

SOME REMARKS ON A q -ANALOGUE OF BERNOULLI NUMBERS

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ABSTRACT. Using the p -adic q -integral due to T. Kim [4], we define a number $B_n^*(q)$ and a polynomial $B_n^*(x; q)$ which are p -adic q -analogue of the ordinary Bernoulli number and Bernoulli polynomial, respectively. We investigate some properties of these. Also, we give slightly different construction of Tsumura's p -adic function $\ell_p(u, s, \chi)$ [14] using the p -adic q -integral in [4].

1. Introduction

Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and \mathbb{C}_p will respectively denote the ring of p -adic integers, the field of p -adic numbers and the completion of the algebraic closure of \mathbb{Q}_p . Let $|\cdot|_p$ be the p -adic valuation of \mathbb{C}_p such that $|p|_p = p^{-1}$. If $q \in \mathbb{C}_p$, one normally assumes $|q - 1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the notation

$$(1.1) \quad [x] = [x; q] = \frac{1 - q^x}{1 - q}.$$

Hence, $\lim_{q \rightarrow 1} [x; q] = x$ for any x with $|x|_p \leq 1$. Let $UD(\mathbb{Z}_p)$ denote the space of all uniformly (or strictly) differentiable \mathbb{C}_p -valued functions on \mathbb{Z}_p . It is well-known that the I_0 -integral of $f \in UD(\mathbb{Z}_p)$ exists and is given by

$$(1.2) \quad I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \frac{1}{p^N},$$

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where μ_0 is the ordinary p -adic distribution defined by $\mu_0(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}$. The q -analogue of μ_0 , denote by μ_q , defined by T. Kim [4] as follows: Let d be a fixed integer and p be a fixed prime number. We set

$$(1.3) \quad X = \varprojlim_N (\mathbb{Z}/dp^N \mathbb{Z}), \quad X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ with $0 \leq a < dp^N$. For any positive integer N ,

$$(1.4) \quad \mu_q(a + dp^N \mathbb{Z}_p) = \frac{q^a}{[dp^N]} = \frac{q^a}{[dp^N; q]}$$

is known as a distribution on X . In the case of $d = 1$, this distribution yields an I_q -integral for $f \in UD(\mathbb{Z}_p)$

$$(1.5) \quad I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \frac{q^x}{[p^N]}.$$

In particular, the relation between the I_0 -integral and I_q -integral is given by

$$(1.5') \quad \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \frac{\log q}{q-1} \int_{\mathbb{Z}_p} q^{-x} f(x) d\mu_q(x) \text{ for } f \in UD(\mathbb{Z}_p).$$

We recall variant Bernoulli numbers given by below in the symbolic form: For $n \geq 0$

- $B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$
(Ordinary Bernoulli numbers)
- $\beta_0(q) = 1, \quad q(q\beta(q)+1)^n - \beta_n(q) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$
(Carlitz's q -Bernoulli numbers (see [1]))
- $B_0(q) = \frac{q-1}{\log q}, \quad (qB(q)+1)^n - B_n(q) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$
(Tsumura's q -Bernoulli numbers (see [15]))
- $\mathcal{B}_0(q) = 0, \quad q(\mathcal{B}(q)+1)^n - \mathcal{B}_n(q) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$
(Kim's q -Bernoulli numbers (see [2], [9])).

It is known that variant Bernoulli numbers are connected with the I_0 - and I_q -integral as follows: For $n \geq 0$

$$(1.6) \quad I_0(x^n) = \int_{\mathbb{Z}_p} x^n d\mu_0(x) = B_n \text{ (Witt's formula);}$$

$$(1.7) \quad I_q([x]^n) = \int_{\mathbb{Z}_p} [x]^n d\mu_q(x) = \beta_n(q) \text{ (see [4]);}$$

$$(1.8) \quad I_q(q^{-x}[x]^n) = \int_{\mathbb{Z}_p} q^{-x}[x]^n d\mu_q(x) = B_n(q) \text{ (see [3]);}$$

$$(1.9) \quad I_q(q^{-x}x^n) \stackrel{(1.5')}{=} \int_{\mathbb{Z}_p} q^{-x}x^n d\mu_q(x) = \frac{q-1}{\log q} B_n.$$

N. Koblitz [11] constructed the p -adic q - L -series which interpolated Carlitz's q -Bernoulli numbers $\beta_n(q)$. J. Satoh [13] constructed the complex q - L -series which interpolated Carlitz's q -Bernoulli numbers $\beta_n(q)$. T. Kim [4] proved that Carlitz's q -Bernoulli numbers $\beta_n(q)$ can be represented as an integral by the q -analogue μ_q of the ordinary p -adic invariant measure. In the complex case, H. Tsumura [15] studied a q -analogue of the Dirichlet L -series which interpolated q -Bernoulli numbers $B_n(q)$. In the p -adic case, T. Kim [3] constructed the p -adic q - L -function using the congruence on q -Bernoulli numbers $B_n(q)$.

