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ORIGINAL ARTICLE

Some remarks on blueprints and $\mathbb{F}_1$-schemes

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Abstract

Over the past two decades several different approaches to defining a geometry over $\mathbb{F}_1$ have been proposed. In this paper, relying on Toën and Vaquié's formalism (J.K-Theory 3: 437–500, 2009), we investigate a new category $\mathsf{Sch}_{\widetilde{\mathsf{B}}}$ of schemes admitting a Zariski cover by affine schemes relative to the category of blueprints introduced by Lorscheid (Adv. Math. 229: 1804–1846, 2012). A blueprint, which may be thought of as a pair consisting of a monoid M and a relation on the semiring $M \otimes_{\mathbb{F}_1} \mathbb{N}$, is a monoid object in a certain symmetric monoidal category B, which is shown to be complete, cocomplete, and closed. We prove that every $\widetilde{\mathsf{B}}$-scheme Σ can be associated, through adjunctions, with both a classical scheme $\Sigma_{\mathbb{Z}}$ and a scheme $\underline{\Sigma}$ over $\mathbb{F}_1$ in the sense of Deitmar (in van der Geer G., Moonen B., Schoof R. (eds.) Progress in mathematics 239, Birkhäuser, Boston, 87–100, 2005), together with a natural transformation $\Lambda \colon \Sigma_{\mathbb{Z}} \to \underline{\Sigma} \otimes_{\mathbb{F}_1} \mathbb{Z}$. Furthermore, as an application, we show that the category of "$\mathbb{F}_1$-schemes" defined by Connes and Consani in (Compos. Math. 146: 1383–1415, 2010) can be naturally merged with that of $\widetilde{\mathsf{B}}$-schemes to obtain a larger category, whose objects we call "$\mathbb{F}_1$-schemes with relations".

Keywords $\mathbb{F}_1$-schemes · Blueprints · Relative algebraic geometry

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1 Introduction

1.1 A quick overview of $\mathbb{F}_1$-geometry

The nonexistent field $\mathbb{F}_1$ made its first appearance in Jacques Tits's 1956 paper *Sur les analogues algébriques des groupes semi-simples complexes* [26].[1] According to Tits, it was natural to call "n-dimensional projective space over $\mathbb{F}_1$" a set of $n+1$ points, on which the symmetric group Σ_{n+1} acts as the group of projective transformations. So, Σ_{n+1} was thought of as the group of $\mathbb{F}_1$-points of SL_{n+1}, and more generally it was conjectured that, for each algebraic group G, one ought to have $W(G) = G(\mathbb{F}_1)$, where $W(G)$ is the Weyl group of G.

A further strong motivation to seek for a geometry over $\mathbb{F}_1$ was the hope, based on the multifarious analogies between number fields and function fields, to find some pathway to attack Riemann's hypothesis by mimicking André Weil's celebrated proof. The idea behind that, as explicitly stated in Yuri Manin's influential 1991–92 lectures [21] and in Kapranov and Smirnov's unpublished paper [13], was to regard Spec $\mathbb{Z}$, the final object of the category of schemes, as an arithmetic curve over the "absolute point" Spec $\mathbb{F}_1$. Manin's work drew inspiration from Kurokawa's paper [14] together with Deninger's results about "representations of zeta functions as regularized infinite determinants [7–9] of certain 'absolute Frobenius operators' acting upon a new cohomology theory". Developing these insights, Manin suggested a conjectural decomposition of the classical complete Riemann zeta function of the form [21, eq. (1.5)]

$$
\begin{aligned}
Z\left(\overline{\operatorname{Spec}\mathbb{Z}}, s\right) &:= 2^{-1/2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{\prod_\rho^{\text{reg}} \frac{s-\rho}{2\pi}}{\frac{s}{2\pi}\frac{s-1}{2\pi}} \\
&\stackrel{?}{=} \frac{\det^{\text{reg}}\left(\frac{1}{2\pi}(s\cdot\mathrm{Id}-\Phi)\,\middle|\,H^1_?\left(\overline{\operatorname{Spec}\mathbb{Z}}\right)\right)}{\det^{\text{reg}}\left(\frac{1}{2\pi}(s\cdot\mathrm{Id}-\Phi)\middle|H^0_?(\overline{\operatorname{Spec}\mathbb{Z}})\right)\det^{\text{reg}}\left(\frac{1}{2\pi}(s\cdot\mathrm{Id}-\Phi)\middle|H^2_?(\overline{\operatorname{Spec}\mathbb{Z}})\right)},
\end{aligned}
\tag{1.1}
$$

where the notation $\prod_\rho^{\text{reg}}$ and $\det^{\text{reg}}$ refers to "zeta regularization" of infinite products and the last hypothetical equality "postulates the existence of a new cohomology theory $H^\bullet_?$, endowed with a canonical 'absolute Frobenius' endomorphism Φ". He conjectured, moreover, that the functions of the form $\frac{s-\rho}{2\pi}$ in Eq. 1.1 could be interpreted as zeta functions according to the definition

$$
Z\left(\mathbb{T}^\rho, s\right) = \frac{s-\rho}{2\pi}, \quad \rho \geq 0,
$$

where "Tate's absolute motive" $\mathbb{T}$ was to be "imagined as a motive of a one-dimensional affine line over the absolute point, $\mathbb{T}^0 = \bullet = \operatorname{Spec}\mathbb{F}_1$".

[1] For a more detailed and exhaustive account of the development of $\mathbb{F}_1$-geometry we refer to [15] and [17].

The first full-fledged definition of variety over "the field with one element" was proposed by Christophe Soulé in the 1999 preprint [24]; five years later such definition was slightly modified by the same author in the paper [25]). Taking as a starting point Kapranov and Smirnov's suggestion that $\mathbb{F}_1$ should have an extension $\mathbb{F}_{1^n}$ of degree n, Soulé insightfully posited that

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[T]/(T^n - 1) =: R_n \, .$$

Let R be the full subcategory of the category Ring of commutative rings generated by the rings $R_n, n \geq 1$ and their finite tensor products. An affine variety X over $\mathbb{F}_1$ is then defined as a covariant functor $\mathsf{R} \to \mathsf{Set}$ plus some extra data such that there exists a unique (up to isomorphism) affine variety $X_{\mathbb{Z}} = X \otimes_{\mathbb{F}_1} \mathbb{Z}$ over Z along with an immersion $X \hookrightarrow X_{\mathbb{Z}}$ satisfying a suitable universal property [25, Définition 3]. In particular, one has a natural inclusion $X(\mathbb{F}_{1^n}) \subset (X \otimes_{\mathbb{F}_1} \mathbb{Z})(R_n)$ for each $n \geq 1$. A notable result proven by Soulé was that smooth toric varieties can always be defined over $\mathbb{F}_1$.

To formalize $\mathbb{F}_1$-geometry Anton Deitmar adopted, in 2005, a different approach, which can be dubbed as "minimalistic" (using the evocative terminology introduced by Manin in [22]). In his terse paper [4], Deitmar associates to each commutative monoid M its "spectrum over $\mathbb{F}_1$" Spec M consisting of all prime ideals of M, i.e. of all submonoids $P \subset M$ such that $xy \in P$ implies $x \in P$ or $y \in P$. The set Spec M can be endowed with a topology and with a structure (pre)sheaf $\mathcal{O}_M$ via localization, just as in the usual case of commutative rings. A topological space X with a sheaf $\mathcal{O}_X$ of monoids is then called a "scheme over $\mathbb{F}_1$", if for every point $x \in X$ there is an open neighborhood $U \subset X$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to (Spec $M, \mathcal{O}_M$) for some monoid M. The forgetuful functor $\mathsf{Ring} \to \mathsf{Mon}$ has a left adjoint given by $M \mapsto M \otimes_{\mathbb{F}_1} \mathbb{Z}$ (in Deitmar's notation), and this functor extend to a functor - $\otimes_{\mathbb{F}_1} \mathbb{Z}$ from the category of schemes over $\mathbb{F}_1$ to the category of classical schemes over $\mathbb{Z}$. Tit's 1957 conjecture stating that $GL_n(\mathbb{F}_1) = \Sigma_n$ can be easily proven in Deitmar's theory. Indeed, since $\mathbb{F}_1$-modules are just sets and $\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z}$ has to be isomorphic $\mathbb{Z}^n$, it turns out that $\mathbb{F}_{1^n}$ can be identified with the set $\{1, \ldots, n\}$ of n elements. Hence

$$GL_n(\mathbb{F}_1) = \mathrm{Aut}_{\mathbb{F}_1}(\mathbb{F}_{1^n}) = \mathrm{Aut}(1, \ldots, n) = \Sigma_n \, .$$

It is not hard to show, moreover, that the functor GL_n on rings over $\mathbb{F}_1$ is represented by a scheme over $\mathbb{F}_1$ [4, Prop. 5.2]. As for zeta functions, Deitmar defines, for a scheme X over $\mathbb{F}_1$ and for a prime p, the formal power series

$$Z_X(p, T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \# X\left(\mathbb{F}_{p^n}\right)\right) ,$$

where $\mathbb{F}_{p^n}$ stands for the field of p^n elements with only its monoidal multiplicative structure and $X(\mathbb{F}_{p^n})$ denotes the set of $\mathbb{F}_{p^n}$-valued points of X, and proves that $Z_X(p, T)$ coincides with the Hasse–Weil zeta function of $X \otimes_{\mathbb{F}_1} \mathbb{F}_{p^n}$ [4, Prop. 6.3]. Albeit elegant, this result is a bit of a letdown, for—as the author himself is ready to admit—it is clear that "this type of zeta function [...] does not give new insights".

A natural and extremely general formalism for $\mathbb{F}_1$-geometry was elaborated by Bertrand Toën and Michel Vaquié in their 2009 paper [27], tellingly entitled *Au dessous de* Spec $\mathbb{Z}$, whose approach appears to be largely inspired by Monique Hakim's work [11]. The authors there showed how to construct an "algebraic geometry" relative to any symmetric monoidal category $\mathsf{C} = (\mathsf{C}, \otimes, \mathbf{1})$, which is supposed to be complete, cocomplete and to admit internal homs. The basic idea is that the category CMon_C of commutative (associative and unitary) monoid objects in C can be taken as a substitute for the category of commutative rings (the monoid objects in the category $\mathsf{Ab} = \mathbb{Z}\text{-}\mathsf{Mod}$ of Abelian groups) to the end of defining a suitable notion of "scheme over C". Each object V of CMon_C gives rise to the category $V\text{-}\mathsf{Mod}$ of V-modules and each morphism $V \to W$ in CMon_C determines a change of basis functor $\text{-} \otimes_V W : V\text{-}\mathsf{Mod} \to W\text{-}\mathsf{Mod}$; the category of commutative V-algebras can be realized as the category of commutative monoids in $V\text{-}\mathsf{Mod}$ and is naturally equivalent to the category $V/\mathsf{CMon}_\mathsf{C}$. An affine scheme over C is, by definition, an object of the opposite category $\mathsf{Aff}_\mathsf{C} = \mathsf{CMon}_\mathsf{C}^{\mathrm{op}}$ and the tautological contravariant functor $\mathsf{CMon}_\mathsf{C} \to \mathsf{Aff}_\mathsf{C}$ is called Spec(-). By means of the pseudo-functor M that maps an object V in CMon_C to the category of V-modules and a morphism Spec $V \to$ Spec W to the functor $\text{-} \otimes_V W : V\text{-}\mathsf{Mod} \to W\text{-}\mathsf{Mod}$, one may introduce the notions of "Zariski cover" and "flat cover" ("M-faithfully flat in Toën and Vaquié's terminology; see Definition 2.4 and Remark 2.5 below) and use such notions to equip Aff_C with two distinct Grothendieck topologies, called, respectively, the flat and the Zariski topology. These topologies determine two categories of sheave on Aff_C, namely $\mathsf{Sh}^{\mathrm{flat}}(\mathsf{Aff}_\mathsf{C}) \subset \mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_\mathsf{C}) \subset \mathsf{Presh}(\mathsf{Aff}_\mathsf{C})$. At this point, mimicking what is done in classical algebraic geometry, a "scheme over C" is defined as a sheaf in $\mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_\mathsf{C})$ that admits an affine Zariski cover (see Definition 2.6 and Definition 2.7 below). If we take as C the category Set of sets endowed with the monoidal structure induced by the Cartesian product, then the category $\mathsf{Aff}_\mathsf{Set}$ is nothing but the category $\mathsf{Mon}^{\mathrm{op}}$ and the objets of the category $\mathsf{Sch}_\mathsf{Set}$ can be thought of — as remarked by Toën and Vaquié — as "schemes over $\mathbb{F}_1$". Actually, as proven by Alberto Vezzani in [28], such schemes, that we shall call *monoidal schemes*, turn out to be equivalent to Deitmar's schemes.

Deitmar's schemes appear therefore to constitute the very core of $\mathbb{F}_1$-geometry, not just because their definition is rooted in the basic notion of prime spectrum of a monoid, but especially because they naturally fit into the categorical framework established by Toën and Vaquié in [27], which admits generalizations in many directions (e.g. towards a derived algebraic geometry over $\mathbb{F}_1$). Nonetheless, they are affected by some intrinsic limitations, which are clearly revealed by a result proven by Deitmar himself in 2008 [6, Thm. 4.1]:

Theorem *Let X be a connected integral $\mathbb{F}_1$-scheme of finite type.*[2] *Then every irreducible component of $X_\mathbb{C} = X_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ is a toric variety. The components of $X_\mathbb{C}$ are mutually isomorphic as toric varieties.*

Since every toric variety is the lift $X_\mathbb{C}$ of an $\mathbb{F}_1$-scheme X, the previous theorem entails that integral $\mathbb{F}_1$-schemes of finite type are essentially the same as toric varieties. Now,

[2] A Deitmar's $\mathbb{F}_1$-scheme X is said to be of finite type, if it has a finite covering by affine schemes U_i = Spec M_i such that each M_i is a finitely generated monoid. Deitmar proved in [5] that an $\mathbb{F}_1$-scheme X is of finite type if and only if $X_\mathbb{Z}$ is a $\mathbb{Z}$-scheme of finite type.

semisimple algebraic groups are not toric varieties, so it is apparent that Deitmar's $\mathbb{F}_1$-schemes are too little flexible to implement Tits's conjectural program.

A possible generalization of Deitmar's geometry over $\mathbb{F}_1$ was proposed by Olivier Lorscheid, who introduced the notions of "blueprint" and "blue scheme" [16]. The basic idea can be illustrated through the following example. The affine group scheme $(SL_2)_{\mathbb{Z}}$ over the integers is defined as

$$(SL_2)_{\mathbb{Z}} = \mathrm{Spec}\,(\mathbb{Z}\,[T_1, T_2, T_3, T_4]\,/\,(T_1T_4 - T_2T_3 - 1))\ .$$

As the relation $T_1T_4 - T_2T_3 = 1$ does not make sense in the monoid $\mathbb{F}_1[T_1, T_2, T_3, T_4]$, any naive attempt to adapt the previous definition to get a scheme over $\mathbb{F}_1$ will necessarily be unsuccessful. The notion of "blueprint" just serves serves the purpose of getting rid of this difficulty:

Definition *A* blueprint *is a pair $B = (R, A)$, where R is a semiring and A is a multiplicative subset of R containing* 0 *and* 1 *and generating R as a semiring. A blueprint morphism $f : B_1 = (R_1, A_1) \to B_2 = (R_2, A_2)$ is a semiring morphism $f : R_1 \to R_2$ such that $f(A_1) \subset A_2$.*

The rationale behind this definition can be explained by considering the following situation: if one is given a monoid A and some relation which does not makes sense in A but becomes meaningful in the semiring $A \otimes_{\mathbb{F}_1} \mathbb{N}$, then one can look at the blueprint $(A \otimes_{\mathbb{F}_1} \mathbb{N}, A)$.

