SOME REMARKS ON ALGEBRAIC GROUPS

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0. In the section 1 we give a Galois correspondence between a family of subfields of the function field of a connected algebraic group G and a family of algebraic subgroups of G. Generally, if the universal domain is of characteristic $p > 0$, any algebraic subalgebras of the Lie algebras of algebraic groups are p-algebras, but the converse is not true. In the section 2 we give a necessary and sufficient condition for p-subalgebra of the Lie algebra $\mathfrak{g}$ of G to be algebraic, and we show that a subalgebra is a p-subalgebra if and only if it is replica closed. If G is affine, the p-subalgebra generated by one element of $\mathfrak{g}$ is not only replica closed but algebraic. We treat p-subalgebras generated by one element in the section 3. In the section 4 we give some examples showing that p-subalgebras of $\mathfrak{g}$ are not generally algebraic and that the global analogy of the characterization of algebraic subalgebras does not hold even if the universal domain is of characteristic 0.

1. Let G be a connected algebraic group defined over an algebraically closed field k; let $\Omega(G)$ be the field of rational functions of G; let $k(G)$ be the subfield of $\Omega(G)$ consisting of rational functions defined over k. For any point p on G, let R_p^* (resp. L_p^*) be the Ω-automorphism of $\Omega(G)$ induced by the right (resp. left) translation R_p (resp. L_p) of G by p. A subset of $k(G)$ is called to be right (resp. left) invariant if, for any rational point p over k on G, R_p^* (resp. L_p^*) maps it into itself. We call a subfield $\mathfrak{F}$ of $k(G)$ a (H)-subfield of $k(G)$ if the following three conditions are satisfied; (i) $\mathfrak{F}$ contains k, (ii) $\mathfrak{F}$ is right invariant, (iii) $k(G)$ is separrably generated over $\mathfrak{F}$.

Let H be an algebraic subgroup of G which is defined over k; let $k_H(G)$ be the subfield of $k(G)$ consisting of all f such that $L_p^* f = f$ for any rational point p over k on H. Then $k_H(G)$ is a (H)-subfield of $k(G)$.[1] We shall denote $k_H(G)$ by $\mathfrak{F}(H)$.

Conversely, for any (H)-subfield $\mathfrak{F}$ of $k(G)$, let H be the algebraic subgroup of G consisting of all x such that $L_x^* f = f$ for any $f \in \mathfrak{F}$. Then H is k-closed and k being algebraically closed, H is defined over k, which we shall denote by $H(\mathfrak{F})$.

Then we have the following theorem:

1) Cf. The proof of the Theorem 2 of [10]

THEOREM *The correspondences $H \to \mathfrak{F}(H)$ and $\mathfrak{F} \to H(\mathfrak{F})$ are the inverses of each other and give one-to-one correspondence between the algebraic subgroups defined over k of G and the (H)-subfields of $k(G)$. Further H is normal if and only if $\mathfrak{F}(H)$ is left invariant, and H is connected if and only if $\mathfrak{F}(H)$ is algebraically closed in $k(G)$.*