In this paper, we consider a uniformly (strictly) differentiable function $f(x) = x^n$ ($n \geq 0$) in the I_q -integral given by (1.5) and put

$$B_n^*(q) = \int_{\mathbb{Z}_p} x^n d\mu_q(x); \quad B_n^*(x; q) = \int_{\mathbb{Z}_p} (x+t)^n d\mu_q(t).$$

The purpose of this paper is to investigate the properties of a number $B_n^*(q)$ and a polynomial $B_n^*(x; q)$. Also, we give slightly different construction of Tsumura's p -adic function $\ell_p(u, s, \chi)$ [14] using the p -adic q -integral in [4].

2. Another p -adic q -Bernoulli number $B_n^*(q)$ and its basic properties

Set $f(x) = x^n \in UD(\mathbb{Z}_p)$ for $n \geq 0$ in the equation (1.5).

Now, for any integer $n \geq 0$ we define a number $B_n^*(q)$ and a polynomial $B_n^*(x; q)$ in the variable $x \in \mathbb{C}_p$ with $|x|_p \leq 1$, respectively, by

$$(2.1) \quad B_n^*(q) \stackrel{\text{def}}{=} \int_{\mathbb{Z}_p} x^n d\mu_q(x); \quad B_n^*(x; q) \stackrel{\text{def}}{=} \int_{\mathbb{Z}_p} (x+t)^n d\mu_q(t).$$

The generating function, denote by $G_q^*(t)$, of $B_n^*(q)$ is given by

$$(2.2) \quad G_q^*(t) = \frac{q-1}{\log q} \left(\frac{\log q + t}{qe^t - 1} \right) = \sum_{n=0}^{\infty} B_n^*(q) \frac{t^n}{n!}.$$

Indeed, for $f(x) = q^x e^{xt} \in UD(\mathbb{Z}_p)$ using the equation $I_0(f_1) = I_0(f) + f'(0)$, where $f_1(x) = f(x+1)$ for all $x \in \mathbb{Z}_p$, we have

$$(2.3) \quad I_0(q^x e^{xt}) = \frac{\log q + t}{qe^t - 1}.$$

From the formula (1.5'), we obtain

$$(2.4) \quad \begin{aligned} G_q^*(t) &= \frac{q-1}{\log q} \left(\frac{\log q + t}{qe^t - 1} \right) = \frac{q-1}{\log q} I_0(q^x e^{xt}) \\ &= \sum_{n=0}^{\infty} \left(\frac{q-1}{\log q} I_0(x^n q^x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} I_q(x^n) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} B_n^*(q) \frac{t^n}{n!}. \end{aligned}$$

We can easily prove the following.

PROPOSITION 2.1. For $n \geq 0$ and $x \in \mathbb{Z}_p$, we have

- (1) $B_0^*(q) = 1$, $q(B^*(q) + 1)^n - B_n^*(q) = \begin{cases} \frac{q-1}{\log q} & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$;
- (2) $\lim_{q \rightarrow 1} B_n^*(q) = B_n$;
- (3) $B_n^*(x; q) = (B^*(q) + x)^n$ and $\lim_{q \rightarrow 1} B_n^*(x; q) = B_n(x)$, where $B_n(x)$ is the ordinary Bernoulli polynomial.

From Proposition 2.1 we may say that a number $B_n^*(q)$ and a polynomial $B_n^*(x; q)$ are another p -adic q -analogue of ordinary Bernoulli number and Bernoulli polynomial, respectively.

PROPOSITION 2.2. For $m, n \geq 0$ and $x \in \mathbb{Z}_p$, we have

- (1) $I_0(x^m q^{m'}) = \frac{n \log q}{q-1} I_{q^n}(x^m) = \frac{\log q}{q-1} B_m^*(q^n)$;
- (2) $I_q(x^m q^{m'}) = \sum_{s=0}^{\infty} \frac{(\log q^n)^s}{s!} B_{m+s}^*(q)$.