In the same vein as Deitmar's approach, Lorscheid [16] associates to each blueprint B its spectrum Spec B, which turns out to be a locally blueprinted space (i.e. a topological space endowed with a sheaf of blueprints, such that all stalks have a unique maximal ideal). An affine blue scheme is then defined as a locally blueprinted space that is isomorphic to the spectrum of a blueprint, and a blue scheme as a locally blueprinted space that has a covering by affine blue schemes. Deitmar's schemes over $\mathbb{F}_1$ and classical schemes over $\mathbb{Z}$ are recovered as special cases of this definition.

1.2 About the present paper

A natural question arises: do blue schemes fit into Toën and Vaquié's framework? This problem was addressed by Lorscheid himself in his 2017 paper [18] and answered in the negative. Nonetheless, it is possible—as already pointed out in [18]—to define a category of schemes (here called B-*schemes*) relative (in Toën and Vaquié's sense) to the category of blueprints. Our first aim is to study these schemes by introducing the category of blueprint *in a purely functorial way*, as the category of monoid objects in a closed, complete and cocomplete symmetric monoidal category B.

There is a natural adjunction $\rho \dashv \sigma : \mathsf{Aff}_{\mathsf{B}} \to \mathsf{Aff}_{\mathsf{Set}_*}$ between the category of affine B-schemes and that of affine monoidal schemes. However, since the functor ρ is not continuous w.r.t. the Zariski topology, this adjunction does not give rise to a geometric morphism between the corresponding category of schemes. This hurdle may be sidestepped by introducing a larger category $\widetilde{\mathsf{B}}$ containing B and by considering the category of those schemes in $\mathsf{Sch}_{\widetilde{\mathsf{B}}}$ that admit a Zariski cover by affine B-schemes.

Such schemes, by a slight abuse of language, will be called $\widetilde{\mathsf{B}}$-*schemes*. It will be proved that the adjunction $\rho \dashv \sigma$ above induce an adjunction $\widehat{\rho} \dashv \widehat{\sigma}$ between the category of $\widetilde{\mathsf{B}}$-schemes and that of affine monoidal schemes. Moreover, it will be shown that every $\widetilde{\mathsf{B}}$-scheme Σ generates a pair $(\underline{\Sigma}, \Sigma_{\mathbb{Z}})$, where $\underline{\Sigma}$ is a monoidal scheme and $\Sigma_{\mathbb{Z}}$ a classical scheme, together with a natural transformation $\Lambda\colon \Sigma_{\mathbb{Z}} \to \underline{\Sigma} \otimes_{\mathbb{F}_1} \mathbb{Z}$.

More in detail the present paper is organized as follows.

After briefly recalling in Sect. 2 the fundamental notions of "relative algebraic geometry" and fixing our notation, in Sect. 3 we define the full subcategory B of the category $\mathbb{N}[\text{-}]/\mathsf{Mon}_0$ (where the functor $\mathbb{N}[\text{-}]\colon \mathsf{Set}_* \to \mathsf{Mon}_0$ is left adjoint to the forgetful functor $|\text{-}|$ from the category Set_* of pointed sets to the category of monoids with "absorbent object"; see Sect. 2.2), whose objects $(X, \mathbb{N}[X] \to M)$ satisfy the conditions:

a) the morphism $\mathbb{N}[X] \to M$ is an epimorphism;
b) the composition $X \to |\mathbb{N}[X]| \to |M|$ is a monomorphism.

As proven in Theorem 3.5, the category B—which corresponds to the category of pointed set endowed with a pre-addition structure introduced in [18, §4]—carries a natural structure of symmetric monoidal category. Moreover, this structure is closed, complete, and cocomplete. So, the category B possesses all the properties necessary to carry out Toën and Vaquié's program.

It is quite straightforward to show (Proposition 3.6) that the category Blp of monoid objects in B coincides with the category of blueprints (this result was already stated, in equivalent terms, in [18, Lemma 4.1], but we provide a detailed and completely functorial proof). Thus, by applying Toën and Vaquié's formalism to the category B, we define *the category* $\mathsf{Aff}_{\mathsf{B}} = \mathsf{Blp}^{\mathrm{op}}$ *of affine* B-*schemes* and then *the category* $\mathsf{Sch}_{\mathsf{B}}$ *of* B-*schemes*.

The core of our paper is Sect. 4. The natural adjunction between the category Mon_0 and the category Set_* gives rise to an adjunction $\mathsf{Aff}_{\mathsf{Mon}_0} \underset{|\text{-}|}{\overset{\text{-}\otimes_{\mathbb{F}_1}\mathbb{N}}{\rightleftarrows}} \mathsf{Aff}_{\mathsf{Set}_*}$ that factorizes as shown in the following diagram

$$
\begin{tikzcd}
\mathsf{Aff}_{\mathsf{Mon}_0} \arrow[r, "|\text{-}|"] \arrow[d, shift left, "G"] & \mathsf{Aff}_{\mathsf{Set}_*} \arrow[l, bend right, "\text{-}\otimes_{\mathbb{F}_1}\mathbb{N}"'] \arrow[dl, shift left, "\sigma"] \\
\mathsf{Aff}_{\mathsf{B}} \arrow[u, bend left, "F"] \arrow[ur, shift left, "\rho"] &
\end{tikzcd}
\tag{1.2}
$$

In Proposition 4.4 it is proven that the functor F in the diagram 1.2 is continuous w.r.t. the Zariski topology and that the induced functor $\widehat{F}\colon \mathsf{Sh}(\mathsf{Aff}_{\mathsf{B}}) \to \mathsf{Sh}(\mathsf{Aff}_{\mathsf{Mon}_0})$ determines a functor $\widehat{F}\colon \mathsf{Sch}_{\mathsf{B}} \to \mathsf{Sch}_{\mathsf{Mon}_0}$ between the category of B-schemes and that of semiring schemes. Similarly, in Proposition 4.5 it is shown that the functor $\sigma\colon \mathsf{Aff}_{\mathsf{Set}_*} \to \mathsf{Aff}_{\mathsf{B}}$ in the diagram 1.2 is continuous w.r.t. the Zariski topology and that the induced functor $\widehat{\sigma}\colon \mathsf{Sh}(\mathsf{Aff}_{\mathsf{Set}_*}) \to \mathsf{Sh}(\mathsf{Aff}_{\mathsf{B}})$ determines a functor $\widehat{\sigma}\colon \mathsf{Sch}_{\mathsf{Set}_*} \to \mathsf{Sch}_{\mathsf{B}}$ between the category of monoidal schemes and that of B-schemes. One would like the functor $\widehat{\sigma}$ to have a left adjoint determined by the functor $\rho\colon \mathsf{Aff}_{\mathsf{B}} \to \mathsf{Aff}_{\mathsf{Set}_*}$ (see diagram 1.2). However, the functor ρ, although it preserves Zariski covers, does not commute with finite limits. This difficulty may be overcome by introducing the categories $\widetilde{\mathsf{B}}$ and $\widetilde{\mathsf{Blp}}$ containing, respectively, B and Blp (Definition 4.7), and by defining the category $\widetilde{\mathsf{Sch}_{\widetilde{\mathsf{B}}}}$ of $\widetilde{\mathsf{B}}$-*schemes* as the subcategory of $\mathsf{Sch}_{\widetilde{\mathsf{B}}}$ whose objects admit a Zariski cover by affine schemes in $\mathsf{Aff}_{\mathsf{B}}$ (Definition 4.15). So, a $\widetilde{\mathsf{B}}$-scheme is locally described by blueprints. In this way, one shows (Theorem 4.14) that there is a geometric morphism

$$\widehat{\rho} \dashv \widehat{\sigma}\colon \widetilde{\mathsf{Sch}_{\widetilde{\mathsf{B}}}} \to \mathsf{Sch}_{\mathsf{Set}_*}\,.$$

It follows (see Definition 4.16 and the ensuing remarks) that each $\widetilde{\mathsf{B}}$-scheme Σ determines the following geometric data:

- a monoidal scheme $\underline{\Sigma} = \widehat{\rho}(\Sigma)$;
- a scheme $\Sigma_{\mathbb{Z}} = \widehat{F}_{\mathbb{Z}}(\Sigma)$ over $\mathbb{Z}$;
- a natural transformation $\Lambda\colon \Sigma_{\mathbb{Z}} \to \underline{\Sigma} \circ |\text{-}| \cong \underline{\Sigma} \otimes_{\mathbb{F}_1} \mathbb{Z}$.

In Sect. 5, as an application of our approach, we investigate the relationship of $\widetilde{\mathsf{B}}$-schemes and $\mathbb{F}_1$-schemes in the sense of Alain Connes and Caterina Consani [1]. According to their definition [1, Def. 4.7], an $\mathbb{F}_1$-scheme is a triple $(\underline{\Xi}, \Xi_{\mathbb{Z}}, \Phi)$, where $\underline{\Xi}$ is a monoidal scheme, $\Xi_{\mathbb{Z}}$ is a scheme over $\mathbb{Z}$, and Φ is natural transformation $\underline{\Xi} \to \Xi_{\mathbb{Z}} \circ (\text{-} \otimes_{\mathbb{F}_1} \mathbb{Z})$, such that the induced natural transformation $\underline{\Xi} \circ |\text{-}| \to \Xi_{\mathbb{Z}}$, when evaluated on fields, gives isomorphisms (of sets). Thus, the category of $\widetilde{\mathsf{B}}$-schemes and that of $\mathbb{F}_1$-schemes can be combined into a larger category, namely their fibered product over the category of monoidal schemes, whose objects will be called $\mathbb{F}_1$-*schemes with relations* (Definition 5.3). In more explicit terms, a $\widetilde{\mathsf{B}}$-scheme Σ determining the pair $(\underline{\Sigma}, \Sigma_{\mathbb{Z}})$ and an $\mathbb{F}_1$-scheme $(\underline{\Sigma}, \Sigma'_{\mathbb{Z}}, \Phi)$ will give rise to a $\mathbb{F}_1$-scheme with relations denoted by the quadruple $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$. The main motivation behind this notion is to combine in a single geometric object both the advantages of blueprint approach and the benefits of Connnes and Consani's definition (cf. Remark 5.4 for a better explanation). Each $\mathbb{F}_1$-scheme with relations $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$ (with a slight modification of our terminology, see Convention 5.5) determines a natural transformation

$$\Psi_1\colon \Sigma_{\mathbb{Z}} \to \Sigma'_{\mathbb{Z}}$$

and a natural transformation

$$\Psi_2\colon \Sigma'_{\mathsf{B}} \to \Sigma'_{\mathbb{Z}}\,,$$

where Σ'_{B} is a certain pullback sheaf on the category Ring (defined by the diagram 5.4). This implies that, given a $\widetilde{\mathsf{B}}$-scheme Σ underlying a $\mathbb{F}_1$-scheme with relations, we can think of its "$\mathbb{F}_{1^{q-1}}$-points" in two different senses, and therefore count them in two different ways, as stated in Proposition 5.6 and in Theorem 5.7. An interesting case is when the $\mathbb{F}_{1^n}$-points of the underlying monoidal scheme $\underline{\Sigma}$ are counted by a polynomial in n. Theorem 4.10 of [1] shows that, if $(\underline{\Sigma}, \Sigma'_{\mathbb{Z}}, \Phi)$ is an $\mathbb{F}_1$-scheme such that the monoidal scheme $\underline{\Sigma}$ is noetherian and torsion-free, then $\#\underline{\Sigma}(\mathbb{F}_{1^n}) = P(\underline{\Sigma}, n)$, where

$$P\left(\underline{\Sigma}, n\right) = \sum_{x\in\underline{\Sigma}} \#\operatorname{Hom}(\mathcal{O}^{\times}_{\underline{\Sigma},x}, \mathbb{F}_{1^n})\,.$$

For an $\mathbb{F}_1$-scheme with relations $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$ such that the underlying $\widetilde{\mathsf{B}}$-scheme Σ is noetherian and torsion-free (Definition 5.11), we introduce the polynomial

$$Q\left(\underline{\Sigma}, n\right) = \sum_{x\in\underline{\Sigma}} \#\operatorname{Hom}_{\mathsf{B}}\left(\mathcal{O}^{\times}_{\Sigma,x}, \mathbb{F}_{1^n}\right)\,,$$

and prove (Proposition 5.14) that $Q(\underline{\Sigma}, n) \leq P(\underline{\Sigma}, n)$.

Finally, we would like to emphasize that our approach to blueprints, being entirely functorial, seems to be appropriate to carry out a “derived version” of the category of B-schemes. In fact, in quite general terms, a definition of “derived B-scheme” could be obtained by replacing, in our definition of B-scheme, the category Set (resp. Set_*) by the category S of spaces (resp. S_* of pointed spaces) and the notion of monoid object by that of $\mathbb{E}_\infty$-algebra. This issue will be the object of future work.

2 The general setting

2.1 Schemes over a monoidal category

For the reader’s convenience, we start by giving a quick résumé of some of the basic constructions of the “relative algebraic geometry” developed in [27, §2].

Let $\mathsf{C} = (\mathsf{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category ($\mathbf{1}$ is the unit object), and denote by $\mathsf{CMon}_{\mathsf{C}}$ the category of commutative (associative and unitary) monoid objects in C.

We assume that C is *complete, cocomplete, and closed* (i.e., for every pair of objects X, Y, the contravariant functor $\operatorname{Hom}_{\mathsf{C}}(\text{-} \otimes X, Y)$ is represented by an “internal hom” set $\underline{\operatorname{Hom}}(X, Y)$).

The assumptions on C imply, in particular, that the forgetful functor

$$|\text{-}|\colon \mathsf{CMon}_{\mathsf{C}} \to \mathsf{C}$$

admits a left adjont

$$L\colon \mathsf{C} \to \mathsf{CMon}_{\mathsf{C}}\,, \tag{2.1}$$

which maps an object X to the free commutative monoid object $L(X)$ generated by X.

For each commutative monoid V in $\mathsf{CMon}_{\mathsf{C}}$ one may introduce the notion of V-module (cf. [12, p. 478]). The category $V\text{-}\mathsf{Mod}$ of such objects has a natural symmetric monoidal structure given by the “tensor product” $\otimes_V$; this structure turns out to be closed. Given a morphism $V \to W$ in $\mathsf{CMon}_{\mathsf{C}}$, there is a change of basis functor

$$\text{-} \otimes_V W\colon\ V\text{-}\mathsf{Mod} \to W\text{-}\mathsf{Mod}\,,$$

whose adjoint is the forgetful functor W - $\mathsf{Mod} \to V$ - Mod. Note that the category of commutative monoids in V - Mod — i.e. the category of *commutative V-algebras* — is naturally equivalent to the category $V/\mathsf{CMon}_\mathsf{C}$.