Let $\mathfrak{F}$ be a (H)-subfield of $k(G)$; put $\mathfrak{F}' = \mathfrak{F}(H(\mathfrak{F}))$, then clearly $\mathfrak{F}'$ contains $\mathfrak{F}$. By the proposition 3 of [12], there exist functions $f_1, \ldots\ldots f_r \in \mathfrak{F}$ such that $\mathfrak{F} = k(f_1, \ldots\ldots, f_r)$. Let x be a generic point over k on G; let V be the locus of $(f_1(x), \ldots\ldots, f_r(x))$ over k on the r-dimensional affine space S^r, then there exists a generically surjective rational mapping τ of G into V such that $\tau(x) = (f_1(x), \ldots\ldots, f_r(x))$. But we have that, for generic points x and y over k on G, $x \in H(\mathfrak{F})\, y$ if and only if $f(x) = f(y)$ for any $f \in \mathfrak{F}$; in fact, suppose that $f(x) = f(y)$ for any $f \in \mathfrak{F}$; $\mathfrak{F}$ being right invarant, $R^*_p f(x) = R^*_p f(y)$ for any rational point p over k on G and $L^*_x\, f(p) = L^*_y f(p)$; since the points of G that are rational over k are dense in G, $L^*_x\, f = L^*_y f$ for any $f \in \mathfrak{F}$; hence $L^*_{y^{-1}} L^*_x f = L^*_{xy^{-1}} f$ and $xy^{-1} \in H(\mathfrak{F})$; the coinverse is trivial. Therefore there exists a generically surjective and generically one-to-one rational mapping $\tilde{\tau}$ of the homogeneous space $G/H(\mathfrak{F})$ into V such that $\tilde{\tau} \circ \varphi = \tau$, where φ is the natural mapping of G into $G/H(\mathfrak{F})$. Let φ^*, τ^* and $\tilde{\tau}^*$ be the mappings of the function fields concerned, which are induced naturally by φ, τ and $\tilde{\tau}$, respectively, and we have $\varphi^* \cdot \tilde{\tau}^*\ f = \tau^*\ f$ for any $f \in k(V)$. From the definitions $\mathfrak{F}' = k_{H(\mathfrak{F})}(G) = \varphi^* k(G/H(\mathfrak{F}))$ and $\mathfrak{F} = \tau^* k(V)$. Since $k(G)$ is separably generated over $\mathfrak{F}$, by the proposition 3 of [12] and the proposition 19 of [13], I_7, $\mathfrak{F}'$ is separably generated over $\mathfrak{F}$. So we have that $k(G/H(\mathfrak{F}))$ is separably generated over $\tilde{\tau}^*\ k(V)$. But $\tilde{\tau}$ is generically one-to-one. Hence $k(G/H(\mathfrak{F})) = \tilde{\tau}^* k(V)$ and $\mathfrak{F}' = \varphi^* k(G/H(\mathfrak{F})) = \varphi^* \tilde{\tau}^* k(V) = \tau^*\ k(V) = \mathfrak{F}$.

Conversely, let H be an algebraic subgroup of G which is defined over k; put $H' = H(\mathfrak{F}(H))$, then $H' \supset H$. Since from what we have seen, $\mathfrak{F}(H(\mathfrak{F}(H))) = \mathfrak{F}(H)$ and $k_{H'}(G) = k_H(G)$. The lemma 2 of [7] shows that $H' = H$.

If H is normal in G, for $f \in \mathfrak{F}(H)$, $x \in H$ and a rational point p over k on G, $L^*_x L^*_p\ f = L^*_{px}\ f = L^*_{yp}\ f = L^*_p L^*_y\ f = L^*_p\ f$ for some $y \in H$. Hence $L^*_p f \in \mathfrak{F}(H)$. Conversely if $\mathfrak{F}(H)$ is left invariant, for $x \in H$ and a rational point p over k on G, $L^*_{pxp^{-1}}\ f = L^*_{p^{-1}}\, L^*_x L^*_p f = L^*_{p^{-1}} L^*_p\ f = f$. Hence $pxp^{-1} \in H$. Since the rational points over k on G are dense in G, H is normal in G.

Now suppose that H is connected. If $f \in k(G)$ is algebraic over $\mathfrak{F}(H)$, $L^*_x\ f$ is a conjugate of f over $\mathfrak{F}(H)$ for any $x \in H$. The subset H_0 of H consisting of all x such that $L^*_x\ f = f$ is an algebraic subgroup of H of finite index. Since H is connected, $H_0 = H$ and $f \in \mathfrak{F}(H)$. Conversely suppose that $\mathfrak{F}(H)$ is algebraically closed in $k(G)$. Let H_0 be the connected component of H containing the unit element. Then there exist finite rational points h_i over k on H such that $H = \bigcup_{i=1}^{s} H_0 h_i$ is a coset-decomposition. Then for $f \in \mathfrak{F}(H_0)$ and $h_0 \in H_0$, $L^*_{h_0 h_i} f = L^*_{h_i} f$. Thus for $h \in H$, L^*_h gives a permutation of $\{L^*_{h_1} f, \ldots, L^*_{h_s} f\}$ and $P(X) = \prod_{i=1}^{s} (X - L^*_{h_i} f)$ is in $\mathfrak{F}(H)[X]$ such that $P(f) = 0$. So $\mathfrak{F}(H_0)$ is algebraic over $\mathfrak{F}(H)$ and $\mathfrak{F}(H_0) = \mathfrak{F}(H)$. Thus $H_0 = H$.