LEMMA 2.3. For $n \geq 0$

$$\int_{\mathbb{Z}_p} x^n d\mu_q(x) = \int_X x^n d\mu_0(x).$$

Proof. Note that $d\mu_q(x) = \frac{(q-1)q^x}{\log q} d\mu_0(x)$ for $|1-q|_p < p^{-1/(p-1)}$ (see [4], [10]). It is well known that $\int_{\mathbb{Z}_p} x^n d\mu_0(x) = \int_X x^n d\mu_0(x)$ for $n \geq 0$ (cf. [4, Lemma 1]) and $q^x = \sum_{s=0}^{\infty} \frac{x^s (\log q)^s}{s!}$ for $|1-q|_p < p^{-1/(p-1)}$. The result now follows easily. \square

LEMMA 2.4. For any positive integer d and $k \geq 0$

$$B_k^*(x; q) = d^k [d]^{-1} \sum_{i=0}^{d-1} q^i B_k^* \left(\frac{x+i}{d}; q^d \right).$$

Proof. By Lemma 2.3 we have

$$\begin{aligned} B_k^*(x; q) &= \lim_{N \rightarrow \infty} \frac{1}{[dp^N]} \sum_{n=0}^{dp^N-1} q^n (x+n)^k \\ &= \lim_{N \rightarrow \infty} \frac{1}{[d]} \frac{1}{[p^N; q^d]} \sum_{i=0}^{d-1} \sum_{n=0}^{p^N-1} q^{i+dn} (x+i+dn)^k \\ &= [d]^{-1} \sum_{i=0}^{d-1} d^k q^i \lim_{N \rightarrow \infty} \frac{1}{[p^N; q^d]} \sum_{n=0}^{p^N-1} (q^d)^n \left(\frac{x+i}{d} + n \right)^k \\ &= [d]^{-1} \sum_{i=0}^{d-1} d^k q^i \int_{\mathbb{Z}_p} \left(\frac{x+i}{d} + t \right)^k d\mu_{q^d}(t) \\ &= d^k [d]^{-1} \sum_{i=0}^{d-1} q^i B_k^* \left(\frac{x+i}{d}; q^d \right). \end{aligned}$$

This completes the proof. \square

THEOREM 2.5. For $k \geq 0$, let $\mu_k^* = \mu_{k;q}^*$ be define by

$$\mu_k^*(a + dp^N \mathbb{Z}_p) \stackrel{\text{def}}{=} (dp^N)^k [dp^N]^{-1} q^a B_k^* \left(\frac{a}{dp^N}; q^{dp^N} \right),$$

where N and d are positive integers. Then μ_k^* is a distribution on X .

Proof. Since $\mu_{0;q}^* = \mu_q$ which is a distribution on X (see [4]), for any positive integer k we show that μ_k^* is a distribution on X . For that, it is suffices to check that

$$\begin{aligned} & \sum_{i=0}^{p-1} \mu_k^*(a + idp^N + dp^{N+1} \mathbb{Z}_p) \\ &= (dp^{N+1})^k q^a [dp^{N+1}]^{-1} \sum_{i=0}^{p-1} q^{idp^N} B_k^* \left(\frac{a + idp^N}{dp^{N+1}}; q^{dp^{N+1}} \right) \\ &= (dp^N)^k q^a [dp^N]^{-1} \left\{ \frac{p^k}{[p; q^{dp^N}]} \sum_{i=0}^{p-1} (q^{dp^N})^i B_k^* \left(\frac{\frac{a}{dp^N} + i}{p}; (q^{dp^N})^p \right) \right\} \\ &= (dp^N)^k [dp^N]^{-1} q^a B_k^* \left(\frac{a}{dp^N}; q^{dp^N} \right) \text{ (using Lemma 2.4)} \\ &= \mu_k^*(a + dp^N \mathbb{Z}_p). \end{aligned}$$

This completes the proof. □

3. A generalized q -Bernoulli numbers $B_{k,\chi}^*(q)$ and related properties

Let χ be a Dirichlet character with conductor d , where d is a positive integer. For $k \geq 0$ we define the k -th generalized q -Bernoulli number belonging to the character χ by

$$(3.1) \quad B_{k,\chi}^*(q) \stackrel{\text{def}}{=} \int_X \chi(x) x^k d\mu_q(x).$$

Using a similar method used in the proof of Lemma 2.4, we may perform the integral in the right hand side of (3.1) to get

$$(3.2) \quad B_{k,\chi}^*(q) = d^k [d]^{-1} \sum_{a=0}^{d-1} q^a \chi(a) B_k^* \left(\frac{a}{d}; q^d \right).$$

PROPOSITION 3.1. For $k \geq 0$, we have

- (1) $\int_X \chi(x) d\mu_k^*(x) = B_{k,\chi}^*(q)$;
- (2) $\int_{pX} \chi(x) d\mu_k^*(x) = \chi(p)p^k [p]^{-1} B_{k,\chi}^*(q^p)$;
- (3) $\int_X \chi(x) d\mu_{k;q^c}^*(\frac{1}{c}x) = \chi(c) B_{k,\chi}^*(q^c)$;
- (4) $\int_{pX} \chi(x) d\mu_{k;q^c}^*(\frac{1}{c}x) = \chi(c)\chi(p)p^k [p]^{-1} B_{k,\chi}^*(q^{pc})$.