The category Aff_C of *affine schemes over* C is, by definition, the category $\mathsf{CMon}_\mathsf{C}^{\mathrm{op}}$. Given an object V in CMon_C the corresponding object in Aff_C will be denoted by $\operatorname{Spec} V$.

To define, in full generality, the category of schemes over C one follows the standard procedure of glueing together affine schemes. To this end, one first endows Aff_C with a suitable Grothendieck topology. Let us recall the general definition.

Definition 2.1 Let G be any category. A *Grothendieck topology* on G is the assignment to each object U of G of a collection of sets of arrows $\{U_i \to U\}$ called *coverings of* U so that the following conditions are satisfied:

i) if $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering;
ii) if $\{U_i \to U\}$ is a covering and $V \to U$ is any arrow, then there exist the fibered products $\{U_i \times_U V\}$ and the collection of projections $\{U_i \times_U V \to V\}$ is a covering;
iii) if $\{U_i \to U\}$ is a covering and for each index i there is a covering $\{V_{ij} \to U_i\}$ (where j varies in a set depending on i), each collection $\{V_{ij} \to U_i \to U\}$ is a covering of U.

A category with a Grothendieck topology is a called a *site*.

Remark 2.2 As it is clear from the definition above, a Grothendieck topology on a category G is introduced with the aim of glueing objects locally defined, and what really matters is therefore the notion of covering. So, in spite of its name, a Grothendieck topology could better be thought of as a generalization of the notion of covering rather than of the notion of topology (notice, for example, that, though the maps $U_i \to U$ in a covering can be seen as a generalization of open inclusions $U_i \subset U$, no condition generalizing the topological requirement about unions of open subsets is prescribed).$\triangle$

Given a site G and a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$, we denote by h_U the presheaf represented by U and by $h_\mathcal{U} \subset h_U$ the subpresheaf of those maps that factorise through some element of $\mathcal{U}$.

Definition 2.3 Let G be a site. A presheaf $F : \mathsf{G}^{\mathrm{op}} \to \mathsf{Set}$ is said to be a sheaf if, for every covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$, the restriction map $\operatorname{Hom}(h_U, F) \to \operatorname{Hom}(h_\mathcal{U}, F)$ is an isomorphism.

Coming back to our symmetric monoidal category C, the associated category of affine schemes Aff_C can be equipped with two different Grothendieck topologies by means of the following ingenious definitions (which, of course, generalize the corresponding usual definitions in "classical" algebraic geometry).

One says [27, Def. 2.9, 1), 2), 3)] that a morphism $f : \operatorname{Spec} W \to \operatorname{Spec} V$ in Aff_C is

- *flat* if the functor - $\otimes_V W : V$ - $\mathsf{Mod} \to W$ - Mod is exact;
- *an epimorphism* if, for any Z in CMon_C, the functor

$$f^*\colon \mathrm{Hom}_{\mathsf{CMon}_\mathsf{C}}(W, Z) \to \mathrm{Hom}_{\mathsf{CMon}_\mathsf{C}}(V, Z)$$

is injective ;

- *of finite presentation* if, for any filtrant diagram $\{Z_i\}_{i\in I}$ in $V/\mathsf{CMon}_\mathsf{C}$, the natural morphism

$$\varinjlim \mathrm{Hom}_{V/\mathsf{CMon}_\mathsf{C}}(W, Z_i) \to \mathrm{Hom}_{V/\mathsf{CMon}_\mathsf{C}}\left(W, \varinjlim Z_i\right)$$

is an isomorphism.

Definition 2.4 [27, Def. 2.9, 4); Def. 2.10] a) A collection of morphisms

$$\{f_j\colon \operatorname{Spec} W_j \to \operatorname{Spec} V\}_{j\in J}$$

in Aff_C is a flat cover if

i) each morphism $f_j\colon \operatorname{Spec} W_j \to \operatorname{Spec} V$ is flat and
ii) there exists a finite subset of indices $J' \subset J$ such that the functor

$$\prod_{j\in J'} \text{-} \otimes_V W_j\colon V\text{-}\mathsf{Mod} \to \prod_{j\in J'} W_j\text{-}\mathsf{Mod}$$

is conservative.

(b) A morphism $f\colon \operatorname{Spec} W \to \operatorname{Spec} V$ in Aff_C is an open Zariski immersion if it is a flat epimorphism of finite presentation.
(c) A collection of morphisms $\{f_j\colon \operatorname{Spec} W_j \to \operatorname{Spec} V\}_{j\in J}$ in Aff_C is a Zariski cover if it is a flat cover and each $f_j\colon \operatorname{Spec} W_j \to \operatorname{Spec} V$ is an open Zariski immersion.

Remark 2.5 The previous definition is actually a particular case of a more general construction. Indeed, as shown in [27], to define a topology on a complete and cocomplete category D is enough to assign a pseudo-functor $M\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Cat}$ satisfying the the following conditions:

i) for each morphism $q\colon X \to Y$ in D, the functor $M(q) = q^*\colon M(Y) \to M(X)$ has a right adjoint $q_*\colon M(X) \to M(Y)$ which is conservative
ii) for each Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{q'} & Y' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ X & \xrightarrow[q]{} & Y \end{array}$$

in D, the natural transformation $q^* r'_* \Longrightarrow r_* q'^*$ is an isomorphism.

In terms of such a functor one can define the notion of M-faithfully flat cover [27, Def. 2.3] and the associated pretopology [27, Prop. 2.4], which induces a topology on D.

In the classical theory of schemes, D is the category $\mathsf{Ring}^{\mathrm{op}}$ of affine schemes and, for each $X = \operatorname{Spec} A$, $M(A)$ is the category of quasi-coherent sheaves on X. When starting with a monoidal category C satisfying our assumptions, D is the category $\mathsf{Aff}_{\mathsf{C}}$ and the pseudo-functor M maps an object V in $\mathsf{CMon}_{\mathsf{C}}$ to the category of V-modules and a morphism $\operatorname{Spec} V \to \operatorname{Spec} W$ to the functor $- \otimes_V W \colon V\text{-}\mathsf{Mod} \to W\text{-}\mathsf{Mod}$. What we have called "flat cover" correspond to Toën-Vaquié's "M-faithfully flat cover" (cf. [27, Def. 2.8, Def. 2.10]).
When D is endowed with a topology, a natural question that arises is how the pseudo-functor M behaves with respect to it. It can be proven ([27, Th. 2.5] that M is a stack with respect to that topology (for the notion of a stack, the reader may consult [29]).$\triangle$

By making use of flat covers and Zariski covers introduced in Definition 2.4 we may equip the category $\mathsf{Aff}_{\mathsf{C}}$ with two distinct Grothendieck topologies, called, respectively, the *flat* and the *Zariski* topology. Correspondingly, there are two categories of sheaves on $\mathsf{Aff}_{\mathsf{C}}$, namely

$$\mathsf{Sh}^{\mathrm{flat}}(\mathsf{Aff}_{\mathsf{C}}) \subset \mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_{\mathsf{C}}) \subset \mathsf{Presh}(\mathsf{Aff}_{\mathsf{C}}) \ .$$

Notice that, for each affine scheme Ξ, the presheaf $Y(\Xi)$ given by the Yoneda embedding $Y(-)\colon \mathsf{Aff}_{\mathsf{C}} \to \mathsf{Presh}(\mathsf{Aff}_{\mathsf{C}})$ is actually a sheaf in $\mathsf{Sh}^{\mathrm{flat}}(\mathsf{Aff}_{\mathsf{C}}) \subset \mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_{\mathsf{C}})$ [27, Cor. 2.11, 1)]; this sheaf will be denoted again by Ξ.

The next and final step is to define the category of schemes over the category C. We first have to introduce the notion of affine Zariski cover in the category $\mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_{\mathsf{C}})$.

Definition 2.6 [27, Def. 2.12] a) Let Ξ be an affine scheme in $\mathsf{Aff}_{\mathsf{C}}$. A subsheaf $\mathcal{F} \subset \Xi$ is said to be a Zariski open of Ξ if there exists a collection of open Zariski immersions $\{\Xi_i \to \Xi\}_{i \in I}$ such that $\mathcal{F}$ is the image of the sheaf morphism $\coprod_{i \in I} \Xi_i \to \Xi$.
(b) A morphism $\mathcal{F} \to \mathcal{G}$ in $\mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_{\mathsf{C}})$ is said to be an open Zariski immersion if, for any affine scheme Ξ and any sheaf morphism $\Xi \to \mathcal{G}$, the induced morphism $\mathcal{F} \times_{\mathcal{G}} \Xi \to \Xi$ is a monomorphism whose image is a Zariski open of Ξ.

(c) Let $\mathcal{F}$ be a sheaf in $\mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_{\mathsf{C}})$. A collection of open Zariski immersions $\{\Xi_i \to \mathcal{F}\}_{i \in I}$, where each Ξ_i is an affine scheme over $\mathsf{Aff}_{\mathsf{C}}$, is said to be an affine Zariski cover of $\mathcal{F}$ if the resulting morphism

$$\coprod_{i \in I} \Xi_i \to \mathcal{F}$$

is a sheaf epimorphism.

It should be noted that, in the case of affine schemes over C, the definition of open Zariski immersion in Definition 2.6, (b) does coincide with that previously introduced in Definition 2.4, (b) [27, Lemma 2.14].

Definition 2.7 A scheme over the category C is a sheaf $\mathcal{F}$ in $\mathsf{Sh}^{\mathrm{Zar}}(\mathsf{Aff}_{\mathsf{C}})$ that admits an affine Zariski cover. The category of schemes over C will be denoted by $\mathsf{Sch}_{\mathsf{C}}$.

2.2 Notation and examples

Primarily to the purpose of fixing our notational conventions, we now briefly describe the basic examples of symmetric monoidal categories we shall work with in the sequel of the present paper.

- The category Set of sets can be endowed with a monoidal product given by the Cartesian product. Then $(\mathsf{Set}, \times, *)$ is a symmetric monoidal category and $\mathsf{CMon}_{\mathsf{Set}} = \mathsf{Mon}$ is the usual category of commutative, associative and unitary monoids.
- The category Set_* of pointed sets can be endowed with a monoidal product given by the smash product $\wedge$; in this case, the unit object is the pointed set $\mathbb{S}^0$ consisting of two elements. Then $(\mathsf{Set}_*, \wedge, \mathbb{S}^0)$ is a symmetric monoidal category and $\mathsf{CMon}_{\mathsf{Set}_*} = \mathsf{Mon}_0$ is the category of commutative, associative and unitary monoids with "absorbent object" (such an object will be denoted by 0 in multiplicative notation and by $-\infty$ in additive notation).
- The category Mon can be endowed with a monoidal product $\otimes$ defined in the following way: $R \otimes R'$ is the quotient of the product $R \times R'$ by the relation $\sim$ such that $(nr, r') \sim (r, nr')$ for each $(n, r, r') \in \mathbb{N} \times R \times R'$. Clearly, the unit object is the additive monoid $(\mathbb{N}, +)$. Then $(\mathsf{Mon}, \otimes, \mathbb{N})$ is a symmetric monoidal category and $\mathsf{CMon}_{\mathsf{Mon}} = \mathsf{SRing}$ is the category of commutative, associative and unitary semirings.
- The category $\mathsf{Ab} = \mathbb{Z}\text{-}\mathsf{Mod}$ of Abelian groups can be endowed with a monoidal product $\otimes_{\mathbb{Z}}$ given by the usual tensor product of $\mathbb{Z}$-modules. Then $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ is a symmetric monoidal category and $\mathsf{CMon}_{\mathsf{Ab}} = \mathsf{Ring}$ is the category of commutative, associative and unitary rings.

For the functor $L\colon \mathsf{C} \to \mathsf{CMon}_{\mathsf{C}}$ defined in Eq. 2.1 as left adjoint to the forgetful functor $|\text{-}|\colon \mathsf{CMon}_{\mathsf{C}} \to \mathsf{C}$ we shall adopt the following special conventions:

- if $\mathsf{C} = \mathsf{Set}$, L will be denoted by
$$\mathbb{N}[\text{-}]\colon \mathsf{Set} \to \mathsf{Mon}\,; \tag{2.2}$$
- if $\mathsf{C} = \mathsf{Mon}$, L will be denoted by
$$\text{-} \otimes_{\mathbb{U}} \mathbb{N}\colon \mathsf{Mon} \to \mathsf{SRing}\,, \tag{2.3}$$
where $\mathbb{U}$ is the monoid consisting of just one element (the notation being motivated by the identity $\mathbb{U} \otimes_{\mathbb{U}} \mathbb{N} = \mathbb{N}$);
- if $\mathsf{C} = \mathsf{Mon}_0$, L will be denoted by
$$\text{-} \otimes_{\mathbb{F}_1} \mathbb{N}\colon \mathsf{Mon}_0 \to \mathsf{SRing}\,, \tag{2.4}$$
where $\mathbb{F}_1$ is the object of Mon_0 consisting of two element, namely $\mathbb{F}_1 = \{0, 1\}$ in multiplicative notation (also in this case, the notation is motivated by the identity $\mathbb{F}_1 \otimes_{\mathbb{F}_1} \mathbb{N} = \mathbb{N}$);

- if $\mathsf{C} = \mathsf{Ab}$, L will be denoted by

$$\mathbb{Z}[\,\text{-}\,]\colon \mathsf{Ab} \to \mathsf{Ring}\,. \tag{2.5}$$

All symmetric monoidal categories Set, Set_*, Mon, Mon_0, Ab described above are complete, cocomplete, and closed, so we can apply the machinery of Toën-Vaquié's theory illustrated in Subsect. 2.1 and define, for each of these categories, the corresponding category of schemes over it. In this way, when $\mathsf{C} = \mathsf{Ab}$, one unsurprisingly recovers the usual notion of *classical scheme*. A more intriguing example is provided by the case of $\mathsf{C} = \mathsf{Set}$.

Example 2.8 **Monoidal schemes** An object of the category $\mathsf{Sch}_{\mathsf{Set}}$ is a "scheme over $\mathbb{F}_1$" in the sense of [4]. The equivalence between the two definitions was proved in [28]. We recall that, if M is a commutative monoid, its "spectrum over $\mathbb{F}_1$" Spec M can be realized as the set of prime ideals of M and given a topological space structure.

In the present paper we shall call an object in $\mathsf{Sch}_{\mathsf{Set}}$ a *monoidal scheme* and use the name of "$\mathbb{F}_1$-scheme" for a different kind of algebro-geometric structures (see Definition 5.1). △

3 The category of blueprints

The notion of *blueprint* was introduced by Olivier Lorscheid in his 2012 paper [16].

Definition 3.1 A *blueprint* is a pair $B = (R, A)$, where R is a semiring and A is a multiplicative subset of R containing 0 and 1 and generating R as a semiring. A blueprint morphism $f\colon B_1 = (R_1, A_1) \to B_2 = (R_2, A_2)$ is a semiring morphism $f\colon R_1 \to R_2$ such that $f(A_1) \subset A_2$.

Notice that, given a blueprint morphism $f\colon B_1 = (R_1, A_1) \to B_2 = (R_2, A_2)$, its restriction $f|_{A_1}\colon A_1 \to A_2$ is a monoid morphism that uniquely determines f on the whole of R_1.