2. The notations being as in the section 1, let k be an algebraically closed field of characteristic $p > 0$; let $\mathfrak{D}$ be the Lie algebra of all derivations of $k(G)$ which is a Lie algebra over $k(G)$ of dimension n, where n is the dimension of G; put $\Gamma = k(G)^p$; for a subfield $\mathfrak{S}$ of $k(G)$ containing Γ, let $\mathfrak{E}(\mathfrak{S})$ be the subalgebra of $\mathfrak{D}$ consisting of all D such that $Df = 0$ for $f \in \mathfrak{S}$, then $\mathfrak{E}(\mathfrak{S})$ is a p-subalgebra of $\mathfrak{D}$. Conversely, for a p-subalgebra $\mathfrak{E}$ of $\mathfrak{D}$, let $\mathfrak{S}(\mathfrak{E})$ be the subset of $k(G)$ consisting of all f such that $Df = 0$ for all $D \in \mathfrak{E}$, then $\mathfrak{S}(\mathfrak{E})$ is a subfield of $k(G)$ containing Γ. The theorem 12 of [6] shows that $\mathfrak{S} \to \mathfrak{E}(\mathfrak{S})$ and $\mathfrak{E} \to \mathfrak{S}(\mathfrak{E})$ are the inverses of each other and give one-to-one correspondence between the p-subalgebras of $\mathfrak{D}$ and the subfields of $k(G)$ containing Γ.

Now let $\mathfrak{g}_k$ be the subset of $\mathfrak{g}$ consisting of all invariant derivations defined over k, then $\mathfrak{g}_k$ is a Lie algebra over k, whose scalar extension to Ω is $\mathfrak{g}$. If $\mathfrak{H}$ is a p-subalgebra of $\mathfrak{D}$, the intersection $\mathfrak{H} \cap \mathfrak{g}_k$ is a p-subalgebra of $\mathfrak{g}_k$. Conversely if $\mathfrak{h}_k$ is a p-subalgebra of $\mathfrak{g}_k$, the scalar extension of $\mathfrak{h}_k$ to $k(G)$ is a p-subalgebra of $\mathfrak{D}$ by the Hochschild's formula

$$(fD)^p = f^p D^p + (fD)^{p-1} fD \qquad \text{for } D \in \mathfrak{D},\ f \in k(G)$$

and the Jacobson's formula

$$(D_1 + D_2)^p = D_1^p + D_2^p + s(D_1, D_2) \qquad \text{for } D_1, D_2 \in \mathfrak{D},$$

where $s(D_1, D_2)$ is a polynomial with respect to the bracket operation. Similarly if $\mathfrak{h}$ is a p-subalgebra of $\mathfrak{g}$, the intersection $\mathfrak{h} \cap \mathfrak{g}_k$ is a p-subalgebra of $\mathfrak{g}_k$, and if $\mathfrak{h}_k$ is a p-subalgebra of $\mathfrak{g}_k$, the scalar extension of $\mathfrak{h}_k$ to Ω is a p-subalgebra of $\mathfrak{g}$. We call a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to be defined over k if $\mathfrak{h}$ has a base consisting of invariant derivations defined over k.

Let H be a connected algebraic subgroup of G defined over k; let $\mathfrak{h}$ be the Lie algebra of H; let $h \times x$ be a generic point over k on $H \times G$; for $f \in \mathfrak{F}(H)$ we have $f(hx) = f(x)$ and therefore $R^*_x f - f(x) \in k(x)(G) \cap \mathfrak{m}_h$, where $\mathfrak{m}_h$ is the maximal ideal of the local ring of h in $\Omega(G)$. If $D \in \mathfrak{h}$ is defined over k, $D(R^*_x f - f(x)) \in k(x)(G) \cap \mathfrak{m}_h$ and $D(R^*_x f - f(x))(h) = 0$.