Proof. Using the definition of $\mu_{k;q}^*$ given by Theorem 2.5 and the formula (3.2), the proofs are clear. □

COROLLARY 3.2. For $k \geq 0$

$$\int_X \chi(x)x^k d\mu_q(x) = \int_X \chi(x) d\mu_k^*(x).$$

Proof. The definition of $B_{k,\chi}^*(q)$ and Proposition 3.1(1) imply

$$\int_X \chi(x)x^k d\mu_q(x) = \int_X \chi(x) d\mu_k^*(x).$$

This completes the proof. □

We set

$$(3.3) \quad p^* = \begin{cases} p & \text{if } p > 2 \\ 4 & \text{if } p = 2. \end{cases}$$

Let $\bar{d} = [d, p^*]$ be the least common multiple of conductor d of χ and p^* . By using the I_q -integral, we have the Witt's type formula in the p -adic cyclotomic field $\mathbb{Q}_p(\chi)$ as follows:

$$(3.4) \quad B_{k,\chi}^*(q) = \lim_{N \rightarrow \infty} \sum_{x=1}^{\bar{d}p^N} \chi(x)x^k \frac{q^x}{[\bar{d}p^N]}, \quad k \geq 0.$$

For any rational integers s and t , let $\chi^s = \chi^{s,k;q}$ be an operator on $f(q)$ as follows:

$$(3.5) \quad \chi^s f(q) \stackrel{\text{def}}{=} s^k [s]^{-1} \chi(s) f(q^s);$$

$$(3.6) \quad \chi^s \chi^t \stackrel{\text{def}}{=} [s]^{-1} [s; q^t] \left(\chi^{s,k;q^t} \circ \chi^{t,k;q} \right).$$

Now we choose a rational integer number c such that $(c, \bar{d}) = 1$ and $c \neq \pm 1$, and we put

$$(3.7) \quad \mu_k^c = \frac{1}{k} \frac{\log q}{q-1} \left(\mu_{k;q}^*(U) - c^{k+1} [c]^{-1} \mu_{k;q^c}^* \left(\frac{1}{c} U \right) \right), \quad k \geq 1,$$

where $U \subset X$ is compact open set. Then μ_k^c must be a distribution on X (μ_k^c is not a measure on X) and, using Proposition 3.1 and the definition of the operator χ^p , χ^c and $\chi^p \chi^c$ given by (3.5) and (3.6), this distribution yields an integral on $X^* = X - pX$ as follows:

$$\begin{aligned} & \int_{X^*} \chi(x) d\mu_k^c(x) \\ &= \int_X \chi(x) d\mu_k^c(x) - \int_{pX} \chi(x) d\mu_k^c(x) \\ &= \frac{1}{k} \frac{\log q}{q-1} \left(\int_X \chi(x) d\mu_{k;q}^*(x) - c^{k+1} [c]^{-1} \int_X \chi(x) d\mu_{k;q^c}^* \left(\frac{1}{c} x \right) \right) \\ & \quad - \frac{1}{k} \frac{\log q}{q-1} \left(\int_{pX} \chi(x) d\mu_{k;q}^*(x) - c^{k+1} [c]^{-1} \int_{pX} \chi(x) d\mu_{k;q^c}^* \left(\frac{1}{c} x \right) \right) \\ &= \frac{1}{k} \frac{\log q}{q-1} \left\{ (B_{k,\chi}^*(q) - p^k [p]^{-1} \chi(p) B_{k,\chi}^*(q^p)) \right. \\ & \quad \left. - \chi(c) c^{k+1} [c]^{-1} (B_{k,\chi}^*(q^c) - p^k [p]^{-1} \chi(p) B_{k,\chi}^*(q^{pc})) \right\} \\ &= \frac{\log q}{q-1} (1 - \chi^p)(1 - c\chi^c) \frac{B_{k,\chi}^*(q)}{k}. \end{aligned}$$

Hence we obtain

$$(3.8) \quad \int_{X^*} \chi(x) d\mu_k^c(x) = \frac{\log q}{q-1} (1 - \chi^p)(1 - c\chi^c) \frac{B_{k,\chi}^*(q)}{k}.$$

THEOREM 3.3. For $k \geq 1$

$$\int_{X^*} \chi(x) d\mu_k^c(x) = \lim_{N \rightarrow \infty} \sum_{x=1}^{\bar{d}p^N} \chi(cx) (cx)^{k-1} \left[-\frac{cx}{\bar{d}p^N} \right]_g q^{cx},$$

where \sum^* means to take sums over the rational integers prime to p in the given range, c is a rational integer number such that $(c, \bar{d}) = 1$ and $c \neq \pm 1$, and $[\cdot]_g$ is Gauss's symbol.