The idea underlying the notion of blueprint can be illustrated as follows. Some equivalence relations that do not make sense in a monoid A may be expressed in the semiring $A \otimes_{\mathbb{F}_1} \mathbb{N}$. Now, any equivalence relation $\mathcal{R}$ on a semiring S induces a projection $S \to S/\mathcal{R}$ and can indeed be recovered by such a map. So, the assignment of a pair $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$ is to be interpreted as the datum of a monoid A plus the relation on $A \otimes_{\mathbb{F}_1} \mathbb{N}$ given by the epimorphism $A \otimes_{\mathbb{F}_1} \mathbb{N} \to R$.

Example 3.2 Consider the monoid $A_T = \mathbb{N} \cup \{-\infty\}$ (in additive notation, corresponding to $\{T^i\}_{i \in \mathbb{N} \cup \{-\infty\}}$ in multiplicative notation) and the corresponding free semiring $A_T \otimes_{\mathbb{F}_1} \mathbb{N}$ of polynomials in T with coefficient in $\mathbb{N}$ (the functor - $\otimes_{\mathbb{F}_1} \mathbb{N}$ has been introduced in eq. 2.4). Notice that Spec A_T has two points, namely the prime ideals $\{-\infty\}$ and $(\mathbb{N} \setminus \{0\}) \cup \{-\infty\}$, which embed in Spec $A_T \otimes_{\mathbb{F}_1} \mathbb{N}$ (we are loosely thinking of Spec $A_T \otimes_{\mathbb{F}_1} \mathbb{N}$ as the underlying topological space).
Now, if one takes a closed subset of Spec $A_T \otimes_{\mathbb{F}_1} \mathbb{N}$ and intersects it with Spec A_T, one could naively think that the intersection is nonempty only when the chosen closed subset is defined by some relation in A_T. However, this is not the case: for instance,

the relation $2T = 1$, which makes the ideal (T) trivial, cannot be expressed in the monoid A_T. According to Lorscheid's idea, one can represent this affine "monoidal scheme" by considering the pair $(A_T, A_T \otimes_{\mathbb{F}_1} \mathbb{N} \to A_T \otimes_{\mathbb{F}_1} \mathbb{N}/(2T = 1))$. △

The category of blueprints can be given a handier description, which makes it easier to characterise it as the category of commutative monoids in a suitable symmetric monoidal category.

Let us consider the functor $-\otimes_{\mathbb{F}_1} \mathbb{N} \colon \mathsf{Mon}_0 \to \mathsf{SRing}$ (introduced in eq. 2.4)

Definition 3.3 The category Blp is the full subcategory of $-\otimes_{\mathbb{F}_1} \mathbb{N}/\mathsf{SRing}$ whose objects $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$ satisfy the conditions:

$$\begin{aligned}&\text{a) the morphism } A \otimes_{\mathbb{F}_1} \mathbb{N} \to R \text{ is an epimorphism;}\\ &\text{b) the composition } A \to |A \otimes_{\mathbb{F}_1} \mathbb{N}| \to |R|\text{, is a monomorphism}\\ &\text{(the first map being the unit of the adjunction).}\end{aligned} \tag{3.1}$$

It is immediate that the category Blp is equivalent to the category of blueprints introduced in Definition 3.1

Consider now the forgetful functor $|-| \colon \mathsf{Mon}_0 \to \mathsf{Set}_*$; for each monoid M with absorbent object 0 (in multiplicative notation), the base point of the associated set $|M|$ is clearly the element corresponding to 0. Its adjoint functor is the functor

$$\mathbb{N}[-] \colon \mathsf{Set}_* \to \mathsf{Mon}_0\,.$$

We can now form the full subcategory B of $\mathbb{N}[-]/\mathsf{Mon}_0$ whose objects $(X, \mathbb{N}[X] \to M)$ are described by conditions formally identical to those in eq. 3.1

$$\begin{aligned}&\text{a) the morphism } \mathbb{N}[X] \to M \text{ is an epimorphism;}\\ &\text{b) the composition } X \to |\mathbb{N}[X]| \to |M| \text{ is a monomorphism.}\end{aligned} \tag{3.2}$$

Remark 3.4 The category B above corresponds to the category of pointed set endowed with a pre-addition structure, as described in [18, §4]. △

Theorem 3.5 *The category B carries a natural structure of symmetric monoidal category. Moreover, this structure is closed, complete, and cocomplete.*

Proof In the category B there is a natural symmetric monoidal product given by

$$(X, \mathbb{N}[X] \to M) \otimes \left(X', \mathbb{N}[X'] \to M'\right) = \left(X \wedge X', \mathbb{N}[X \wedge X'] \to M \otimes M'\right), \tag{3.3}$$

where the map $\mathbb{N}[X \wedge X'] \to M \otimes M'$ is the composition

$$\mathbb{N}[X \wedge X'] \to \mathbb{N}[X] \otimes \mathbb{N}[X'] \to M \otimes M'\,;$$

the first morphism maps $n(x, x')$ to $nx \otimes x'$ and is an isomorphism (in other words, the functor $\mathbb{N}[-]$ is monoidal).

Since $M \otimes M'$ is generated as a monoid by elements of the form $x \otimes x'$, and since the two maps $\mathbb{N}[X] \to M$ and $\mathbb{N}[X'] \to M'$ are surjective, the map $\mathbb{N}[X \wedge X'] \to M \otimes M'$

is also surjective. Moreover, by the definition of tensor product in the category Mon, for any $x, y \in X \setminus \{*\}$ and $x', y' \in X' \setminus \{*\}$ one has $x \otimes x' = y \otimes y'$ if and only if $(x, x') = (y, y')$, so that the map

$$X \wedge X' \to |M \otimes M'|$$

is a monomorphism. Conditions 3.2 are therefore satisfied.

We now show that the monoidal category B is closed. Let us define the internal hom functor by setting

$$\begin{aligned}&\underline{\mathrm{Hom}}\left((X, \mathbb{N}[X] \to M), (Y, \mathbb{N}[Y] \to N)\right)\\ &= \left(Y^X \times_{|N|^X} |N^M|, \mathbb{N}\left[Y^X \times_{|N|^X} |N^M|\right] \to \widetilde{N^M}\right),\end{aligned} \tag{3.4}$$

where $\widetilde{N^M}$ is the image of the map

$$\mathbb{N}\left[Y^X \times_{|N|^X} |N^M|\right] \to \mathbb{N}\left[|N^M|\right] \to N^M$$

(the second map above is the counit of the adjunction). Let us check the adjunction property. For each map

$$\begin{aligned}&(X, \mathbb{N}[X] \to M) \otimes (Y, \mathbb{N}[Y] \to N)\\ &= (X \wedge Y, \mathbb{N}[X \wedge Y] \to M \otimes N) \to (Z, \mathbb{N}[Z] \to L),\end{aligned} \tag{3.5}$$

the first component corresponds, by the exponential law in Set_*, to a map $X \to Z^Y$, while the second component is given by a commutative square

$$\begin{tikzcd} \mathbb{N}[X \wedge Y] \arrow[r] \arrow[d] & \mathbb{N}[Z] \arrow[d] \\ M \otimes N \arrow[r] & L \end{tikzcd} \tag{3.6}$$

where the arrow on the left is the product map $\mathbb{N}[X] \otimes \mathbb{N}[Y] \to M \otimes N$ and the top arrow is the image of the map in the first component through the functor $\mathbb{N}[\,\text{-}\,]$. By using the property that $\mathbb{N}[\,\text{-}\,]$ is the left adjoint to the forgetful functor and by noticing that the bottom arrow in 3.6 corresponds to a map $M \to L^N$, it is immediate that assigning the commutative diagram 3.6 is equivalent to assigning the two commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Z^Y \arrow[d] \\ & |L|^Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d] \arrow[dr] & \\ |M| \arrow[r] & |L^N| \end{tikzcd}$$

together with the condition that the diagonal morphism of the first coincides with the composition of the diagonal morphism of the second and the morphisms $|L^N| \hookrightarrow$

$|L|^{|N|} \to |L|^Y$ (the second map being induced by the map $Y \to |N|$). Summing up, a map as in eq. 3.5 is equivalent to a map from X to the pullback defined by the diagram

$$\begin{tikzcd} & Z^Y \arrow[d] \\ |L^N| \arrow[r] & |L|^Y \end{tikzcd}$$

along with a compatible map $M \to L^N$ in such a way that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[rr, bend left] & |L^N| \times_{|L|^Y} Z^Y \arrow[r] \arrow[d] & Z^Y \arrow[d] \\ |M| \arrow[r] & |L^N| \arrow[r] & |L|^Y \end{tikzcd}$$

This shows that the internal hom functor in eq. 3.4 is indeed a right adjoint to the monoidal product functor in eq. 3.3.

We wish now to show that the category B is complete and cocomplete. First we prove that it admits colimits. Given a diagram whose objects are $(X_i, \mathbb{N}[X_i] \to M_i)$, we claim that its colimit is the object

$$B = \left(\widetilde{\varinjlim X_i}, \mathbb{N}\left[\widetilde{\varinjlim X_i}\right] \to \varinjlim M_i\right) ,$$

where $\widetilde{\varinjlim X_i}$ denotes the image of the natural map $\varinjlim X_i \to |\varinjlim M_i|$; the maps from the diagram to B are the obvious ones. It is immediate that B is an object of B. The injectivity condition is satisfied by definition. As for the surjectivity condition, one has that, since the functor $\mathbb{N}[\text{-}]$ preserves colimits (being a left adjoint), the map $\mathbb{N}[\varinjlim X_i] \to \varinjlim M_i$ is surjective (by [20], Theorem V.2.1, it is enough to show that for arbitrary coproducts and coequalizers, in which cases it is a consequence of the surjectivity of the maps $\mathbb{N}[X_i] \to M_i$), so that the image of $\varinjlim X_i$ generates $\varinjlim M_i$; hence, the map $\mathbb{N}[\widetilde{\varinjlim X_i}] \to \varinjlim M_i$ is surjective.

Consider a map from the given diagram to an object C of B. In the category $\mathbb{N}[\text{-}]/\mathsf{Mon}_0$ such a map factorises in a unique way through the object $(\varinjlim X_i, \mathbb{N}[\varinjlim X_i] \to \varinjlim M_i)$ because of the colimit properties in the categories Set_* and Mon_0 and because the functor $\mathbb{N}[\text{-}]$ preserves colimits. If two elements $x, y \in \varinjlim X_i$ have the same image $m \in \varinjlim M_i$, then their images in the first component of C are mapped by the morphism in the second component to the same element. So, the images of x and y do coincide, just because C is an object of B. It follows that the map from the diagram in C uniquely factorises through B, so that our claim is proved.

Second we prove that B admits limits. Given a diagram as above, we claim that its limit is the object

$$B' = \left(\varprojlim X_i, \mathbb{N}\left[\varprojlim X_i\right] \to \widetilde{\varprojlim M_i}\right) ,$$

where $\widetilde{\varprojlim M_i}$ is the image of the natural map $\mathbb{N}[\varprojlim X_i] \to \varprojlim M_i$, which is adjoint to the map $\varprojlim X_i \to \varprojlim |M_i| \cong |\varprojlim M_i|$ (the last isomorphism holds since $|\text{-}|$ preserves limits, being a right adjoint) induced by the maps $X_i \to |M_i|$; the maps from B' to the diagram are the obvious ones. It is clear that B' is an object of B: the surjectivity condition holds by definition, while for the injectivity condition it is enough to note that it holds when the limit is either an arbitrary product or an equalizer (see [20], Theorem V.2.1). Consider now a map from an object C to the given diagram. In the category $\mathbb{N}[\text{-}]/\mathsf{Mon}_0$ such a map uniquely factorises through the object $(\varprojlim X_i, \mathbb{N}[\varprojlim X_i] \to \varprojlim M_i)$, because of the limit properties in the categories Set_* and Mon_0. Since the second component of C is a surjective morphism, this map uniquely factorises through B'. Thus, B' satisfies the limit condition, as claimed. □

Proposition 3.6 *The category* Blp *of blueprints is equivalent to the category* CMon_B *of monoids in the symmetric monoidal category* B*.*

Proof To begin with, notice that, for each monoid object $((X, \mathbb{N}[X] \to M), \mu)$ in B, the domain of the multiplication map

$$\mu\colon\ (X, \mathbb{N}[X] \to M) \otimes (X, \mathbb{N}[X] \to M) \to (X, \mathbb{N}[X] \to M)$$

is defined in eq. 3.3 as

$$(X, \mathbb{N}[X] \to M) \otimes (X, \mathbb{N}[X] \to M) := (X \wedge X, \mathbb{N}[X \wedge X] \to M \otimes M)\,.$$

So the first component of μ is a map

$$m : X \wedge X \to X$$

which defines a (multiplicative) monoid structure on the set X, while the second component of μ yields a commutative diagram

$$\begin{tikzcd} \mathbb{N}[X \wedge X] \arrow[r, "\mathbb{N}[m]"] \arrow[d] & \mathbb{N}[X] \arrow[d] \\ M \otimes M \arrow[r] & M \end{tikzcd}$$

whose bottom arrow induces an associative and commutative multiplication on the monoid M compatible with its monoidal sum; in other words, it induces a semiring structure on M.

Similarly, the top arrow induces a semiring structure one the monoid $\mathbb{N}[X]$. In this case, since the multiplication is given by the application of the free monoid functor $\mathbb{N}[\text{-}]$ to the multiplication m of X, the resulting semiring is nothing but the free semiring $X \otimes_{\mathbb{F}_1} \mathbb{N}$ generated by the monoid (X, m). The commutativity of the diagram ensures that the multiplication on X is consistent with that on M, so that X can still be seen as a subobject of $|M|$.

In conclusion, a monoid object in the category B is a blueprint, and it is also obvious that any blueprint can be obtained this way. □

Remark 3.7 Theorem 3.5 and Proposition 3.6 should hopefully provide a full elucidation of [18, Lemma 4.1]. △

We have shown that the category of blueprints fits in with the general framework proposed by Toën and Vaquié, so we can apply the formalism of Subsection 2.1 to define the category of schemes over B.

Definition 3.8 An affine B-scheme is an object of the category $\mathsf{Aff}_{\mathsf{B}} = \mathsf{Blp}^{\mathrm{op}}$, a B-scheme an object of the category $\mathsf{Sch}_{\mathsf{B}}$ *(see Definition 2.7)*.

Remark 3.9 A "B-scheme" corresponds to what is called a "subcanonical blue scheme" in [18]. △

4 Adjunctions

4.1 B-schemes

This sections aims to show that the natural adjunction between the categories $\mathsf{Aff}_{\mathsf{Mon}_0}$ and $\mathsf{Aff}_{\mathsf{Set}_*}$ factorizes through an adjunction between the categories $\mathsf{Aff}_{\mathsf{Mon}_0}$ and $\mathsf{Aff}_{\mathsf{B}}$ and an adjunction between the categories $\mathsf{Aff}_{\mathsf{Set}_*}$ and $\mathsf{Aff}_{\mathsf{B}}$, whose right adjoints induce functors between the corresponding categories of relative schemes.