But $D(R_x^* f - f(x))(h) = (DR_x^* f)(h) = (R_x^* Df)(h) = (Df)(hx)$. Since hx is generic over k on G and Df is defined over k, we have $Df = 0$. Thus, $\mathfrak{h}$ being defined over k, we obtain that $Df = 0$ for $D \in \mathfrak{h}$ and $f \in \mathfrak{F}(H)$. Now put $\mathfrak{S}$ be the compositum of Γ and $\mathfrak{F}(H)$, then $\mathfrak{E}(\mathfrak{S}) \cap \mathfrak{g}_k \supset \mathfrak{h}_k$, where $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{g}$. By the proposition 1 of [10] the dimension of $\mathfrak{F}(H)$ over k is $n - r$ if the dimension of H is r. Since $k(G)$ is separably generated over $\mathfrak{F}(H)$, the derivation algebra of $k(G)$ over $\mathfrak{F}(H)$ which is in fact $\mathfrak{E}(\mathfrak{S})$ itself is r-dimensional over $k(G)$. Since the elements of $\mathfrak{g}_k$ which are linearly indepenent over k are linearly independent over $k(G)$, we have that the dimension of $\mathfrak{E}(\mathfrak{S}) \cap \mathfrak{g}_k$ over k is at most r. Thus $\mathfrak{G}(\mathfrak{S}) \cap \mathfrak{g}_k = \mathfrak{h}_k$ and the scalar extension of $\mathfrak{E}(\mathfrak{S}) \cap \mathfrak{g}_k$ to Ω is $\mathfrak{h}$.

Conversely let $\mathfrak{h}$ be a p-subalgebra of $\mathfrak{g}$ defined over k for which there exists a (H)-subfield $\mathfrak{F}$ of $k(G)$ such that for $D \in \mathfrak{g}$ to be $D \in \mathfrak{h}$ it is necessary and sufficient $Df = 0$ for $f \in \mathfrak{F}$. Then from the theorem and what we have known it is easily seen that $\mathfrak{h}$ is the Lie algebra of $H(\mathfrak{F})$.

Thus we obtain a proposition which gives a method to take algebraic subalgepra of $\mathfrak{g}$ from non-algebraic p-subalgebras of $\mathfrak{g}$;

PROPOSITION 1 *Let* $\mathfrak{h}$ *be a p-subalgebra of* $\mathfrak{g}$ *defined over k. Then* $\mathfrak{h}$ *is algebraic if and only if* $\mathfrak{S}((\mathfrak{h} \cap \mathfrak{g}_k)^{k(G)})$ *is the compositum of* Γ *and some* (H)*-subfield* $\mathfrak{F}$ *of* $k(G)$, *and then* $\mathfrak{h}$ *is the Lie algebra of* $H(\mathfrak{F})$, *where* $(\mathfrak{h} \cap \mathfrak{g}_k)^{k(G)}$ *is the scalar extension of* $\mathfrak{h} \cap \mathfrak{g}_k$ *to* $k(G)$.

We shall give an example of non algebraic p-subalgebra of $\mathfrak{g}$ in the section 4.

Further,for an algebraic p-subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exist generally infinitely many subgroups of G whose Lie algebras are the same $\mathfrak{h}$. For example, let G be the group of all diagonal matrices in $GL(2, \Omega)$, where Ω is of characteristic $p > 0$, then the subgroups H_s consisting of the diagonal matrices (z, z^{p^s}), $z \in \Omega^*$ have the same Lie algebra consisting of the diagonal matrices $(a, 0)$, $a \in \Omega$ where $s = 0, 1, 2, \ldots\ldots$. As a corollary to the proposition 1, we have

COROLLARY *Let* H_1 *and* H_2 *be algebraic subgroups of* G *defined over k. Then the Lie algebras of* H_1 *and* H_2 *are same if and only if* $\mathfrak{F}(H_1)\Gamma = \mathfrak{F}(H_2)\Gamma$, *where* $\mathfrak{F}(H_i)\Gamma$ *means the compositum of* Γ *and* $\mathfrak{F}(H_i)$ *for* $i = 1, 2$.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called an r-subalgebra of $\mathfrak{g}$ if $\mathfrak{h}$ contains all replicas of any elements of itself.[2] From the definition, for any D of $\mathfrak{g}$, D^p is a replica of D. Hence any r-subalgebras of $\mathfrak{g}$ are p-subalgebras of $\mathfrak{g}$. Conversely if $\mathfrak{h}$ is a p-subalgebra of $\mathfrak{g}$ defined over k, the intersection $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{g}_k$ and

2) Cf. the section 1 of [7].

