Some remarks on Banach spaces in which martingale difference sequences are unconditional

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Introduction

This note deals with Banach spaces X which have so-called UMD-property, which means that X-valued martingale difference sequences are unconditional in $L_X^p(1<p<\infty)$. These spaces were recently studied in [2], [3], [4] and we refer the reader to them for details not presented here. Let us recall following fact (see [2]).

Theorem. *For a Banach space X, following conditions are equivalent:*
(i) *X has* UMD,
(ii) *There exists a symmetric biconvex function ζ on $X\times X$ satisfying $\zeta(0,0)>0$ and $\zeta(x,y)\leqq\|x+y\|$ if $\|x\|\leqq 1\leqq\|y\|$.*

If X has UMD, then the same holds for subspaces and quotients of X, X^* and for the spaces L_X^p $(1<p<\infty)$. It is shown in [1] that if $1<p<\infty$ and $L_X^p(0,1)$ has an unconditional basis, then X is UMD. Conversely, it is not difficult to see that if X is a UMD-space possessing an unconditional basis, then the spaces $L_X^p(0,1)$ $(1<p<\infty)$ have unconditional basis.

In [3], it is proved that if X is UMD, then certain singular integrals such as the Hilbert transform are bounded operators on L_X^p $(1<p<\infty)$. Our first purpose will be to show the converse, i.e. Hilbert transfoım boundedness implies UMD.

From [1], we know that UMD implies super-reflexivity. Another, more direct argument will be given in the remarks below. In [7], an example is described of a superreflexive space failing UMD. We will show that superreflexivity does not imply UMD also for lattices, a question left open by [7].

At this point, the class UMD seems rather small, in the sense that the only basis examples we know about are spaces appearing in classical analysis.

1. Hilberttransform and martingale difference sequences

Denote D the Cantor group and Π the circle group (equipped with their respective Haar measure). Let $\mathcal{H}$ be the Hilbert transform acting on $L^p(\Pi)$. It will be convenient to introduce following definition:

For $1<p<\infty$, say that the Banach space X has property (h_p) provided $\mathcal{H}$ acts boundedly on $L^p_X(\Pi)$, i.e.

$$\|\mathcal{H}(f)\|_p \leqq C\|f\|_p \quad \text{for} \quad f\in L^p_X(\Pi).$$

In [3], a classical approach is used to derive (h_p) from the p-boundedness of the martingale transforms acting on $L^p_X(D)$. We will explain here a reverse procedure.

As a consequence, each of the properties (h_p) is equivalent to UMD. The main point is following fact

Lemma 1. *Denote for* $k=1,2,\ldots$ $\Pi^k=\underbrace{\Pi\times\ldots\times\Pi}_{k}$. *Assume given for each* $k=1,2,\ldots$ *a function* $\Phi_k\in L^p_X(\Pi^k)$ *and a scalar function* $\varphi_k\in L^\infty(\Pi)$, $\int\varphi_k=0$. *If* X *satisfies* (h_p), *one has the inequality*

$$\|\Sigma'\Phi_k(\theta_1,\ldots,\theta_k)\mathcal{H}(\varphi_k)(\theta_{k+1})\|_p \leqq C\|\Sigma'\Phi_k(\theta_1,\ldots,\theta_k)\varphi_k(\theta_{k+1})\|_p,$$

$(\Sigma'=\sum_{k=1}^n$ *for some integer* n).

Proof. By an approximation argument, we can assume the Φ_k-functions to be X-valued polynomials, say

$$|\gamma| = |\gamma_1|+\ldots+|\gamma_k| \leqq N_k \quad \text{if} \quad \gamma\in\operatorname{Spec}\Phi_k\subset \mathbf{Z}^k$$

where N_k is some positive integer.

Define inductively an increasing sequence (n_k) of integers, taking

$$n_1 = 0,$$
$$n_{k+1} = n_kN_k+1.$$

For fixed $(\theta_1,\theta_2,\ldots)$, notice that

$$\mathcal{H}_\psi(\Phi_k(\theta_1+n_1\psi,\ldots,\theta_k+n_k\psi)\varphi_k(\theta_{k+1}+n_{k+1}\psi))$$
$$=\Phi_k(\theta_1+n_1\psi,\ldots,\theta_k+n_k\psi)\mathcal{H}(\varphi_k)(\theta_{k+1}+n_{k+1}\psi)$$

since it concerns the product of a function with spectrum contained in $]-n_{k+1}, n_{k+1}[$ and a function with spectrum contained in $n_{k+1}(\mathbf{Z}\setminus\{0\})$. So, if X has (h_p), we get

$$\int\|\Sigma'\Phi_k(\theta_1+n_1\psi,\ldots,\theta_k+n_k\psi)\mathcal{H}(\varphi_k)(\theta_{k+1}+n_{k+1}\psi)\|^p m(d\psi)$$
$$\leqq c^p\int\|\Sigma'\Phi_k(\theta_1+n_1\psi,\ldots,\theta_k+n_k\psi)\varphi_k(\theta_{k+1}+n_{k+1}\psi)\|^p m(d\psi)$$

and integration on ψ clearly leads to the required conclusion.