Lemma 4.1 *The functor $\tilde{F}\colon \mathbb{N}[\,\text{-}\,]/\mathit{Mon}_0 \to \mathit{Mon}_0$ mapping an object $(X, \mathbb{N}[X] \to M)$ to the monoid M admits a right adjoint*

$$\tilde{G}\colon \mathit{Mon}_0 \to \mathbb{N}[\,\text{-}\,]/\mathit{Mon}_0\,, \tag{4.1}$$

mapping a monoid M to the object $(|M|, \mathbb{N}[|M|] \to M)$, where the second component is the counit of the adjunction $\mathbb{N}[\,\text{-}\,] \dashv |\,\text{-}\,|$. The adjunction $\tilde{F} \dashv \tilde{G}$ induces an adjunction between the associated categories of monoids

$$\begin{tikzcd} \mathit{SRing} \arrow[r, bend right, "G"'] & \text{-} \otimes_{\mathbb{F}_1} \mathbb{N}/\mathit{SRing} \arrow[l, bend right, "F"'] \end{tikzcd}\,, \tag{4.2}$$

where F maps an object $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$ to the semiring R and its right adjoint G maps a semiring R to the object $(|R|, |R| \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$, where the second component is the counit of the adjunction $\text{-} \otimes_{\mathbb{F}_1} \mathbb{N} \dashv |\,\text{-}\,|$.

Proof Let $(X, \mathbb{N}[X] \to M)$ be an object of $\mathbb{N}[\,\text{-}\,]/\mathsf{Mon}_0$ and N a monoid. Let us consider a morphism

$$(X, \mathbb{N}[X] \to M) \to (|N|, \mathbb{N}[|N|] \to N)$$

in the category $\mathbb{N}[\,\text{-}\,]/\mathsf{Mon}_0$ and denote by $f\colon X \to |N|$ the induced set morphism. In the commutative square

$$\begin{tikzcd} \mathbb{N}[X] \arrow[r, "\mathbb{N}[f]"] \arrow[d] & \mathbb{N}[|N|] \arrow[d] \\ M \arrow[r] & N \end{tikzcd} \tag{4.3}$$

the map $\mathbb{N}[f]$, because of the property of the vertical arrow on the right (which is the counit of the adjunction), amounts to the same as a map $\mathbb{N}[X] \to N$. Such a map, by adjunction, must be induced by the map $f\colon X \to |N|$. Thus, the assignment of the map f and the commutative square 4.3 are equivalent to the assignment of the commutative triangle

$$\begin{tikzcd} \mathbb{N}[X] \arrow[d] \arrow[dr] & \\ M \arrow[r] & N \end{tikzcd}$$

But this diagram is equivalent to the assignment of a map $M \to N$, since the vertical map is given. We have therefore the adjunction $\tilde{F} \dashv \tilde{G}$, as claimed. The last statement is now straightforward. □

Since image of the functor $\tilde{G}\colon \mathsf{Mon}_0 \to \mathbb{N}[\,\text{-}\,]/\mathsf{Mon}_0$ is contained in the subcategory B, the adjunction 4.2 restricts to the adjunction

$$\begin{tikzcd} \mathsf{SRing} \arrow[r, bend right, "G"'] & \mathsf{Blp} \arrow[l, bend right, "F"'] \end{tikzcd} . \tag{4.4}$$

It is immediate that the adjunction $\begin{tikzcd} \mathsf{SRing} \arrow[r, bend right, "|\,\text{-}\,|"'] & \mathsf{Mon}_0 \arrow[l, bend right, "\text{-}\otimes_{\mathbb{F}_1}\mathbb{N}"'] \end{tikzcd}$ factorises through the adjunction 4.4 and the adjunction

$$\begin{tikzcd} \mathsf{Mon}_0 \arrow[r, bend right, "\sigma"'] & \mathsf{Blp} \arrow[l, bend right, "\rho"'] \end{tikzcd} , \tag{4.5}$$

where $\rho(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R) = A$ and $\sigma(A) = (A,\ A \otimes_{\mathbb{F}_1} \mathbb{N} \xrightarrow{=} A \otimes_{\mathbb{F}_1} \mathbb{N}\ )$.

The adjunctions above induce opposite adjunctions between the corresponding categories of affine schemes. We have therefore the following diagram

$$\begin{tikzcd} \mathsf{Aff}_{\mathsf{Mon}_0} \arrow[r, "|\text{-}|"] \arrow[d, "G"] & \mathsf{Aff}_{\mathsf{Set}_*} \arrow[l, bend right, "\text{-}\otimes_{\mathbb{F}_1}\mathbb{N}"'] \arrow[dl, shift left, "\sigma"] \\ \mathsf{Aff}_{\mathsf{B}} \arrow[u, bend left, "F"] \arrow[ur, shift left, "\rho"] & \end{tikzcd} \tag{4.6}$$

associated to the diagram

$$\begin{tikzcd} \mathsf{Mon}_0 \arrow[r, "|\text{-}|"] \arrow[d, "\tilde{G}"] & \mathsf{Set}_* \arrow[l, bend right, "\mathbb{N}[\text{-}]"'] \arrow[dl, shift left, "\tilde{\sigma}"] \\ \mathsf{B} \arrow[u, bend left, "\tilde{F}"] \arrow[ur, shift left, "\tilde{\rho}"] & \end{tikzcd} \tag{4.7}$$

We now wish to show that the functors in diagram 4.7 satisfy the conditions that are required to apply [27, Prop. 2.1, Cor. 2.2]. Of course, it will be enough to check that for the adjunctions $\tilde{F} \dashv \tilde{G}$ and $\tilde{\rho} \dashv \tilde{\sigma}$.

Lemma 4.2 *In the adjunction* $\begin{tikzcd} Mon_0 \arrow[r, bend right, "\tilde{G}"'] & B \arrow[l, bend right, "\tilde{F}"'] \end{tikzcd}$

(1) *the left adjoint $\tilde{F}$ is monoidal;*
(2) *the right adjoint $\tilde{G}$ is conservative;*
(3) *the functor $\tilde{G}$ preserves filtered colimits.*

Proof (1) and (2) are straightforward.

As for (3), we have to show that the right adjoint preserves filtered colimits, which is also quite obvious. The colimit of a filtered diagram $(X_i, \mathbb{N}[X_i] \to M_i)$ is indeed given by

$$\left(\varinjlim X_i, \mathbb{N}\left[\varinjlim X_i\right] \to \varinjlim M_i\right)$$

provided that it belongs to our category (notice that $\mathbb{N}[\varinjlim X_i] \cong \varinjlim \mathbb{N}[X_i]$ since $\mathbb{N}[\text{-}]$ is a left adjoint). But it does, because the map $N[\varinjlim X_i] \to \varinjlim M_i$ is surjective due to the fact that so are the maps $\mathbb{N}[X_i] \to M_i$ and the injectivity condition is satisfied since the diagram is filtrant. □

Lemma 4.3 *In the adjunction*
$$\begin{tikzcd} B \arrow[r, bend right, "\tilde{\rho}"'] & Set_* \arrow[l, bend right, "\tilde{\sigma}"'] \end{tikzcd}$$

(1) *the left adjoint $\tilde{\sigma}$ is monoidal;*
(2) *the right adjoint $\tilde{\rho}$ is conservative;*
(3) *the functor $\tilde{\rho}$ preserves filtered colimits.*

Proof The functors $\tilde{\sigma}, \tilde{\rho}$ are defined as follows: $\tilde{\sigma}(X) = (X,\ \mathbb{N}[X] \xrightarrow{=} \mathbb{N}[X]\ )$ and $\tilde{\rho}(X, \mathbb{N}[X] \to M) = X$. (1) is then straightforward. As for (2), we know that a map $(X, \mathbb{N}[X] \to M) \to (Y, \mathbb{N}[Y] \to N)$ is determined by the first component, so that $\tilde{\rho}$ is conservative. Finally, (3) is proved by proceeding as in the proof of Lemma 4.2. □

Proposition 4.4 *The functor $F: Aff_B \to Aff_{Mon_0}$ is continuous w.r.t. the Zariski and the flat topology; morevover, the functor*

$$\widehat{F}: Sh\,(Aff_B) \to Sh\left(Aff_{Mon_0}\right) \tag{4.8}$$

preserves the subcategories of schemes and so induces a functor

$$\begin{aligned} \widehat{F}: Sch_B &\to Sch_{Mon_0} \\ \Sigma &\mapsto \widehat{F}(\Sigma) \end{aligned} \tag{4.9}$$

Proof a) We first note that, given objects $X_M = (X, \mathbb{N}[X] \to M)$, $X'_M = (X, \mathbb{N}[X] \to M')$ in B, if $X_M \to X_{M'}$ is a flat morphism in B, then in the associated diagram

$$\begin{tikzcd} M\text{ - Mod} \arrow[r] \arrow[d] & X_M\text{ - Mod} \arrow[d] \\ M'\text{ - Mod} \arrow[r] & X_{M'}\text{ - Mod} \end{tikzcd}$$

the natural transformation between the two compositions is an isomorphism. We wish to prove that an analogous property holds when one considers a flat morphism in the category Blp. As usual, it will be enough to work in the category $- \otimes_{\mathbb{F}_1} \mathbb{N}/\mathsf{SRing}$. Let $A_R = (A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$ and $A_S = (A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to S)$ be objects in this category, and consider a flat morphism $A_R \to A_S$. An A_R-module is given by a pair

$$(N, M) \in \mathsf{Set}_* \times \mathsf{Mon}_0$$

such that N is a subset of $|M|$ and generates it as a module, together with an action of A on N and an action of R on M, such that the former is the restriction of the latter. If M is an R-module M, its associated A_R-module is the $(R, R \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$-module $(|M|, M)$, whose A_R-module structure is induced by the map

$$A_R \to \left(R, R \otimes_{\mathbb{F}_1} \mathbb{N} \to R\right)$$

given by the pair of immersions $\iota\colon A \hookrightarrow R$ and $\iota \otimes_{\mathbb{F}_1} \mathrm{id}\colon A \otimes_{\mathbb{F}_1} \mathbb{N} \to R \otimes_{\mathbb{F}_1} \mathbb{N}$, where the latter fits in the commutative square

$$\begin{tikzcd} A \otimes_{\mathbb{F}_1} \mathbb{N} \arrow[r, "\iota\otimes_{\mathbb{F}_1}\mathrm{id}"] \arrow[d] & R \otimes_{\mathbb{F}_1} \mathbb{N} \arrow[d] \\ R \arrow[r, "\mathrm{id}_R"'] & R \end{tikzcd}$$

The category R - Mod can therefore be identified with the full subcategory of the category of

$$\left(A \otimes_{\mathbb{F}_1} \mathbb{N}, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R\right) \text{ - Mod}$$

whose underlying objects in $\mathsf{Mon}_0/\mathsf{Mon}_0$ are of the kind $(M, M = M)$.

We have now to show that, for any flat morphism $A_R \to A_S$ in - $\otimes_{\mathbb{F}_1} \mathbb{N}/\mathsf{SRing}$, in the associated diagram

$$\begin{tikzcd} R\text{ - Mod} \arrow[r] \arrow[d] & A_R\text{ - Mod} \arrow[d] \\ S\text{ - Mod} \arrow[r] & A_S\text{ - Mod} \end{tikzcd}$$

the natural transformation between the two compositions is an isomorphism. As for the first component, the commutativity up isomorphism of the above diagram is straightforward. As for the second component, that can be easily shown by adapting the argument in proof of Prop. 3.6 of [27]. The statement then follows from [27, Cor. 2.22]. □

Proposition 4.5 *The functor $\sigma : \mathit{Aff}_{\mathsf{Set}_*} \to \mathit{Aff}_{B}$ is continuous w.r.t. the Zariski and the flat topology; morevover, the functor*

$$\widehat{\sigma} : \mathit{Sh}\left(\mathit{Aff}_{\mathsf{Set}_*}\right) \to \mathit{Sh}\left(\mathit{Aff}_{B}\right) \tag{4.10}$$

preserves the subcategories of schemes and so induces a functor

$$\begin{aligned} \widehat{\sigma} : \mathit{Sch}_{\mathsf{Set}_*} &\to \mathit{Sch}_{B} \\ \Xi &\mapsto \widehat{\sigma}(\Xi) \end{aligned} \tag{4.11}$$

Proof Consider a flat morphism $A \to B$ in the category Mon_0, and denote by $A_{A\otimes_{\mathbb{F}_1}\mathbb{N}}$ the object $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} = A \otimes_{\mathbb{F}_1} \mathbb{N})$ in - $\otimes_{\mathbb{F}_1} \mathbb{N}/\mathsf{SRing}$. Each $A_{A\otimes_{\mathbb{F}_1}\mathbb{N}}$-module is given by a pair $(N, M) \in \mathsf{Set}_* \times \mathsf{Mon}_0$ together with an action of A on N and an action of

$A \otimes_{\mathbb{F}_1} \mathbb{N}$ on M, the two actions being compatible in the obvious sense. In the diagram

$$\begin{tikzcd} A_{A\otimes_{\mathbb{F}_1}\mathbb{N}}\text{-}\mathsf{Mod} \arrow[r] \arrow[d] & A\text{-}\mathsf{Mod} \arrow[d] \\ B_{B\otimes_{\mathbb{F}_1}\mathbb{N}}\text{-}\mathsf{Mod} \arrow[r] & B\text{-}\mathsf{Mod} \end{tikzcd}$$

the horizontal map sends an object (N, M) to the set N endowed with an action of the monoid A. Since tensor products are defined “componentwise”, the diagram commutes. □

4.2 $\widetilde{\mathsf{B}}$-schemes

By Proposition 4.5 there is an induced functor $\widehat{\sigma}\colon \mathsf{Sch}_{\mathsf{Set}_*} \to \mathsf{Sch}_{\mathsf{B}}$. One would like this functor to have a left adjoint determined by the functor $\rho\colon \mathsf{Aff}_{\mathsf{Set}_*} \to \mathsf{Aff}_{\mathsf{B}}$. The functor ρ may be easily shown to preserve Zariski covers, but it does not commute with finite limits (in other words, it is not continuous w.r.t. the Zariski topology, according to the usual terminology).

Example 4.6 Let us consider the free monoid $M = \langle X, Y\rangle$ and the blueprint B defined by the free monoid $\langle T, T_1, T_2, S, S_1, S_2\rangle$ with the relations $T = T_1 + T_2$ and $S = S_1 + S_2$. Let $f, g\colon M \to B$ be the morphisms mapping (X, Y), respectively, into (T_1, T_2) and (S_1, S_2). The coequalizer of f and g is the blueprint B' defined by the free monoid $\langle X, Y, Z\rangle$ with the relation $Z = X + Y$, while the coequalizer of ρf and ρg is the the free monoid $\langle T, S, Z_1, Z_2\rangle$. The latter is obviously different from $\rho B'$. △

This drawback may be sidestepped by proceeding as follows: 1) omit the requirement that the map $A \to |A \otimes_{\mathbb{F}_1} \mathbb{N}| \to |R|$ is a monomorphism in Definition 3.3 and define a category $\widetilde{\mathsf{Blp}}$ that contains the category Blp of blueprints; analogously, by omitting the second condition in eq. 3.2, define a category $\widetilde{\mathsf{B}}$ containing B; 2) prove that there is a functor $\rho\colon \mathsf{Aff}_{\mathsf{Set}_*} \to \mathsf{Aff}_{\widetilde{\mathsf{B}}}$ that is continuous w.r.t. the Zariski topology; 3) define the category of schemes $\mathsf{Sch}_{\widetilde{\mathsf{B}}}$ associated to this new category; 4) restrict our attention to the subcategory of $\mathsf{Sch}_{\widetilde{\mathsf{B}}}$ consisting of schemes that admit a cover by affine schemes in the category $\mathsf{Aff}_{\mathsf{B}}$.