its scalar extension $\mathfrak{h}_k^{k(G)}$ to $k(G)$ are p-subalgebras of $\mathfrak{g}_k$ and $\mathfrak{D}$ respectively. From what we have seen it follows that $\mathfrak{E}(\mathfrak{S}(\mathfrak{h}_k^{k(G)})) \cap \mathfrak{g}_k = \mathfrak{h}_k$. Let $D_1, \ldots\ldots D_r, D_1', \ldots\ldots, D_s'$ be a base for $\mathfrak{g}_k$ such that $D_1, \ldots\ldots, D_r$ is a base for $\mathfrak{h}_k$; let $D = \sum_{i=1}^{r} \alpha_i D_i + \sum_{j=1}^{s} \beta_j D_j' \in \mathfrak{g}$ for some $\alpha_i, \beta_j \in \Omega$. Then if $Df = 0$ for any $f \in \mathfrak{S}(\mathfrak{h}_k^{k(G)})$, D is contained in $\mathfrak{h}$; in fact, there exist $\gamma_1, \ldots\ldots, \gamma_t \in \Omega$ such that $\gamma_1, \ldots\ldots, \gamma_t$ are linearly independent over k and $\sum_{j=1}^{s} \beta_j D_j' = \sum_{j=1}^{t} \gamma_j D_j''$ for some $D_j'' \in kD_1' + \ldots\ldots + kD_s'$; if $f \in \mathfrak{S}(\mathfrak{h}_k^{k(G)})$, then $Df = \sum \alpha_i D_i f + \sum \beta_j D'_j f = \sum \gamma_j D_j'' f = 0$ and therefore $\gamma_j = 0$ for j such that $D_j'' f \neq 0$, $k(G)$ and Ω being linearly disjoint over k; since for D_j'' there exists $f_j \in \mathfrak{S}(\mathfrak{h}_k^{k(G)})$ such that $D_j'' f_j \neq 0$, we have $D = \sum_{i=1}^{r} \alpha_i D_i$. Any $D \in \mathfrak{h}$ annihilating $\mathfrak{S}(\mathfrak{h}_k^{k(G)})$, we have that $\mathfrak{h}$ consists of all elements of $\mathfrak{g}$ annihilating $\mathfrak{S}(\mathfrak{h}_k^{k(G)})$ and that $\mathfrak{h}$ is an r-subalgebra of $\mathfrak{g}$. Thus we obtain

PROPOSITION 2 *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a p-subalgebra if and only if $\mathfrak{h}$ is an r-subalgebra.*

In the next section we shall show directly this fact in the case of algebraic groups of matrices.

3. In this section we use the definitions by Chevalley. Let k be an algebraically closed field of characteristic $p > 0$. For $X \in \mathfrak{gl}(n, k)$ let $\mathfrak{p}(X)$ be the p-subalgebra generated by X and let $\mathfrak{r}(X)$ be the r-subalgebra generated by X. Then we have

PROPOSITION 3 *For any $X \in \mathfrak{gl}(n, k)$, $\mathfrak{p}(X) = \mathfrak{r}(X)$ and this subalgebra is algebsaic.*

Let $X = S + N$ be the canonical decomposition, i. e. S is semi-simple and N is nilpotent such that $NS = SN$. Since $X^{p^i} = S^{p^i} + N^{p^i}$, we have $\mathfrak{p}(X) = \mathfrak{p}(S) + \mathfrak{p}(N)$[3]. But by the theorem 5 of [3], for $X' \in \mathfrak{gl}(n, k)$ to be a replica of X it is necessary and sufficient that S' and N' are replicas of S and N, respectively, where $X' = S' + N'$ is the canonical decomposition of X'. Hence we have $\mathfrak{r}(X) = \mathfrak{r}(S) + \mathfrak{r}(N)$. Thus to show $\mathfrak{p}(X) = \mathfrak{r}(X)$ it is sufficient to show it for the case where X is semisimple or nilpotent. The well known theorem[4] of Chevalley and Tuan shows $\mathfrak{p}(X) = \mathfrak{r}(X)$ for the nilpotent case. If X is a semi-simple S, we may suppose that S is diag. $(\alpha_1, \ldots\ldots, \alpha_n)$ for some

3) Cf. the proof of the theorem 2 of [5].
4) Cf. [11].