Lemma 2. *Let* X *be* (h_p) *and consider for each* $k=1, 2, \ldots$ *a function* $\Delta_k \in L_X^p(D)$ *depending on the first* k *Rademachers* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$. *Then*

$$\|\Sigma' \alpha_{k+1} \Delta_k(\varepsilon_1, \ldots, \varepsilon_k)\varepsilon_{k+1}\|_p \leqq C^2 \|\Sigma' \Delta_k(\varepsilon_1, \ldots, \varepsilon_k)\varepsilon_{k+1}\|_p,$$

for all sequences $\alpha_k = \pm 1$. *Consequently,* X *possesses* UMD.

Proof. Considering again Π^{N}, one can replace D by Π^N, writing

$$\varepsilon_k = \text{sign} \cos \theta_k \quad (\text{sign} = \text{sign function}).$$

So, define

$$\Phi_k(\theta_1, \ldots, \theta_k) = \Delta_k(\text{sign} \cos \theta_1, \ldots, \text{sign} \cos \theta_k)$$

and let

$$\varphi_k(\theta) = \text{sign} \cos \theta$$

for each k.

Thus Φ_k is even in $\theta_1, \ldots, \theta_k$ and $\mathscr{H}(\varphi_k)$ is an odd function. Thus, applying Lemma 1 and replacing θ_k by $\alpha_k \theta_k$, it follows

$$\|\Sigma' \alpha_{k+1} \Phi_k(\theta_1, \ldots, \theta_k) \mathscr{H}(\varphi_k)(\theta_{k+1})\|_p \leqq C \|\Sigma' \Phi_k(\theta_1, \ldots, \theta_k) \varphi_k(\theta_{k+1})\|_p.$$

But, again by Lemma 1

$$\|\Sigma' \alpha_{k+1} \Phi_k(\theta_1, \ldots, \theta_k) \varphi_k(\theta_{k+1})\|_p \leqq C \|\Sigma' \alpha_{k+1} \Phi_k(\theta_1, \ldots, \theta_k) \mathscr{H}(\varphi_k)(\theta_{k+1})\|_p.$$

Thus, the desired inequality is obtained.

Remark that the method extends to more variables and allows to translate inequalities for polydisc in inequalities for multiindexed martingales.

2. An example

From [9] we know that each superreflexive lattice can be obtained as complex interpolation space between a Hilbert space and some lattice. Therefore, one could hope to prove UMD for this class of spaces. The next example shows however that this is not possible.

Proposition. *For* $1 < p < q < \infty$, *there is a lattice* X *satistying an upper-p and lower-q estimate and failing* UMD.

The reader is referred to [6] for definitions and basic facts. We will construct finite dimensional lattices X with upper-p and lower-q constant 1 and for which the bound for martingale transforms acting on $L_x^p(D)$ goes to infinity. The final lattice is then obtained as l^p-direct sum (again D stands for the Cantor group or a finite Cantor group). The following definition will be useful.

Say that a collection $\mathfrak{A}$ of subsets of D is a translation invariant paving iff
(i) $A\in\mathfrak{A},\ B\subset A\Rightarrow B\in\mathfrak{A}$,
(ii) $A\in\mathfrak{A},\ g\in D\Rightarrow A_g\in\mathfrak{A}$ (A_g=g-translate of A).
Let $1<p<q<\infty$ and define following function lattice $X=X_{p,q}(\mathfrak{A})$ on D

$$\|f\|_X = \sup\,(\Sigma\|f\chi_{A_i}\|_p^q)^{1/q}.$$

Here the supremum is taken over all disjoint collection $\{A_i\}$ of $\mathfrak{A}$-members. (χ_A denotes the indicator function of the set A.) The proof of following facts is standard and left as exercice to the reader.

Lemma 3.
(i) *X has upper-p and lower-q estimates with constant* 1,
(ii) $\|f\|_X=\|f_g\|_X$ *for all* $g\in D$,
(iii) $\|f\|_X\leqq\|f\|_p^{p/q}\sup_{\mathfrak{A}}\|f\chi_A\|_p^{1-p/q}$.

Denote $\tilde{}$ some transform. For a fixed $\varphi\in X$, define $\Phi\in L_X^p(D)$ by $\Phi(g)=\varphi_g$. Then $\tilde{\Phi}(g)=(\tilde{\varphi})_g$ and the norm of $\tilde{}$ acting on $L_X^p(D)$ is thus minorated by the ratio $\|\tilde{\varphi}\|_X\,\|\varphi\|_X^{-1}$. In order to introduce $\mathfrak{A}$ and φ, we need following additional lemma

Lemma 4. *For each* $\varepsilon>0$, *there exist* $\varphi\in L^p(D)$ *and a measurable subset* $M\subset D$ *satisfying*
(i) $\|\varphi\|_p=1$,
(ii) $\|\varphi_g\chi_M\|_p<\varepsilon$ *for each* $g\in D$,
(iii) $\|S(\varphi)\chi_M\|_p\geqq 1/2$
(denoting S the Walsh—Paley square function).