Proof. From (3.8) we must show that

$$\lim_{N \rightarrow \infty} \sum_{x=1}^{\bar{d}p^N} \chi(cx)(cx)^{k-1} \left[-\frac{cx}{\bar{d}p^N} \right]_g q^{cx} = \frac{\log q}{q-1} (1-\chi^p)(1-c\chi^c) \frac{B_{k,\chi}^*(q)}{k}.$$

We can rewrite $B_{k,\chi}^*(q)$, $k \geq 1$, given by (3.4) as

$$\begin{aligned} & B_{k,\chi}^*(q) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[\bar{d}p^N]} \sum_{x=1}^{\bar{d}p^N} \chi(x)x^k q^x + \lim_{N \rightarrow \infty} \frac{1}{[\bar{d}p^{N-1}; q^p]} \frac{1}{[p]} \sum_{y=1}^{\bar{d}p^{N-1}} \chi(py)(py)^k q^{py} \\ &= \lim_{N \rightarrow \infty} \frac{1}{[\bar{d}p^N]} \sum_{x=1}^{\bar{d}p^N} \chi(x)x^k q^x + p^k [p]^{-1} \chi(p) B_{k,\chi}^*(q^p). \end{aligned}$$

That is,

$$(A) \quad \lim_{N \rightarrow \infty} \frac{1}{[\bar{d}p^N]} \sum_{x=1}^{\bar{d}p^N} \chi(x)x^k q^x = B_{k,\chi}^*(q) - p^k [p]^{-1} \chi(p) B_{k,\chi}^*(q^p).$$

We choose a rational integer number c such that $(c, \bar{d}) = 1$ and $c \neq \pm 1$. Let x and x_N be the rational integers such that $1 \leq x, x_N \leq \bar{d}p^N$ and $(x, p) = (x_N, p) = 1$, and determine a rational integer number $r_N(x)$ by $x_N = cx + r_N(x)\bar{d}p^N$, i.e.,

$$(B) \quad r_N(x) = -\frac{cx}{\bar{d}p^N} + \frac{x_N}{\bar{d}p^N} = \left[-\frac{cx}{\bar{d}p^N} \right]_g,$$

where $[\cdot]_g$ is Gauss's symbol. Then we have

$$\begin{aligned} & \frac{1}{[\bar{d}p^N]} \sum_{x_N=1}^{\bar{d}p^N} \chi(x_N)(x_N)^k q^{x_N} \\ &= \sum_{x=1}^{\bar{d}p^N} \chi(cx) \left\{ (cx)^k + k(cx)^{k-1} (r_N(x)\bar{d}p^N) + \dots + (r_N(x)\bar{d}p^N)^k \right\} \frac{q^{x_N}}{[\bar{d}p^N]} \\ &\equiv \frac{1}{[\bar{d}p^N]} \sum_{x=1}^{\bar{d}p^N} \chi(cx)(cx)^k q^{x_N} + k \frac{\bar{d}p^N}{[\bar{d}p^N]} \sum_{x=1}^{\bar{d}p^N} \chi(cx)(cx)^{k-1} r_N(x) q^{x_N} \end{aligned}$$

$$(\text{mod } (\bar{d}p^N)^2 [\bar{d}p^N]^{-1}).$$

Since

$$\lim_{N \rightarrow \infty} q^{r_N(x)\bar{d}p^N} = 1 \text{ for } |1 - q|_p < p^{-1/(p-1)}; \quad \lim_{N \rightarrow \infty} \frac{\bar{d}p^N}{[\bar{d}p^N]} = \frac{q - 1}{\log q},$$

using the formula (A) we find that

$$\begin{aligned} & B_{k,\chi}^*(q) - p^k [p]^{-1} \chi(p) B_{k,\chi}^*(q^p) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[\bar{d}p^N]} \sum_{x_N=1}^{\bar{d}p^N} \chi(x_N) (x_N)^k q^{x_N} \\ &= \lim_{N \rightarrow \infty} \frac{1}{[\bar{d}p^N]} \sum_{x=1}^{\bar{d}p^N} \chi(cx) (cx)^k q^{cx} \\ &\quad + k \frac{q - 1}{\log q} \lim_{N \rightarrow \infty} \sum_{x=1}^{\bar{d}p^N} \chi(cx) (cx)^{k-1} r_N(x) q^{cx} \\ &= \chi(c) c^{k+1} [c]^{-1} \lim_{N \rightarrow \infty} \frac{1}{[\bar{d}p^N; q^c]} \sum_{x=1}^{\bar{d}p^N} \chi(x) x^k (q^c)^x \\ &\quad + k \frac{q - 1}{\log q} \lim_{N \rightarrow \infty} \sum_{x=1}^{\bar{d}p^N} \chi(cx) (cx)^{k-1} r_N(x) q^{cx} \\ &= \chi(c) c^{k+1} [c]^{-1} \{ B_{k,\chi}^*(q^c) - p^k [p]^{-1} \chi(p) B_{k,\chi}^*(q^{p^c}) \} \\ &\quad + k \frac{q - 1}{\log q} \lim_{N \rightarrow \infty} \sum_{x=1}^{\bar{d}p^N} \chi(cx) (cx)^{k-1} r_N(x) q^{cx}, \end{aligned}$$