More precisely, the categories $\widetilde{\mathsf{B}}$ and $\widetilde{\mathsf{Blp}}$ are defined in the following way.

Definition 4.7 The category $\widetilde{\mathsf{B}}$ is the full subcategory of $\mathbb{N}[\text{-}]/\mathsf{Mon}_0$ whose objects

$$(X, \mathbb{N}[X] \to M)$$

satisfy the condition that the morphism $\mathbb{N}[X] \to M$ is an epimorphism.
The category $\widetilde{\mathsf{Blp}}$ is the category $\mathsf{CMon}_{\widetilde{\mathsf{B}}}$ of monoids in the symmetric monoidal category $\widetilde{\mathsf{B}}$.

We denote again by $\rho\colon \widetilde{\mathsf{Blp}} \to \mathsf{Mon}_0$ the forgetful functor, $\rho(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R) = A$; analogously to adjunction 4.5, there is an adjunction

$$\begin{tikzcd} \mathsf{Mon}_0 \arrow[r, bend right, "\sigma"'] & \widetilde{\mathsf{Blp}} \arrow[l, bend right, "\rho"'] \end{tikzcd}\ , \tag{4.12}$$

where $\sigma(A) = (A,\ A \otimes_{\mathbb{F}_1} \mathbb{N} \xrightarrow{=} A \otimes_{\mathbb{F}_1} \mathbb{N}\ )$.

Lemma 4.8 (a) *Given an object $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$ of $\widetilde{\mathsf{Blp}}$, any diagram $X\colon I \to A - \mathsf{Mod}$ can be lifted to a diagram $I \to (A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R) - \mathsf{Mod}$.*

(b) *Given a diagram $X\colon I \to \mathsf{Mon}_0$ and a sieve I_0 of I, any lift of $X_{|I_0}$ to a diagram $I_0 \to \widetilde{\mathsf{Blp}}$ can be extended to a diagram $I \to \widetilde{\mathsf{Blp}}$.*

Proof (a) Let $X\colon I \to A - \mathsf{Mod}$ be a diagram. For each object i of I, consider the $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$-module $(X_i, \mathbb{N}[X_i] \to M_i^0)$, where M_i^0 is the quotient of $\mathbb{N}[X_i]$ by the equivalence relation generated by $am = bm$, for each $m \in \mathbb{N}[X_i]$ and for each pair (a, b) in the relation defining the quotient R.

By induction, the $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$-module $(X_i, \mathbb{N}[X_i] \to M_i^{\alpha+1})$ is defined by setting $M_i^{\alpha+1}$ to be the quotient of $\mathbb{N}[X_i]$ by the equivalence relation generated by the equations defining M_i^{α} and by the equations $\mathbb{N}[f]m = \mathbb{N}[f]n$, where $f\colon X_j \to X_i$ is any map in the diagram and where $m = n$ w.r.t. the relation defining M_j^{α}. When α is a limit ordinal, M_i^{α} is defined as the obvious colimit $\varinjlim_{\beta<\alpha} M_i^{\beta}$. Finally, let $M_i = \varinjlim_{\alpha} M_i^{\alpha}$. It is clear that the diagram X can be lifted in a unique way to a diagram $(X_i, \mathbb{N}[X_i] \to M_i)$.

(b) The proof is analogous to that of point (a). □

Remark 4.9 A particular case of Lemma 4.8(b) is the following. Given an object $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$ of $\widetilde{\mathsf{Blp}}$, any diagram $A \overset{f}{\underset{g}{\rightrightarrows}} B$ can be lifted (w.r.t. ρ) to a diagram $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R) \rightrightarrows (B, B \otimes_{\mathbb{F}_1} \mathbb{N} \to S)$. △

Remark 4.10 Should one admit the existence of the zero monoid and of the zero ring (i.e. the possibility that $0 = 1$), in the proof of Lemma 4.8 it would be enough to set $M_i = 0$ and $S = 0$, respectively △

Proposition 4.11 *The functor $\rho\colon \mathsf{Aff}_{\mathsf{Set}_*} \to \mathsf{Aff}_{\widetilde{\mathsf{B}}}$ preserves Zariski covers.*

Proof Let

$$\left\{\mathrm{Spec}\left(A_i, A_i \otimes_{\mathbb{F}_1} \mathbb{N} \to R_i\right) \to \mathrm{Spec}(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)\right\}_{i \in I}$$

be any Zariski cover in the category $\mathsf{Aff}_{\widetilde{\mathsf{B}}}$. We have to prove that $\{\mathrm{Spec}\, A_i \to \mathrm{Spec}\, A\}_{i \in I}$ is a Zariski cover in $\mathsf{Aff}_{\mathsf{Set}_*}$. To do that, by taking into account [27, Déf. 2.10], we have to check the following four points:

(1) To show that, for each i, $\operatorname{Spec} A_i \to \operatorname{Spec} A$ is flat, that is that

$$\text{-} \otimes_A A_i \colon A - \mathsf{Mod} \to A_i - \mathsf{Mod}$$

is exact. By applying Lemma 4.8(a) to any finite diagram, this follows from the flatness of the morphism $\operatorname{Spec}(A_i, A_i \otimes_{\mathbb{F}_1} \mathbb{N} \to R_i) \to \operatorname{Spec}(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$ and from the fact that ρ preserves limits, being a right adjoint.

(2) To show that there is a finite subset $J \subset I$ such that

$$\prod_{j \in J} \text{-} \otimes_A A_j \colon A - \mathsf{Mod} \to \prod_{j \in J} A_j - \mathsf{Mod}$$

is conservative. This follows from Lemma 4.8(a) in the case where I is the category $\bullet \to \bullet$.

(3) To show that ρ preserves epimorphisms. This is consequence of Lemma 4.8(b) (see Remark 4.9).

(4) To show that ρ preserves the finite presentation property. This fact follows from Lemma 4.8(b).

□

Proposition 4.12 *The functor* $\rho \colon \mathsf{Aff}_{\mathsf{Set}_*} \to \mathsf{Aff}_{\widetilde{\mathsf{B}}}$ *preserves finite limits.*

Proof We will show the equivalent statement that the opposite functor from $\rho \colon \widetilde{\mathsf{B}} \to \mathsf{Set}_*$ preserves finite colimits. As usual, it is enough to show that it preserves finite coproducts and coequalizers.

Let $(A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R)$ and $(B, B \otimes_{\mathbb{F}_1} \mathbb{N} \to S)$ be objects in $\widetilde{\mathsf{B}}$ and take the coproduct $(A \coprod B, (A \coprod B) \otimes_{\mathbb{F}_1} \mathbb{N} \to R \oplus S)$ in the category $\text{-} \otimes_{\mathbb{F}_1} \mathbb{N}/\mathsf{SRing}$: we have to show that the second component is surjective. This follows from the fact that, being $\text{-} \otimes_{\mathbb{F}_1} \mathbb{N}$ a left adjoint, one has $(A \coprod B) \otimes_{\mathbb{F}_1} \mathbb{N} \cong (A \otimes_{\mathbb{F}_1} \mathbb{N}) \oplus (B \otimes_{\mathbb{F}_1} \mathbb{N})$.

Let $f, g \colon (A, A \otimes_{\mathbb{F}_1} \mathbb{N} \to R) \to (B, B \otimes_{\mathbb{F}_1} \mathbb{N} \to S)$. Analogously as above, the domain of the second component of the coequalizer C of f, g in $\text{-} \otimes_{\mathbb{F}_1} \mathbb{N}/\mathsf{SRing}$ is the coequalizer of

$$f \otimes_{\mathbb{F}_1} \mathbb{N}, g \otimes_{\mathbb{F}_1} \mathbb{N} \colon A \otimes_{\mathbb{F}_1} \mathbb{N} \to B \otimes_{\mathbb{F}_1} \mathbb{N}.$$

Because of the universal property of colimits, there is a commutative diagram giving rise to a commutative diagram

$$\begin{tikzcd}
A \otimes_{\mathbb{F}_1} \mathbb{N} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & B \otimes_{\mathbb{F}_1} \mathbb{N} \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d] \\
R \arrow[r, shift left] \arrow[r, shift right] & S \arrow[r, two heads] & T
\end{tikzcd}$$

in SRing, whose rows are coequalizers and where the map $C \to T$ is the second component of the coequalizer of f, g in the category $\text{-} \otimes_{\mathbb{F}_1} \mathbb{N}/\mathsf{SRing}$. As the middle vertical map and the bottom right one are surjective, so is the map $C \to T$. □

Proposition 4.11 and Proposition 4.12 entail the following result.

Corollary 4.13 *The functor $\rho\colon \mathit{Aff}_{\mathsf{Set}_*} \to \mathit{Aff}_{\widetilde{\mathsf{B}}}$ is continuous w.r.t. the Zariski topology, and the adjunction 4.12 gives rise to a geometric morphism*

$$\begin{tikzcd} \mathit{Sh}(\mathit{Aff}_{\widetilde{\mathsf{B}}}) \arrow[r, bend right, "\widehat{\rho}"'] & \mathit{Sh}(\mathit{Aff}_{\mathsf{Set}_*}) \arrow[l, bend right, "\widehat{\sigma}"'] \end{tikzcd} \tag{4.13}$$

Theorem 4.14 *The functor $\widehat{\rho}\colon \mathit{Sh}(\mathit{Aff}_{\widetilde{\mathsf{B}}}) \to \mathit{Sh}(\mathit{Aff}_{\mathsf{Set}_*})$ preserves the subcategories of schemes and so induces a functor*

$$\widehat{\rho}\colon \mathit{Sch}_{\widetilde{\mathsf{B}}} \to \mathit{Sch}_{\mathsf{Set}_*}\,. \tag{4.14}$$

Hence, the adjunction 4.13 induces an adjunction $\widehat{\rho} \dashv \widehat{\sigma}\colon \mathit{Sch}_{\widetilde{\mathsf{B}}} \to \mathit{Sch}_{\mathsf{Set}_*}$.

Proof We already proved that $\widehat{\sigma}$ preserves the relevant subcategory of schemes in Proposition 4.5. So all we have to prove is that $\widehat{\rho}$ preserves the relevant subcategory of schemes. In view of [27, Proposition 2.18], it suffices to observe that the following properties of $\widehat{\rho}$ are satisfied:

- it preserves coproducts (for it is a left adjoint), and affine schemes;
- it preserves finite limits (by Proposition 4.12) and Zariski opens of affine schemes (by Lemma 4.8(b) and by the fact that $\widehat{\rho}$ preserves finite limits);
- it preserves images (since it preserves finite limits and colimits) and diagonal morphisms;
- it preserves quotients, since it preserves colimits.

□

Definition 4.15 A scheme Σ in $\mathsf{Sch}_{\widetilde{\mathsf{B}}}$ that admits a Zariski cover by affine schemes in $\mathsf{Aff}_{\mathsf{B}}$ will be called (by a slight abuse of language) a $\widetilde{\mathsf{B}}$-scheme. The category of such schemes will be denoted by $\widetilde{\mathsf{Sch}_{\widetilde{\mathsf{B}}}}$.

The rationale behind this definition is that, while $\widetilde{\mathsf{B}}$-schemes retain all good local properties of B-schemes (namely, the properties of blueprints), one gains the advantages of working in the wider and more comfortable environment of the category $\mathsf{Sch}_{\widetilde{\mathsf{B}}}$.

Notice that the adjunction in Theorem 4.14 obviously restrict to an adjunction

$$\widehat{\rho} \dashv \widehat{\sigma}\colon \widetilde{\mathsf{Sch}_{\widetilde{\mathsf{B}}}} \to \mathsf{Sch}_{\mathsf{Set}_*}\,. \tag{4.15}$$

Morevover, one can define a functor

$$\mathsf{Sch}_{\mathsf{B}} \xrightarrow{\widehat{F_{\mathbb{Z}}}} \mathsf{Sch}_{\mathsf{Ab}}\;, \tag{4.16}$$

obtained by composing the functor $\widehat{F}\colon \mathsf{Sch}_{\mathsf{B}} \to \mathsf{Sch}_{\mathsf{Mon}_0}$ in eq. 4.9 with the functor

$$\text{-} \otimes_{\mathbb{N}} \mathbb{Z}\colon \mathsf{Sch}_{\mathsf{Mon}_0} \to \mathsf{Sch}_{\mathsf{Ab}}$$

defined in [27, Prop. 3.4]. Of course, this functor restricts to a functor

$$\widetilde{\mathsf{Sch}_{\mathsf{B}}} \xrightarrow{\widehat{F}_{\mathbb{Z}}} \mathsf{Sch}_{\mathsf{Ab}} \;. \tag{4.17}$$

A $\widetilde{\mathsf{B}}$-scheme gives rise, through the functors $\widehat{\rho}$ and $\widehat{F}_{\mathbb{Z}}$, to a pair consisting of a monoidal scheme and a classical scheme.

Definition 4.16 Given a $\widetilde{\mathsf{B}}$-scheme Σ, we set

- $\Sigma_{\mathbb{Z}} := \widehat{F}_{\mathbb{Z}}(\Sigma)$, which is an object of $\mathsf{Sch}_{\mathsf{Ab}}$ (i.e. a classical scheme);
- $\underline{\Sigma} := \widehat{\rho}(\Sigma)$, which is an object of $\mathsf{Sch}_{\mathsf{Set}_*}$ (i.e. a monoidal scheme).

There is a natural transformation $\Sigma_{\mathbb{Z}} \to \underline{\Sigma} \otimes_{\mathbb{F}_1} \mathbb{Z}$, which is obtained via the unit of the adjunction $\widehat{\rho} \dashv \widehat{\sigma}$ and by applying the functor $\widehat{F}_{\mathbb{Z}}$. By definition, there is indeed a map

$$\widehat{F}_{\mathbb{Z}}\Sigma \to \widehat{F}_{\mathbb{Z}}\widehat{\sigma}\,\widehat{\rho}\,\Sigma \cong \underline{\Sigma} \otimes_{\mathbb{F}_1} \mathbb{Z}\,,$$

where the isomorphism is given by the natural isomorphism $\widehat{F}_{\mathbb{Z}} \circ \widehat{\sigma} = \text{-} \otimes_{\mathbb{F}_1} \mathbb{Z}$.

In the affine case, such a map is simply realized as the bottom arrow of the map between arrows

$$\begin{tikzcd} A \otimes_{\mathbb{F}_1} \mathbb{Z} \arrow[r] \arrow[d] & A \otimes_{\mathbb{F}_1} \mathbb{Z} \arrow[d] \\ A \otimes_{\mathbb{F}_1} \mathbb{Z} \arrow[r] & R \otimes_{\mathbb{N}} \mathbb{Z} \end{tikzcd}$$

where the top and the left map are identities.

Summing up, a $\widetilde{\mathsf{B}}$-scheme Σ induces therefore the following objects:

- a monoidal scheme $\underline{\Sigma}$; (4.18)
- a (classical) scheme $\Sigma_{\mathbb{Z}}$ over $\mathbb{Z}$; (4.19)
- a natural transformation $\Lambda\colon \Sigma_{\mathbb{Z}} \to \underline{\Sigma} \circ |\text{-}| \cong \underline{\Sigma} \otimes_{\mathbb{F}_1} \mathbb{Z}\,.$ (4.20)

We shall say that the $\widetilde{\mathsf{B}}$-scheme Σ *generates* the pair $(\underline{\Sigma}, \Sigma_{\mathbb{Z}})$, the natural transformation 4.20 being omitted.

