $y_n + i_n^* \to 0$ in X. Certainly the sequence (i_n^*) must be bounded and, therefore, (i_n) is bounded since the involution is continuous on I. Hence

$$(y_n i_n)^* (y_n i_n) = i_n^* y_n^* y_n i_n \to 0$$
.

Since I is a B*-equivalent algebra, $y_n i_n \to 0$. But now

$$i_n^* i_n = \left(y_n + i_n^* \right) i_n - y_n i_n \to 0$$

(for (i_n) is bounded and $y_n + i_n^* \to 0$). Therefore, $i_n \to 0$, since *I* is a *B**-equivalent algebra.

However, it is impossible that $i_n^* \to 0$ because $||y_n|| = 1$ and $y_n + i_n^* \to 0$. So it must be the case that X is a B*-equivalent algebra, which proves Theorem 1.1.

2. Now let X be a commutative Banach algebra with a unit element. X is called a *function algebra* if the norm in X is equivalent to the spectral norm $\|\cdot\|_{sp}$, defined by

$$||x||_{\text{sp}} = \lim_{n \to \infty} ||x^n||^{1/n} = \sup_{f \in \mathfrak{M}(X)} |f(x)|$$

(where $\mathfrak{M}(X)$ is the structure space of X, i.e. the space of all nonzero multiplicative linear functionals).

Lemma 2.1. The following conditions are equivalent:

(i) X is a function algebra,

(ii) if $x_n \in X$, $n = 1, 2, ..., and x_n^2 \to 0$, then $x_n \to 0$,

(iii) if $x_n \in X$, $n = 1, 2, ..., and x_n^2 \to 0$, then (x_n) is bounded,

(iv) there exists C > 0 such that $||x||^2 \leq C ||x^2||$ for every $x \in X$.

Proof. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are trivial. Suppose that the condition (iii) holds. If (iv) is not true, then there exists a sequence (x_n) in X such that $||x_n|| = 1$ and $||x_n||^2 > n^3 ||x_n^2||$, n = 1, 2, ... Then $||x_n^2 \cdot n^2/||x_n||^2 || < 1/n$ and $(x_n \cdot n/||x_n||)^2 \rightarrow 0$, but the sequence $(x_n \cdot n/||x_n||)$ is not bounded. It remains to show (iv) \Rightarrow (i), but this is obvious.

Theorem 2.2. Let X be a commutative^{*}) Banach algebra with a unit element and I a closed ideal in X. If I and X|I are function algebras, so is X. (In other words, the property of being a function algebra is regular.)

^{*)} Reviewer's remark: this assumption is superfluous.

Proof. We show that the condition (iii) of Lemma 2.1. holds. Suppose instead that $x_n^2 \to 0$ and the sequence (x_n) is not bounded. We may assume that $||x_n|| \to \infty$, and let $y_n = x_n/||x_n||$, (n = 1, 2, ...). Then certainly $y_n^2 \to 0$. If $\varkappa : X \to X/I$ is the quotient homomorphism, then

$$\varkappa(y_n^2) = (\varkappa(y_n))^2 \to 0 .$$

Therefore, since X/I is a function algebra, $\varkappa(y_n) \to 0$. Consequently, there exists a sequence (i_n) in I such that $y_n + i_n \to 0$, and so necessarily (i_n) is bounded. Thus $(i_n y_n)^2 = i_n^2 y_n^2 \to 0$, and since I is a function algebra, $i_n y_n \to 0$. Hence

$$i_n^2 = i_n(y_n + i_n) - i_n y_n \rightarrow 0$$
 and $i_n \rightarrow 0$.

But this is absurd, as $y_n + i_n \to 0$ and $||y_n|| = 1$ for all n.

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Reference

[1] V. Pták: Banach algebras with involution, Manuscripta math. 6 (1972), p. 245-290.

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