that is, using (B) and the definition of the operators χ^p , χ^c and $\chi^p \chi^c$ given by (3.5) and (3.6) we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \sum_{x=1}^{\bar{d}p^N} \chi(cx) (cx)^{k-1} \left[-\frac{cx}{\bar{d}p^N} \right]_g q^{cx} \\ &= \frac{1}{k} \frac{\log q}{q - 1} \left\{ (B_{k,\chi}^*(q) - p^k [p]^{-1} \chi(p) B_{k,\chi}^*(q^p)) \right. \\ &\quad \left. - \chi(c) c^{k+1} [c]^{-1} (B_{k,\chi}^*(q^c) - p^k [p]^{-1} \chi(p) B_{k,\chi}^*(q^{p^c})) \right\} \\ &= \frac{\log q}{q - 1} (1 - \chi^p)(1 - c\chi^c) \frac{B_{k,\chi}^*(q)}{k}. \end{aligned}$$

This completes the proof. □

Now, we will consider a q -analogue of Nasybullin's lemma (see [8, Theorem 1]; we follow the notation of [8]).

Let $B_n^*(x; q)$ be the n th q -Bernoulli polynomials in (2.2). The n th q -Bernoulli functions $P_n(x)$ are define by $P_n(x) = P(x; q) = B_n^*(x; q)$ for $0 \leq x < 1$. They are periodic with period 1 and agree with the q -Bernoulli polynomials $B_n^*(x; q)$ in the interval $0 \leq x < 1$.

By Lemma 2.4 we have

$$d^n [d]^{-1} \sum_{i=0}^{d-1} q^i B_n^* \left(\frac{x+i}{d}; q^d \right) = B_n^*(x; q).$$

Hence for any real number x

$$d^n [d]^{-1} \sum_{i=0}^{d-1} q^i P_n \left(\frac{x+i}{d}; q^d \right) = P_n(x; q).$$

From the above that the function $P_n(x; q)$ satisfies the property of q -Nasybullin's lemma with constants $A = d^{-n}[d]$, $B = 0$. Then $\rho \neq 0$ is equal to $d^{-n}[d]$, as $\rho^2 = A\rho + B\rho$ reduces simply to $\rho^2 = d^{-n}[d]\rho$. Thus we define the function $\mu_n = \mu_{n,q}$ on $a + \bar{d}p^N \mathbb{Z}_p$ by

$$\mu_n(a + \bar{d}p^N \mathbb{Z}_p) := (\bar{d}p^N)^n [\bar{d}p^N]^{-1} q^a P_n \left(\frac{a}{\bar{d}p^N}; q^{\bar{d}p^N} \right).$$

This can be extended to a measure on $\varprojlim_N (\mathbb{Z}/\bar{d}p^N \mathbb{Z})$ for $N \geq 0$.

Let χ be a primitive Dirichlet character with conductor \bar{d} . Then the generalized q -Bernoulli number in (3.4) is defined by

$$B_{k,\chi}^*(q) = \lim_{N \rightarrow \infty} \frac{1}{[\bar{d}p^N]} \sum_{n=0}^{\bar{d}p^N-1} \chi(n) n^k q^n = \frac{\bar{d}^k}{[\bar{d}]} \sum_{a=0}^{\bar{d}-1} q^a \chi(a) B_k^* \left(\frac{a}{\bar{d}}; q^{\bar{d}} \right).$$