$\alpha_i \in k$. Let k_0 be the prime field of k; let $\xi_1, \xi_2, \ldots\ldots, \xi_n, \ldots\ldots$ be a base of k over k_0, then $S = \sum_{j=1}^{l} \xi_j S_j$ for some $S_j = \text{diag.}\ (\alpha_{j1}, \ldots\ldots, \alpha_{jn})$ for $\alpha_{ji} \in k$. It is easily seen that $\mathfrak{r}(S)$ is the vector space spanned by $S_1, \ldots\ldots, S_l$ over k; in fact, for $B = \text{diag.}\ (\beta_1, \ldots\ldots, \beta_n)$, we may suppose that $B = \sum_{j=1}^{l} \xi_j B_j$ for some $B_j = \text{diag.}(\beta_{j1}, \ldots\ldots, \beta_{jn})$ for $\beta_{ji} \in k_0$; B is a replica of S if and only if B is a linear specialization of S, i. e. $\sum_{j=1}^{l} n_j\beta_j = 0$ for integers n_j such that $\sum_{j=1}^{l} n_j\alpha_j = 0$ i. e. $\sum_{j=1}^{l} n_j\beta_{ji} = 0$ for integers n_j such that $\sum_{j=1}^{l} n_j\alpha_{ji} = 0$; thus B is a replica of S if and only if B is contained in the vector space spanned by $S_1, \ldots\ldots, S_l$. Hence $S^{p^i} = \sum_{j=1}^{l} \xi_j^{p^i} S_j^{p^i} = \sum_{j=1}^{l} \xi_j^{p^i} S_j$ implies $S^{p^i} \in \mathfrak{r}(\mathfrak{S})$ and $\mathfrak{p}(S) \subseteqq \mathfrak{r}(S)$.

Let m be the integer such that $S, S^p, \ldots\ldots, S^{p^{m-1}}$ are linearly independent over k and S^{p^m} is linearly dependent upon $S, S^p, \ldots\ldots, S^{p^{m-1}}$ over k, then we have $\dim \mathfrak{p}(S) = m$. Let $f_p(x) = x^{p^m} + \beta_1 x^{p^{m-1}} + \ldots\ldots + \beta_m x \in k[x]$ such that $f_p(S) = 0$, then $\beta_m \neq 0$, and $f_p(S) = \text{diag.}\ (f_p(\alpha_1), \ldots\ldots, f_p(\alpha_n)) = 0$ implies $f_p(\alpha_j) = 0$, $1 \leqq j \leqq n$. But as $\beta_m \neq 0$, by the theorem 6 of [9], the set of all roots of $f_p(x)$ in k is a vector space of dimension m over k_0, and therefore the elements $\alpha_j = \sum_{i=1}^{l} \alpha_{ij}\xi_i$, $1 \leqq j \leqq n$, generate a vector space whose dimension over k_0 is at most m. So, put $A = (\alpha_{ij})_{1 \leqq i \leqq l, 1 \leqq j \leqq n}$, and we have rank $A \leqq m$. Thus $\dim \mathfrak{r}(S) \leqq \dim \mathfrak{p}(S)$ and $\mathfrak{r}(S) = \mathfrak{p}(S)$.

Now we shall show that $\mathfrak{p}(X)$ is algebraic. Let $X = S + N$ and $S = \sum_{j=1}^{l} \xi_j S_j$ as above. The subgroup of $GL(n, k)$ consisting of all diag. $(t^{\alpha_{j1}}, \ldots\ldots t^{\alpha_{jn}})$ for $t \in k^*$ is a connected algebraic group $G(S_j)$ of dimension 1 whose Lie algebra is kS_j. Put $G(S) = \prod_{j-1}^{l} G(S_j)$, then $G(S)$ is the connected commutative algebraic group of dimension m whose Lie algebra is $\mathfrak{p}(S)$. Let q be the integer such that $N^q = 0$ and $N^{q-1} \neq 0$; let r be the maximal integer such that $p^r < q$; let $U_0, \ldots\ldots U_r$ be indeterminates. Then we have exp $(U_0 N + U_1 N^p + \ldots\ldots + U_r N^{p^r}) = \sum_{h=0}^{q-1} E_h(T_0, \ldots\ldots, T_r) N^h$, where T_i is the coefficient of N^{p^i} in this series and $E_h(T_0, \ldots\ldots, T_r)$ is a polynomial of $T, \ldots\ldots T_r$ with coefficients in the p-adic integer ring.[5] Putting $E(T_0, \ldots\ldots, T_r; N) =$

5) Cf. the lemma 6 of [5].