5 An application: $\widetilde{\mathsf{B}}$-schemes and $\mathbb{F}_1$-schemes

The geometric data 4.18, 4.19, 4.20 appear to be similar to (but different from) those used by A. Connes and C. Consani [1] in their definition of $\mathbb{F}_1$-scheme, which is as follows.

Definition 5.1 [1, Def. 4.7] An $\mathbb{F}_1$-scheme is a triple $(\underline{\Xi}, \Xi_{\mathbb{Z}}, \Phi)$, where

(1) $\underline{\Xi}$ is a monoidal scheme;
(2) $\Xi_{\mathbb{Z}}$ is a (classical) scheme;
(3) Φ is a natural transformation $\underline{\Xi} \to \Xi_{\mathbb{Z}} \circ (\text{-} \otimes_{\mathbb{F}_1} \mathbb{Z})$, such that the induced natural transformation $\underline{\Xi} \circ |\text{-}| \to \Xi_{\mathbb{Z}}$, when evaluated on fields, gives isomorphisms (of sets).[3]

A manifest difference between $\widetilde{\mathsf{B}}$-schemes and $\mathbb{F}_1$-schemes is, of course, the direction of the natural transformation linking the monoidal scheme and the classical scheme. Moreover, the condition on Φ in Definition 5.1(3) may fail to be fulfilled in the case of $\widetilde{\mathsf{B}}$-schemes, as shown by the following example.

Example 5.2 Consider a pair $(A, R \to A \otimes_{\mathbb{F}_1} \mathbb{Z})$ defining an affine $\mathbb{F}_1$-scheme in the sense Definition 5.1. Notice that, in this case, the natural transformation Φ calculated on a field k corresponds to mapping a prime ideal $\mathfrak{p}$ of $A \otimes_{\mathbb{F}_1} \mathbb{Z}$ plus an immersion $A \otimes_{\mathbb{F}_1} \mathbb{Z}/\mathfrak{p} \hookrightarrow k$ to their restrictions to R; the requirement is that this is a bijection.

On the other hand, according to the general idea underlying the notion of blueprint, if the pair (A, R) is associated with an affine B-scheme (which is, of course, the same thing as an affine $\widetilde{\mathsf{B}}$-scheme), then the ring R encodes the information of a relation $\mathcal{R}$ intended to *reduce* the number of ideals of A. Take for instance the case $(A, A \otimes_{\mathbb{F}_1} \mathbb{Z} \to R)$, with $A = \mathbb{N} \cup \{-\infty\}$ (additive notation) and $R = A \otimes_{\mathbb{F}_1} \mathbb{Z}/(2T-1)$. Then, $\mathbb{N}$ is an ideal not coming from any ideal of R, since T is invertible (in more algebraic terms, we are saying that the map to any field k sending T to 0 can not be lifted to a map from R to k). $\triangle$

The category $\widetilde{\mathsf{Sch}}_{\widetilde{\mathsf{B}}}$ and that of $\mathbb{F}_1$-schemes may be combined into a larger category.

Definition 5.3 The category of $\mathbb{F}_1$-schemes with relations is the fibered product of the category $\widetilde{\mathsf{Sch}}_{\widetilde{\mathsf{B}}}$ of $\widetilde{\mathsf{B}}$-schemes and that of $\mathbb{F}_1$-schemes over the category of monoidal schemes. Thus, a $\widetilde{\mathsf{B}}$-scheme Σ generating the pair $(\underline{\Sigma}, \Sigma_{\mathbb{Z}})$ and an $\mathbb{F}_1$-scheme $(\underline{\Sigma}, \Sigma'_{\mathbb{Z}}, \Phi)$ will determine a $\mathbb{F}_1$-scheme with relations denoted by the quadruple $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$.

Remark 5.4 Recall that the aim of Lorscheid's definition of blueprint is to increase the amount of closed subschemes of a monoidal scheme. If we loosely refer to the features of the underlying topological space as "shape" of the scheme, we could say that the category of B-schemes (or that of $\widetilde{\mathsf{B}}$-schemes) adds "extra shapes" to Deitmar's category of monoidal schemes.

Consider now $\mathbb{F}_1$-schemes, and let us restrict our attention to the affine case. So, we just have a ring R, a monoid M, and a map $R \to M \otimes_{\mathbb{F}_1} \mathbb{Z}$. Since it is required, by definition, that points remain the same, the monoid is not enriched with "extra shapes". However, if we think of the given map as a restriction map between the spaces of functions of the affine schemes $M \otimes_{\mathbb{F}_1} \mathbb{Z}$ and R, we can interpret the datum of the $\mathbb{F}_1$-scheme as an enlargement of the space of functions of the affine monoidal scheme M.

[3] In [1] the functor $\text{-} \otimes_{\mathbb{F}_1} \mathbb{Z}$ is denoted by β and its right adjoint $|\text{-}|$ by β^*.

In conclusion, an $\mathbb{F}_1$-scheme with relation, according to the definition 5.3, allows us both to add "extra shapes" to the underlying monoidal scheme and to enlarge its space of functions.

As an example, consider the affine $\mathbb{F}_1$-scheme with relation given by the free monoid on four generators and the data

$$\begin{aligned}\mathbb{Z}[T_1, T_2, T_3, T_4, \varepsilon]/(\varepsilon^2) &\to \mathbb{Z}[T_1, T_2, T_3, T_4] \\ &\to \mathbb{Z}[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3 - 1)\ .\end{aligned}$$

The B-scheme component on the right has been already taken into consideration in the Introduction; the $\mathbb{F}_1$-scheme component on the left adds a nilpotent component to the ring of functions. △

Notice that the classical scheme $\Sigma_{\mathbb{Z}}$ is derived from the $\widetilde{\mathsf{B}}$-scheme Σ via the functor $\widehat{F_{\mathbb{Z}}}\colon \mathsf{Blp} \to \mathsf{Ring}$ (Definition 4.16). This means, in particular, that the affine B-scheme $\Sigma = (M, M \otimes_{\mathbb{F}_1} N \to R)$ generates the affine classical scheme $\Sigma_{\mathbb{Z}} = R \otimes_{\mathbb{N}} \mathbb{Z}$. So Definition 5.3 indicates that, as long as we wish to investigate a relationship between this affine B-scheme with an $\mathbb{F}_1$-scheme and its associated affine classical scheme $\Sigma'_{\mathbb{Z}}$, we are no longer concerned with the "monoid relations" given the map $M \otimes_{\mathbb{F}_1} \mathbb{N} \to R$, but only with the "ring relations" given by the map $M \otimes_{\mathbb{F}_1} \mathbb{Z} \to R \otimes_{\mathbb{N}} \mathbb{Z}$ (cf. eq. 4.20).

From this viewpoint it appears more natural to work with blueprints with "ring relations". More precisely, consider the functor

$$\text{-} \otimes_{\mathbb{F}_1} \mathbb{Z} : \mathsf{Mon}_0 \to \mathsf{Ring}$$

which is the left adjoint to the forgetful functor, and consider the category $(\text{-} \otimes_{\mathbb{F}_1} \mathbb{Z})/\mathsf{Ring}$. We shall denote by $\mathbb{Z}\text{-}\mathsf{Blp}$ the full subcategory of $(\text{-} \otimes_{\mathbb{F}_1} \mathbb{Z})/\mathsf{Ring}$ formally defined in the same way as the subcategory blueprints Blp of $\text{-} \otimes_{\mathbb{F}_1} \mathbb{N}/\mathsf{SRing}$. Analogously, one defines the category $\mathbb{Z}\text{-}\widetilde{\mathsf{Blp}}$. A $\mathbb{Z}\text{-}\widetilde{\mathsf{B}}$-scheme is then a *scheme in* $\mathsf{Sch}_{\mathbb{Z}\text{-}\widetilde{\mathsf{Blp}}}$ *that admits a Zariski cover by affine schemes in* $(\mathbb{Z}\text{-}\mathsf{Blp})^{\mathrm{op}}$. We shall adopt hereafter the following terminological convention.

Convention 5.5 *In what follows, by $\widetilde{\mathsf{B}}$-scheme we mean a $\mathbb{Z}\text{-}\widetilde{\mathsf{B}}$-scheme, and by $\mathbb{F}_1$-scheme with relations we mean the combination of a $\mathbb{Z}\text{-}\widetilde{\mathsf{B}}$-scheme and an $\mathbb{F}_1$-scheme in the sense of Definition 5.3.*

Now, Definition 4.16 and Definition 5.1 imply that, for every $\mathbb{F}_1$-scheme with relations $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$, there is a natural transformation $\Psi_1\colon \Sigma_{\mathbb{Z}} \to \Sigma'_{\mathbb{Z}}$ given by the composition

$$\Sigma_{\mathbb{Z}} \xrightarrow{\Lambda} \underline{\Sigma} \circ |\text{-}| \xrightarrow{\Phi} \Sigma'_{\mathbb{Z}}\ , \tag{5.1}$$

which will be called the *first transferring map* determined by the given $\mathbb{F}_1$-scheme with relations. As its name would suggest, the natural transformation Ψ_1, loosely speaking, conveys information on about how many "points" of $\Sigma'_{\mathbb{Z}}$ are compatible with the $\widetilde{\mathsf{B}}$-scheme that generates the pair $(\underline{\Sigma}, \Sigma_{\mathbb{Z}})$. Actually, there is a different way to "transfer" this information from the $\widetilde{\mathsf{B}}$-scheme to the $\mathbb{F}_1$-scheme associated with the fibered object $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$.

The counit of the adjunction $-\otimes_{\mathbb{F}_1}\mathbb{Z} \dashv |-|$ induces a map

$$\underline{\Sigma}\circ|-| \to \underline{\Sigma}\circ||-|\otimes_{\mathbb{F}_1}\mathbb{Z}|\,. \tag{5.2}$$

Moreover, the natural transformation 4.20 induces a map

$$\Sigma'_{\mathbb{Z}}\circ\left(|-|\otimes_{\mathbb{F}_1}\mathbb{Z}\right) \to \underline{\Sigma}\circ||-|\otimes_{\mathbb{F}_1}\mathbb{Z}|\,. \tag{5.3}$$

Let Σ'_{B} be the sheaf on the category Ring obtained as the pullback of the maps 5.2 and 5.3, i.e.

$$\begin{tikzcd} \Sigma'_{\mathsf{B}} \arrow[r] \arrow[d] & \Sigma'_{\mathbb{Z}}\circ(|-|\otimes_{\mathbb{F}_1}\mathbb{Z}) \arrow[d] \\ \underline{\Sigma}\circ|-| \arrow[r] & \underline{\Sigma}\circ||-|\otimes_{\mathbb{F}_1}\mathbb{Z}| \end{tikzcd} \tag{5.4}$$

By composing the vertical arrow on the left with Φ, we get a natural transformation

$$\Psi_2\colon \Sigma'_{\mathsf{B}} \to \Sigma'_{\mathbb{Z}}\,, \tag{5.5}$$

which will be called the *second transferring map* determined by the $\mathbb{F}_1$-scheme $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$.

In the case of an $\mathbb{F}_1$-scheme $(\underline{\Sigma}, \Sigma'_{\mathbb{Z}}, \Phi)$, the natural transformation Φ induces an isomorphism $\underline{\Sigma}(|\mathbb{K}|) \simeq \Sigma'_{\mathbb{Z}}(\mathbb{K})$ for every field $\mathbb{K}$. Since for the finite field $\mathbb{F}_q$, one has $|\mathbb{F}_q| = \mathbb{F}_{1^{q-1}}$, it immediately follows, as observed in [1], that there is a bijective correspondence between the set of $\mathbb{F}_q$-points of $\Sigma'_{\mathbb{Z}}$ and the set of $\mathbb{F}_{1^{q-1}}$-points of $\underline{\Sigma}$; in others words, one has

$$\#\Sigma'_{\mathbb{Z}}(\mathbb{F}_q) = \#\underline{\Sigma}\left(\mathbb{F}_{1^{q-1}}\right)\,. \tag{5.6}$$

This result can be extended to our setting in two different ways, because, for a $\widetilde{\mathsf{B}}$-scheme underlying an $\mathbb{F}_1$-scheme with relations, we can think of its "$\mathbb{F}_{1^{q-1}}$-points" in two different senses.

On the one hand, the forgetful functor $|-| : \mathsf{Ring} \to \mathsf{Mon}_0$ admits the obvious factorization

$$\mathsf{Ring} \xrightarrow{G_{\mathbb{Z}}} \mathbb{Z}\text{-}\mathsf{Blp} \xrightarrow{\rho} \mathsf{Mon}_0\ , \tag{5.7}$$

(cf. eq. 4.5). Clearly, one has

$$G_{\mathbb{Z}}(\mathbb{F}_q) = \left(\mathbb{F}_{1^{q-1}}, \mathbb{F}_{1^{q-1}}\otimes_{\mathbb{F}_1}\mathbb{Z} \to \mathbb{F}_{1^{q-1}}\right)$$

and $\rho(G_{\mathbb{Z}}(\mathbb{F}_q)) = |\mathbb{F}_q| = \mathbb{F}_{1^{q-1}}$. Now, by definition, the first transferring map Ψ_1 factorises as $\Psi_1 = \Phi\circ\Lambda$. Since Φ gives isomorphisms (of sets) when evaluated on fields and Λ is always locally injective, it is immediate to prove the following result.

Proposition 5.6 *Let $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$ be an $\mathbb{F}_1$-scheme with relations. The first transferring map $\Psi_1\colon \Sigma_{\mathbb{Z}} \to \Sigma'_{\mathbb{Z}}$, when evaluated on a field, gives an injective map (of sets). In particular, the set of $G_{\mathbb{Z}}(\mathbb{F}_q)$-points of the underlying $\widetilde{\mathsf{B}}$-scheme naturally* *injects into the set of* $\mathbb{F}_q$*-points of the scheme* $\Sigma''_{\mathbb{Z}}$ *(which is isomorphic to the set of* $\mathbb{F}_{1^{q-1}}$*-points of the monoidal scheme* $\underline{\Sigma}$*).*

On the other hand, one has the immersion $\sigma : \mathsf{Mon}_0 \hookrightarrow \mathbb{Z}\text{-}\mathsf{Blp}$, with

$$\sigma\left(\mathbb{F}_{1^{q-1}}\right) = \left(\mathbb{F}_{1^{q-1}}, \mathbb{F}_{1^{q-1}} \otimes_{\mathbb{F}_1} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{F}_{1^{q-1}} \otimes_{\mathbb{F}_1} \mathbb{Z}\right).$$

Notice that $G_{\mathbb{Z}}(\mathbb{F}_q) \neq \sigma(\mathbb{F}_{1^{q-1}})$, while $|G_{\mathbb{Z}}(\mathbb{F}_q)| = |\sigma(\mathbb{F}_{1^{q-1}})| = \mathbb{F}_{1^{q-1}}$.