Let

$$\begin{aligned} L(\mu_n, \chi) &= \lim_{N \rightarrow \infty} \sum_{a=0}^{\bar{d}p^N-1} \chi(a) \mu_n(a + \bar{d}p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \sum_{\substack{a \pmod{\bar{d}p^N} \\ (a,p)=1}} \chi(a) \mu_n(a + \bar{d}p^N \mathbb{Z}_p), \end{aligned}$$

where \sum^* means to take sums over the rational integers prime to p in the given range. Then since the character χ is constant on $a + \bar{d}\mathbb{Z}_p$,

$$\begin{aligned} L(\mu_n, \chi) &= \lim_{N \rightarrow \infty} \sum_{a \pmod{\bar{d}p^N}} \chi(a) \mu_n(a + \bar{d}p^N \mathbb{Z}_p) \\ &\quad - \lim_{N \rightarrow \infty} \sum_{\substack{a \pmod{\bar{d}p^N} \\ p|a}} \chi(a) \mu_n(a + \bar{d}p^N \mathbb{Z}_p) \\ &= B_{n, \chi}^*(q) - p^n [p]^{-1} \chi(p) B_{n, \chi}^*(q^p), \end{aligned}$$

where $B_{n, \chi}^*(q)$ is the n th q -Bernoulli number containing χ . Thus we obtain

$$L(\mu_n, \chi \omega^{-n}) = B_{n, \chi \omega^{-n}}^*(q) - p^n [p]^{-1} \chi \omega^{-n}(p) B_{n, \chi \omega^{-n}}^*(q^p)$$

where $n \geq 1$ and ω is the Teichmüller character mod p^* .

4. I_q -integral and Tsumura's p -adic function

Let $z \in \mathbb{C}_p$ be such that $z^{dp^N} \neq 1$ for all N . In [10], N. Koblitz defined

$$(4.1) \quad E_z(a + dp^N \mathbb{Z}_p) = \frac{z^a}{1 - z^{dp^N}}.$$

He obtained

PROPOSITION 4.1 ([10]). *E_z is a distribution on X . Let $D_1 = \{x \in \mathbb{C}_p \mid |x - 1|_p < 1\}$, and let $\bar{D}_1 = \mathbb{C}_p \setminus D_1$ be the complement of the open unit disc around 1. Then E_z is a measure if and only if $z \in \bar{D}_1$.*

Note that if $q \in \bar{D}_1$ and $\text{ord}_p(1 - q) \neq -\infty$, then $\mu_q(a + dp^N \mathbb{Z}_p) = (1 - q)E_q(a + dp^N \mathbb{Z}_p)$. Thus $\mu_q(a + dp^N \mathbb{Z}_p) = \frac{q^a}{[dp^N]}$ in $q \in \bar{D}_1$ and $\text{ord}_p(1 - q) \neq -\infty$ is the similar measure as Koblitz measure.

Hereafter, we assume that $q \in \bar{D}_1$ and $\text{ord}_p(1 - q) \neq -\infty$.

Now, for $t \in \mathbb{C}_p$ with $\text{ord}_p t > \frac{1}{p-1}$, we define a number $H_m^*(q)$ by

$$(4.2) \quad \frac{q - 1}{qe^t - 1} = \sum_{m=0}^{\infty} H_m^*(q) \frac{t^m}{m!}.$$

Note that $H_m^*(q^{-1}) = H_m(q)$ where the number $H_m(q)$ defined by $\frac{1-q}{e^t-q} = \sum_{m=0}^{\infty} \frac{H_m(q)}{m!} t^m$ is called the m -th Euler number belonging to q , which lies in an algebraic closure of \mathbb{Q}_p .

We can express the numbers $H_m^*(q)$ as an integral over \mathbb{Z}_p , for $d = 1, X = \mathbb{Z}_p$, by using the measure μ_q , that is,

$$(4.3) \quad \int_{\mathbb{Z}_p} x^m d\mu_q(x) = H_m^*(q) \quad \text{for } m \geq 0.$$

Indeed, we find that

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) &= \lim_{N \rightarrow \infty} \frac{1-q}{1-q^{p^N}} \sum_{a=0}^{p^N-1} e^{at} q^a \\ &= \frac{1-q}{1-qe^t} \lim_{N \rightarrow \infty} \frac{1-e^{tp^N} q^{p^N}}{1-q^{p^N}} = \frac{q-1}{qe^t-1}, \end{aligned}$$

since e^{tp^N} approaches 1 as $N \rightarrow \infty$, the limit is 1. Let $t \in \mathbb{C}_p$ with $\text{ord}_p t > \frac{1}{p-1}$. Then we obtain

$$(4.4) \quad \sum_{m=0}^{\infty} H_m^*(q) \frac{t^m}{m!} = \int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) = \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} x^m d\mu_q(x) \frac{t^m}{m!}.$$

Hence, comparing the above formulas, $\int_{\mathbb{Z}_p} x^m d\mu_q(x) = H_m^*(q)$ for $m \geq 0$.

Note that if $q \in D_1$ then $\int_{\mathbb{Z}_p} x^m d\mu_q(x) = B_m^*(q)$ (see Section 2).