$\exp(U_0N + \dots\dots + U_rN^{p^r})$, let $G(N)$ be the set of all $E(t_0, \dots\dots, t_r; N)$ for $t_i \in k$, then it is known that $G(N)$ is a connected algebraic group of dimension $r+1$ whose Lie algebra is $\mathfrak{p}(N)$[6]. Let $G(X)$ be the connected algebraic group generated by $G(S)$ and $G(N)$. N being supposed to be in $\mathfrak{gl}(n, k_0)$, $SN = NS$ implies that $S_jN = NS_j$. Hence from the definitions of $G(S)$ and $G(N)$ it follows that $G(S)$ and $G(N)$ are elementwise commutative and therefore $G(X) = G(S)G(N)$. But $G(S) \cap G(N) = \{I\}$, where I is the unit matrix. Thus $G(X)$ is the direct product of $G(S)$ and $G(N)$ and $\dim G(X) = m + r + 1$. Let $\mathfrak{g}(X)$ be the Lie algebra of $G(X)$, then $\mathfrak{g}(X) \supseteq \mathfrak{p}(S)$ and $\mathfrak{g}(X) \supseteq \mathfrak{p}(N)$. Compairing the dimensions, we have $\mathfrak{g}(X) = \mathfrak{p}(S) + \mathfrak{p}(N)$. Thus we have $\mathfrak{g}(X) = \mathfrak{p}(X)$.

Since any algebraic subalgebras of $\mathfrak{gl}(n, k)$ are p-subalgebras, this proposition shows that for $X \in \mathfrak{gl}(n, k)$ there exists the minimal algebraic subalgebra of $\mathfrak{gl}(n, k)$ containing X. But it is not generally true that for $X \in \mathfrak{gl}(n, k)$ there exists the minimal algebraic subgroup of $GL(n, k)$ whose Lie algebra contains X.

4. We wish to show an example of an algebraic group whose Lie algebra contains a p-subalgebra which is not algebraic.

Let G be the group of Chevalley constructed from a simple Lie algebra $\mathfrak{g}$ of the type (A_l), $l > 3$, over the universal domain Ω whose characteristic p is a divisor of $l+1$. By the theorem 2 and the corollary 1 to proposition 7 of [8], G is a connected algebraic group of dimension n, where $n = l(l+2)$ is the dimension of $\mathfrak{g}$.

Let $H_1, \dots\dots, H_l, X_r$, r roots, be a canonical base of $\mathfrak{g}$, from which G is constructed[7]; let $\mathfrak{g}_\Omega$, $\mathfrak{h}_\Omega$ be the tensor product of Ω and the additive group generated by $(H_1, \dots\dots, H_l, X_r, r \text{ roots})$, $(H_1, \dots\dots, H_l)$ respectively. $\mathfrak{g}_\Omega$ is the Lie aegebra over Ω, and we denote also by H_i, X_r etc. for $1_\Omega \otimes H_i$, $1_\Omega \otimes X_r$ etc.; let $\mathfrak{H}_\Omega$ be the connected algebraic subgroup of G consisting of the automorphisms $h(\chi)$ of $\mathfrak{g}_\Omega$ such that $H_i \to H_i$, $(1 \leqq i \leqq l)$, $X_r \to \chi(r) X_r$, where χ are homomorphisms of the additive group generated by the roots into Ω^*; let $\mathfrak{H}_{r,\Omega}$ be the 1-dimensional connected algebraic subgroup of $\mathfrak{H}_\Omega$ consisting of the elements $h(\chi_{r,z})$, where $\chi_{r,z}(s) = z^{r(H_s)}$, $z \in \Omega^*$; let $\mathfrak{X}_{r,\Omega}$ be the 1-dimensional connected algebraic subgroup of G consisting of the automorphisms $x_r(t)$, $t \in \Omega$, of $\mathfrak{g}_\Omega$, which are obtained from the automorphisms of $\mathfrak{g}$ of the form $\exp t \operatorname{ad} X_r$, t being the complex numbers, by reducing them mod p. Since $\bar{\mathfrak{x}}_{r,\Omega} = \Omega\cdot\operatorname{ad} X_r$ and $\bar{\mathfrak{h}}_{r,\Omega} = \Omega\cdot\operatorname{ad} H_r$ are the Lie algebras of $\mathfrak{X}_{r,\Omega}$ and $\mathfrak{H}_{r,\Omega}$ respectively[8], we may identify them with the subalgebra of