Theorem 5.7 *Let* $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$ *be an* $\mathbb{F}_1$*-scheme with relations. The set of* $\sigma(\mathbb{F}_{1^{q-1}})$*-points of the underlying* $\widetilde{B}$*-scheme is in natural bijection with the set of* $\mathbb{F}_q$*-points of the subpresheaf of* $\Sigma'_{\mathbb{Z}}$ *given by the image of* $\Psi_2 : \Sigma'_B \to \Sigma'_{\mathbb{Z}}$*.*

Proof Since we can work locally, we assume that the underlying scheme is given by a monoid M, a ring R, and a map $M \otimes_{\mathbb{F}_1} \mathbb{Z} \to R$ satisfying the usual conditions. An $\mathbb{F}_{1^{q-1}}$-point is given by a commutative square

$$\begin{array}{ccc} M \otimes_{\mathbb{F}_1} \mathbb{Z} & \longrightarrow & \mathbb{F}_{1^{q-1}} \otimes_{\mathbb{F}_1} \mathbb{Z} \cong |\mathbb{F}_q| \otimes_{\mathbb{F}_1} \mathbb{Z} \\ \downarrow & & \downarrow \text{id} \\ R & \longrightarrow & \mathbb{F}_{1^{q-1}} \otimes_{\mathbb{F}_1} \mathbb{Z} \cong |\mathbb{F}_q| \otimes_{\mathbb{F}_1} \mathbb{Z} \end{array}$$

such that the arrow on the top is induced by a map $M \to \mathbb{F}_{1^{q-1}}$.
The datum of a generic commutative square as above is equivalent to the datum of an $\mathbb{F}_q$-point in $\operatorname{Spec} R \circ (|\text{-}| \otimes_{\mathbb{F}_1} \mathbb{Z})$.
The fact that the map on the top has the required property is equivalent to the fact that the image of the point above through the restriction map

$$\operatorname{Spec} R\left(|\mathbb{F}_q| \otimes_{\mathbb{F}_1} \mathbb{Z}\right) \to \operatorname{Spec}(M \otimes_{\mathbb{F}_1} \mathbb{Z})\left(|\mathbb{F}_q| \otimes_{\mathbb{F}_1} \mathbb{Z}\right)$$

is in the image of the map

$$\operatorname{Spec} M\left(|\mathbb{F}_q|\right) \to \operatorname{Spec}(M \otimes_{\mathbb{F}_1} \mathbb{Z})\left(|\mathbb{F}_q| \otimes_{\mathbb{F}_1} \mathbb{Z}\right)$$

induced by the functor $\text{-} \otimes_{\mathbb{F}_1} \mathbb{Z}$. □

We are now interested in the case where the $\mathbb{F}_{1^n}$-points of the underlying monoidal scheme $\underline{\Sigma}$ are counted by a polynomial in n. Some preliminary definitions and results are in order.

A monoidal scheme Σ is said to be *noetherian* if it admits a finite open cover by representable subfunctors $\{\operatorname{Spec}(A_i)\}$, with each A_i a noetherian monoid. Recall that, as it is proved in [10, Theorem 5.10 and 7.8], a monoid is noetherian if and only if it is finitely generated. This immediately implies that, for any prime ideal $\mathfrak{p} \subset M$, the localized monoid $M_{\mathfrak{p}}$ is noetherian and the abelian group $M_{\mathfrak{p}}^{\times}$ of invertible elements in $M_{\mathfrak{p}}$ is finitely generated.

Remark 5.8 Notice that, given an $\mathbb{F}_1$-scheme $(\underline{\Sigma}, \Sigma'_{\mathbb{Z}}, \Phi)$, the fact that the monoidal scheme $\underline{\Sigma}$ is noetherian does not entail that the scheme $\Sigma'_{\mathbb{Z}}$ is noetherian as well. Let us consider, for instance, the affine $\mathbb{F}_1$-scheme given by $\mathbb{Z}[X, \varepsilon_i]/(\varepsilon_i^2) \to \mathbb{Z}[X]$, with $i \in \mathbb{N}$. The monoidal scheme is noetherian, while the ascending chain of ideals $\ldots \subset (\varepsilon_0, \ldots, \varepsilon_i) \subset (\varepsilon_0, \ldots, \varepsilon_{i+1}) \subset \ldots$ does not have a maximal element. Observe that, as for the points of the classical scheme, the presence of the ε_i's is immaterial; hence, one has the required isomorphism $\mathbb{Z}[X](|\mathbb{K}|) \simeq \mathbb{Z}[X, \varepsilon_i]/(\varepsilon_i^2)(\mathbb{K})$ for any field $\mathbb{K}$. △

Let $\widetilde{\underline{\Sigma}}$ the geometrical realization of the monoidal scheme Σ. Following Connes-Consani's definition [1, p. 25], we shall say that $\underline{\Sigma}$ is *torsion-free* if, for any $x \in \widetilde{\underline{\Sigma}}$, the abelian group $\mathcal{O}^{\times}_{\underline{\Sigma},x}$ is torsion-free.

Lemma 5.9 *A noetherian monoidal scheme $\underline{\Sigma}$ is torsion-free if and only if, for any finite group G with $\#G = n$ and for any point $x \in \widetilde{\underline{\Sigma}}$, the number $\#\operatorname{Hom}(\mathcal{O}^{\times}_{\underline{\Sigma},x}, G)$ is polynomial in n.*

Proof Since $\underline{\Sigma}$ is noetherian, the abelian group $\mathcal{O}^{\times}_{\underline{\Sigma},x}$ is finitely generated by the remark above. So, if $\underline{\Sigma}$ is also torsion-free, then $\mathcal{O}^{\times}_{\underline{\Sigma},x}$ is free of rank $N(x)$, and, for any finite group G with $\#G = n$, we have $\#\operatorname{Hom}(\mathcal{O}^{\times}_{\underline{\Sigma},x}, G) = n^{N(x)}$.

For the converse, suppose there is a point x such that $\mathcal{O}^{\times}_{\underline{\Sigma},x}$ is not torsion-free. Being noetherian, $\mathcal{O}^{\times}_{\underline{\Sigma},x}$ decomposes as a product $\mathbb{Z}^n \times \prod_{i\in\{1,\ldots m\}} \mathbb{Z}_{n_i}$. For each prime number p_0 not dividing any of the $n_1, \ldots, n_m$, say $p_0 > \operatorname{LCM}(n_1, \ldots, n_m)$, the number of elements of $\operatorname{Hom}(\mathcal{O}^{\times}_{\underline{\Sigma},x}, \mathbb{Z}_{p_0})$ is then p_0^n. Since there are infinitely many such prime numbers, were $\#\operatorname{Hom}(\mathcal{O}^{\times}_{\underline{\Sigma},x}, \mathbb{Z}_p)$ a polynomial in p, it would be the polynomial p^n. Take now a prime number p_1 dividing n_1; in that case, the number of elements of $\operatorname{Hom}(\mathcal{O}^{\times}_{\underline{\Sigma},x}, \mathbb{Z}_{p_1})$ is greater than p_1^n. In conclusion, $\#\operatorname{Hom}(\mathcal{O}^{\times}_{\underline{\Sigma},x}, \mathbb{Z}_p)$ cannot be a polynomial in p. □

By Lemma 5.9, for each noetherian and torsion-free monoidal scheme $\underline{\Sigma}$, one can define the polynomial

$$P\left(\underline{\Sigma}, n\right) = \sum_{x\in\widetilde{\underline{\Sigma}}} \#\operatorname{Hom}\left(\mathcal{O}^{\times}_{\underline{\Sigma},x}, \mathbb{F}_{1^n}\right). \tag{5.8}$$

The following result is proved in [1] (Theorem 4.10, (1) and (2)).

Theorem 5.10 *Let $(\underline{\Sigma}, \Sigma'_{\mathbb{Z}}, \Phi)$ be an $\mathbb{F}_1$-scheme such that the monoidal scheme $\underline{\Sigma}$ is noetherian and torsion-free. Then*

(1) $\#\underline{\Sigma}(\mathbb{F}_{1^n}) = P(\underline{\Sigma}, n)$;
(2) *for each finite field $\mathbb{F}_q$ the cardinality of the set of points of the scheme $\Sigma'_{\mathbb{Z}}$ that are rational over $\mathbb{F}_q$ is equal to $P(\underline{\Sigma}, q-1)$.*

Note that the last statement immediately follows from eq. 5.6, which holds true without any additional assumption on the monoidal scheme.

For each $\widetilde{\mathsf{B}}$-scheme Σ and each abelian group G (in multiplicative notation, with absorbing element 0), we denote by

$$\mathrm{Hom}_{\mathsf{B}}\left(\mathcal{O}_{\Sigma,x}^{\times}, G\right)$$

the subset of $\mathrm{Hom}(\mathcal{O}_{\underline{\Sigma},x}^{\times}, G)$ given by the morphisms satisfying the relations encoded in the blueprint structure of Σ. Lemma 5.9 prompts us to introduce the following definition.

Definition 5.11 A $\widetilde{\mathsf{B}}$-scheme Σ is said to be noetherian if the monoidal scheme $\underline{\Sigma}$ is noetherian. A noetherian $\widetilde{\mathsf{B}}$-scheme Σ is said to be torsion-free if for any finite group G, the number $\#\,\mathrm{Hom}_{\mathsf{B}}(\mathcal{O}_{\Sigma,x}^{\times}, G)$ is polynomial in $\#G$.

Remark 5.12 While in the case of a noetherian torsion-free monoidal scheme $\underline{\Sigma}$ the polynomial $\#\,\mathrm{Hom}_{\mathsf{B}}(\mathcal{O}_{\underline{\Sigma},x}^{\times}, G)$ is always a monic monomial, this is not always the case for a noetherian torsion-free $\widetilde{\mathsf{B}}$-scheme. The next example illustrates this point. △

Example 5.13 Consider the affine B-scheme Σ given by the free monoid $M = \langle T_1, T_2, T_3, T_4\rangle$ generated by four elements with relations given by the natural projection

$$\mathbb{Z}\,[T_1, T_2, T_3, T_4] \to \mathbb{Z}\,[T_1, T_2, T_3, T_4]\,/\,(T_1 - T_3 + T_2 - T_4)\ .$$

Let G be a finite group (in multiplicative notation, with absorbing element 0); we look for maps $f\colon M \to G$ together with compatible maps

$$\begin{array}{ccc} \mathbb{Z}[T_1, T_2, T_3, T_4] & \longrightarrow & G \otimes_{\mathbb{F}_1} \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Z}[T_1, T_2, T_3, T_4]/(T_1 - T_3 + T_2 - T_4) & \longrightarrow & G \otimes_{\mathbb{F}_1} \mathbb{Z} \end{array}$$

Since $G \otimes_{\mathbb{F}_1} \mathbb{Z}$ is free, to ensure the compatibility of f with the relation $T_1 + T_2 = T_3 + T_4$ one must have that either $f(T_1) = f(T_3)$ and $f(T_2) = f(T_4)$ or $f(T_1) = f(T_4)$ and $f(T_2) = f(T_3)$. There are therefore only 3 possible cases for the polynomial expressing the cardinality of $\mathrm{Hom}_{\mathsf{B}}(\mathcal{O}_{\Sigma,x}^{\times}, G)$:

- $f(T_1) = f(T_2) = f(T_3) = f(T_4) = 0$; in this case the polynomial is the constant polynomial 1;
- either $f(T_1) = 0$ and $f(T_2) \neq 0$ or $f(T_1) \neq 0$ and $f(T_2) = 0$, each case giving rise to two possible cases; therefore, in each of the four possible cases the polynomial is n;
- $f(T_1) \neq 0$ and $f(T_2) \neq 0$; in this case the polynomial is $2n^2 - n$ (the term $2n^2$ accounts for 2 possible free nonzero choices on $f(T_1)$ and $f(T_2)$, that have to be counted twice since either $f(T_1) = f(T_3)$ or $f(T_1) = f(T_4)$, and the term $-n$ accounts for the case $f(T_1) = f(T_2)$).

△

Let $(\underline{\Sigma}, \Sigma_{\mathbb{Z}}, \Sigma'_{\mathbb{Z}}, \Phi)$ be an $\mathbb{F}_1$-scheme with relations such that the underlying $\widetilde{\mathsf{B}}$-scheme Σ is noetherian and torsion-free. We define the polynomial

$$Q\left(\underline{\Sigma}, n\right) = \sum_{x \in \underline{\Sigma}} \# \operatorname{Hom}_{\mathsf{B}}\left(\mathcal{O}^{\times}_{\Sigma,x}, \mathbb{F}_{1^n}\right) .$$

Proposition 5.14 *In the above hypotheses one has the inequality* $Q(\underline{\Sigma}, n) \leq P(\underline{\Sigma}, n)$.

Proof It is clear that $\operatorname{Hom}_{\mathsf{B}}(\mathcal{O}^{\times}_{\Sigma,x}, \mathbb{F}_{1^n}) \subset \operatorname{Hom}(\mathcal{O}^{\times}_{\underline{\Sigma},x}, \mathbb{F}_{1^n})$, since the first set contains only the monoid morphisms that are compatible with the blueprint structure locally defined around x.

Remark 5.15 Connes and Consani developed, in their papers [2] and [3], an approach based on Γ-sets, that generalizes their previous theory of $\mathbb{F}_1$-schemes. Since the category of Γ-sets is endowed with a natural monoidal closed structure, one can apply to that framework the general formalism introduced by Toën and Vaquié in [27]. In [3], a notion of scheme is defined; this notion is compared to that arising from [27], and the two are shown to be different. The situation is thus analogous to that occurring in the case of blue schemes, as described in [18]. Therefore, it seems worth briefly commenting upon Connes-Consani's construction.

Recall that a Γ-set is a functor

$$M(\text{-})\colon \Gamma^{\mathrm{op}} \to \mathsf{Set}$$

from the category Γ^{op} of pointed finite sets (denoted $0^+, 1^+, 2^+, \ldots$) to Set. Such a notion was first introduced by Segal [23], who used spaces instead of sets, in order to model commutative monoid and group structures up to homotopy. To this aim, he restricts to the case of "special Γ-spaces", that is, those functors such that the natural map

$$M\left(n^+\right) \to M\left(1^+\right) \times \ldots \times M\left(1^+\right)$$

is an equivalence. This way, the image of 1^+ corresponds to the object $M := M(1^+)$ one is interested in, the map $2^+ \to 1^+$ extending the final map $2 \to 1$ corresponds to an operation (say, additive)

$$M \times M \to M\,,$$

and all the other data correspond to associativity and commutativity conditions. On the other hand, by using the monoidal structure of the category of Γ-spaces, one can define a second operation (say, multiplicative) on M.

Connes-Consani's basic idea is that it is possible to obtain more general structures on M by dropping the "special" condition; in particular, one can get a multiplicative monoid structure as above (but without the addition defined in Segal's setting). In order to model such a monoid structure, the only relevant information in the Γ-set has to be the image M of 1^+; technically, this is implemented by asking the image of an object

n^+, which is just the n fold coproduct of 1^+ with itself, to be just the n fold coproduct of M with itself. It is shown in [3] that, by following a procedure of this kind, the categories Mon and Ring embed in the category Γ-sets, as well as does the category $\mathfrak{MR}$ as defined in [1]. In the case of rings (realized as objects in the category Γ-sets), the corresponding schemes, according to Connes-Consani's definition, coincide with the classical ones [3, Prop. 7.9], whilst this is not the case for the schemes obtained by applying Toën-Vaquié's formalism [3, Lemma 8.1].

As a general consideration, we could say that the notion of scheme defined [27] places greater emphasis on the overall category, while that defined in [3] focuses more on the intrinsic geometric properties of each single object. $\triangle$

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