Let ω denote the Teichmüller character mod p^* . For $x \in X^*$, we set $\langle x \rangle = x/\omega(x)$. For $s \in \mathbb{Z}_p$, we define

$$(4.5) \quad \ell_{p,q}(s) \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \frac{1-q}{1-q^{p^N}} \sum_{m=0}^{p^N-1} \frac{q^m}{m^s}.$$

Then we obtain $\ell_{p,q}(-k) = \lim_{N \rightarrow \infty} \frac{1-q}{1-q^{p^N}} \sum_{m=0}^{p^N-1} q^m m^k = H_k^*(q)$ for $k \geq 0$.

Let χ be a primitive Dirichlet character with conductor d . For $k \geq 0$, the generalized numbers $H_{k,\chi}^*(q)$ is defined by

$$(4.6) \quad H_{k,\chi}^*(q) = \int_X \chi(x) x^k d\mu_q(x).$$

For $s \in \mathbb{Z}_p$, we define the function $\ell_{p,q}$ by

$$(4.7) \quad \ell_{p,q}(s, \chi) = \int_{X^*} \langle x \rangle^{-s} \chi(x) d\mu_q(x)$$

which is slightly different from the one in [14]. The value of this function at non-positive integers are given by

PROPOSITION 4.2. *For any $k \geq 0$, we have*

$$\ell_{p,q}(-k, \chi\omega^k) = H_{k,\chi}^*(q) - p^k [p]^{-1} \chi(p) H_{k,\chi}^*(q^p).$$

Proof. Since $\mu_q(pU) = \mu_{q^p}(U)$ for $U \subset X$, $\int_{pX} \chi(x)x^k d\mu_q(x) = [p]^{-1} \int_X \chi(px)(px)^k d\mu_{q^p}(x) = p^k [p]^{-1} \chi(p) H_{k,\chi}^*(q^p)$. The proof now follows directly. \square

For $\alpha, \beta \in \mathbb{C}_p$ and any function $f(q)$, we set

$$(4.8) \quad (\alpha + \beta p^k) \otimes f(q) := \alpha f(q) + \beta p^k f(q^p).$$

We have the following Kummer congruences.

COROLLARY 4.3. *If $k \equiv k' \pmod{(p-1)p^N}$, then*

$$(1 - \chi(p)p^k) \otimes \frac{H_{k,\chi}^*(q)}{1-q} \equiv (1 - \chi(p)p^{k'}) \otimes \frac{H_{k',\chi}^*(q)}{1-q} \pmod{p^N}.$$

Proof. Note that (see [10, Proposition 2])

$$\left| \frac{\mu_q(a + dp^N \mathbb{Z}_p)}{1-q} \right|_p = \left| \frac{q^a}{(1-q)[dp^N]} \right|_p = \left| \frac{q^a}{1-q^{dp^N}} \right|_p \leq 1,$$

where we use the assumption $q \in \overline{D}_1$. By [12, Chapter II, §2], if $k \equiv k' \pmod{(p-1)p^N}$, then we have

$$|x^k - x^{k'}|_p \leq \frac{1}{p^N} \quad \text{for } x \in X^*.$$

Using the corollary at the end of [12, Chapter II, §5], we easily see that

$$\begin{aligned} \frac{\ell_{p,q}(-k, \chi\omega^k)}{1-q} &= \int_{X^*} \langle x \rangle^k \chi\omega^k(x) \frac{d\mu_q(x)}{1-q} \\ &= \int_{X^*} \chi(x)x^k \frac{d\mu_q(x)}{1-q} \\ &\equiv \int_{X^*} \chi(x)x^{k'} \frac{d\mu_q(x)}{1-q} \pmod{p^N} \\ &= \frac{\ell_{p,q}(-k', \chi\omega^{k'})}{1-q}. \end{aligned}$$

By Proposition 4.2 and (4.8), the result now follows easily. \square

REMARK. By the definition (2.2) and (4.2), we obtain that

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^*(q) \frac{t^n}{n!} &= \frac{q-1}{\log q} \left(\frac{\log q + t}{qe^t - 1} \right) \\ &= \frac{q-1}{qe^t - 1} + \frac{t}{\log q} \frac{q-1}{qe^t - 1} \\ &= \sum_{n=0}^{\infty} H_n^*(q) \frac{t^n}{n!} + \frac{t}{\log q} \sum_{n=0}^{\infty} H_n^*(q) \frac{t^n}{n!}. \end{aligned}$$

Equating the coefficient of t^n , we obtain the following relation between the q -analogue Bernoulli numbers $B_n^*(q)$ and the number $H_n^*(q)$

$$B_n^*(q) = H_n^*(q) + \frac{n}{\log q} H_{n-1}^*(q) \quad (n \geq 1).$$

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