Let us first show how to conclude.

Define $\mathfrak{A}$ as the class of measurable subsets A of D contained in some translate M_g of M. By Lemma 3 (iii) and Lemma 4 (ii)

$$\|\varphi\|_X\leqq\varepsilon^{1-p/q}$$

while from Lemma 4 (iii), for some transform $\tilde{}$, one has

$$\|\tilde{\varphi}\|_X\geqq\|\tilde{\varphi}\chi_M\|_p\geqq\tfrac{1}{2}.$$

So $\|\tilde{}\|_p\gtrsim\varepsilon^{p/q-1}\to\infty$ for $\varepsilon\to 0$.

Proof of Lemma 4. Fix a positive integer n and consider $D=\{1,-1\}^{2n}$. Define for $k=1,2,\dots,n$

$$R_k^+=(1+\varepsilon_1)\dots(1+\varepsilon_k)(1-\varepsilon_{k+1})\dots(1-\varepsilon_n)(1+\varepsilon_{n+1})\dots(1+\varepsilon_{n+k-1})(1+\varepsilon_{n+k}),$$

$$R_k^-=(1+\varepsilon_1)\dots(1+\varepsilon_k)(1-\varepsilon_{k+1})\dots(1-\varepsilon_n)(1+\varepsilon_{n+1})\dots(1+\varepsilon_{n+k-1})(1-\varepsilon_{n+k}).$$

Take

$$\varphi = n^{-1/p} \sum_{k=1}^{n} 2^{-\frac{n+k}{p'}} R_k^+ ,$$

$$\chi_M = \sum_{k=1}^{n} 2^{-(n+k)} R_k^- .$$

Thus $\|\varphi\|_p = 1$. One also checks easily that

$$\|S(\varphi)\chi_M\|_p^p = \Sigma \|S(\varphi) 2^{-(n+k)} R_k^-\|_p^p \geqq \Sigma \frac{1}{n} 2^{-\frac{p}{p'}(n+k)} 2^{-p} 2^{(p-1)(n+k)}$$

and thus

$$\|S(\varphi)\chi_M\|_p \geqq \tfrac{1}{2}$$

To verify (ii) of Lemma 4, fix $g \in D$ and distinguish following cases

(a) $g_k \neq 1$ for some coordinate $k = 1, 2, \ldots, n$.

Then it is easy to see that $(R_k^+)_g R_l^- \neq 0$ for at most 2 pairs (k, l).

(b) $g_k = 1$ for all $k = 1, 2, \ldots, n$.

Then $(R_k^+)_g R_l^- = 0$ for $k \neq l$ and $(R_k^+)_g R_l^- \neq 0$ for at most 1 value of k.

Therefore $\|\varphi_g \chi_M\|_p \leqq 2n^{-1/p} \to 0$ for $n \to \infty$.

3. Some further remarks

Assuming X a UMD-space and denoting $\|\mathscr{H}\|_{\infty,1}$ the $L_X^\infty \to L_X^1$ norm of the Hilbert-transform, one obtains in terms of the Hilbert-matrix

$$\left| \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\langle x_i, x_j^* \rangle}{i-j} \right| \leqq n \|\mathscr{H}\|_{\infty,1} \max \|x_i\| \max \|x_j^*\|$$

for each n and all sequences $(x_i)_{1 \leqq i \leqq n}$, $(x_j^*)_{1 \leqq j \leqq n}$ in X and X^* (resp.).

Fixing $\delta > 0$, define N_δ as the largest positive integer for which there exists a sequence $(x_i)_{1 \leqq i \leqq n = N_\delta}$ in the unit ball of X such that

$$\operatorname{dist}(\operatorname{conv}(x_1, \ldots, x_j), \operatorname{conv}(x_{j+1}, \ldots, x_n)) \geqq \delta$$

for each $j = 1, \ldots, n$.

From the preceding, we get

$$\delta \log N_\delta \leqq \|\mathscr{H}\|_{\infty,1}. \quad (*)$$

Since in particular $N_\delta < \infty$ for each $\delta > 0$, X must be superreflexive (cfr. [5]).

In [7], interpolation is used to construct a superreflexive space for which left hand side of $(*)$ is unbounded for $\delta \to 0$. It might be interesting to determine the worse bound on the Hilbert transform for $\dim X = d < \infty$. In particular, what is

$$\sup_{\dim X = d} \sup_{\delta > 0} (\delta \log N_\delta)?$$

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