6) Cf. the section 6 of [5].
7) Cf. § 1 of [8].
8) Cf. the proof of the propositions 2 and 3 of [1].

the Lie algebra $\bar{\mathfrak{g}}$ of G and that of the Lie algebra $\bar{\mathfrak{h}}$ of $\mathfrak{H}_\Omega$ respectively.

Let $a_1, \ldots\ldots, a_l$ be a fundamental system of roots of A_l; put $\alpha_{ij} = a_i(H_{a_j})$, then, ad $\mathfrak{h}_\Omega$ being generated by ad $H_{a_i}(1 \leqq i \leqq l)$, the dimension of ad $\mathfrak{h}_\Omega$ is equal to the rank of the matrix $(\bar{\alpha}_{ij})$, where $\bar{\alpha}_{ij}$ is the class of the integer α_{ij} modulo p. Since $\det(\alpha_{ij}) = l + 1 \equiv 0$ modulo p and any principal minor of degree $l - 1$ of (α_{ij}) is $l \not\equiv 0$ modulo p, we have dim ad $\mathfrak{h}_\Omega = l - 1$. Therefore the center of $\mathfrak{g}_\Omega$ being contained in $\mathfrak{h}_\Omega$, it is 1 dimensional and dim ad $\mathfrak{g} = \dim \mathfrak{g} - 1 = n - 1$. We have $[\bar{\mathfrak{x}}_{r,\Omega}, \bar{\mathfrak{x}}_{s,\Omega}] \subset \bar{\mathfrak{x}}_{r+s,\Omega}$ if $r + s$ is a root, $[\bar{\mathfrak{x}}_{r,\Omega}, \bar{\mathfrak{x}}_{s,\Omega}] \equiv \{0\}$ if $r + s$ is not a root and $[\bar{\mathfrak{x}}_{r,\Omega}, \bar{\mathfrak{x}}_{-r,\Omega}] \subseteq \bar{\mathfrak{h}}_{r,\Omega}$. Since ad $\mathfrak{g}_\Omega$ is generated by algebraic subgroups $\bar{\mathfrak{x}}_{r,\Omega}$, the Jacobson's formula shows that ad $\mathfrak{g}_\Omega$ is a p-subalgebra of $\mathfrak{g}$. By the proposition 2 of [1], ad $\mathfrak{g}_\Omega$ is simple. Thus ad $\mathfrak{g}_\Omega$ is simple p-subalgebra of dimension $n - 1$ of the Lie algebra of G.

Suppose that ad $\mathfrak{g}_\Omega$ were algebraic. Let G' be an algebraic subgroup of G, whose Lie algebra is ad $\mathfrak{g}_\Omega$. Then G' is semi-simple and has no non-trivial connected normal algebraic subgroup. Therefore the dimension of G' must be equal to the dimension of some simple Lie algebra over the complex number field.9) But it is impossible for some l (for example $l=4$), since dim $G' = n - 1$. Thus the p-subalgebra ad $\mathfrak{g}_\Omega$ is not algebraic.

Now we shall give an example showing that the global analogy of the characterization of algebraic subalgebras does not hold even if the characteristic of the universal domain is 0.10)

Let G be $GL(1, \Omega)$; let H be the subgroup of G consisting of all elements of finite order ; then, since for $x \in H$ there exists a positive integer r such that $x^r = 1$, the minimal algebraic subgroup of G containing x is contained in the algebraic subgroup of G consisting of all y such that $y^r = 1$; hence H is closed with respect to "s-s-replica operation"; but it is easily seen that H is not algebraic.

9) Cf. Exposé XVII Proposition 1, 2 and Exposé XIX of 4.
10) Cf. 2 §8, p. 48.

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