

SOME REMARKS ON FINSLER MANIFOLDS WITH CONSTANT FLAG CURVATURE

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This article is dedicated to Shiing-Shen Chern, whose beautiful works on Finsler geometry have inspired so much progress in the subject.

ABSTRACT. This article is an exposition of four loosely related remarks on the geometry of Finsler manifolds with constant positive flag curvature.

The first remark is that there is a canonical Kähler structure on the space of geodesics of such a manifold.

The second remark is that there is a natural way to construct a (not necessarily complete) Finsler n -manifold of constant positive flag curvature out of a hypersurface in suitably general position in $\mathbb{C}P^n$.

The third remark is that there is a description of the Finsler metrics of constant curvature on S^2 in terms of a Riemannian metric and 1-form on the space of its geodesics. In particular, this allows one to use any (Riemannian) Zoll metric of positive Gauss curvature on S^2 to construct a global Finsler metric of constant positive curvature on S^2 .

The fourth remark concerns the generality of the space of (local) Finsler metrics of constant positive flag curvature in dimension $n+1 > 2$. It is shown that such metrics depend on $n(n+1)$ arbitrary functions of $n+1$ variables and that such metrics naturally correspond to certain torsion-free $S^1 \cdot GL(n, \mathbb{R})$ -structures on $2n$ -manifolds. As a by-product, it is found that these groups do occur as the holonomy of torsion-free affine connections in dimension $2n$, a hitherto unsuspected phenomenon.

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1. INTRODUCTION

The purpose of this article is to explain some new results in the theory of Finsler manifolds with constant flag curvature, particularly constant positive flag curvature.

For general background in the subject, the reader can consult [2, 21, 24] and for articles dealing specifically with the case of constant flag curvature, the reader may consult [1, 18, 20, 25].

1.1. The main results. Though the discussion in this article will hold for a wider notion of Finsler structure than is usually considered, the statements made in this introduction will be focussed on the case of a classical (though not necessarily reversible) Finsler structure on a manifold.

Suppose that M is an $(n+1)$ -manifold endowed with a Finsler structure, regarded as being specified by its unit tangent bundle $\Sigma \subset TM$ (often referred to as the *tangent indicatrix*). Suppose further that M is geodesically simple, i.e., that the quotient Q of Σ by the geodesic flow can be given the structure of a smooth $2n$ -manifold in such a way that the quotient map $q : \Sigma \rightarrow Q$ is a smooth submersion.¹

As is well-known in symplectic geometry, the space Q , which can be thought of as the space of oriented geodesics of the Finsler structure, inherits a canonical symplectic structure.

According to Theorem 1, when the Finsler structure has constant positive flag curvature, Q also inherits a natural Riemannian metric with respect to which the symplectic form is parallel. In other words, Q is naturally a Kähler manifold.

It turns out that Q has a yet finer structure. For each $x \in M$, the set $Q_x \subset Q$ consisting of the geodesics that pass through x is a totally real submanifold of Q . For a fixed geodesic $q \in Q$, the set of manifolds Q_x as $x \in M$ varies on q defines a 1-parameter family of totally real submanifolds of Q passing through q . In the case that the Finsler structure has constant flag curvature 1, the totally real tangent planes $T_q Q_x \subset T_q Q$ as x varies over q turn out to differ by multiplication by complex numbers of the form $e^{i\theta}$, i.e., there is a canonical circle of totally real n -planes passing through each point of Q . This defines a canonical $S^1 \cdot O(n)$ -structure on Q . This $S^1 \cdot O(n)$ -structure is *not* torsion-free except in the trivial case where M is a Riemannian manifold of constant positive sectional curvature.

However, as is shown in §3.4.3, this $S^1 \cdot O(n)$ -structure on Q underlies a canonical $S^1 \cdot GL(n, \mathbb{R})$ -structure that *is* torsion-free. This is surprising, since, for $n > 2$, the group $S^1 \cdot GL(n, \mathbb{R}) \subset GL(2n, \mathbb{R})$ was not previously recognized to be possible as holonomy of a torsion-free connection on a $2n$ -manifold. Nevertheless, as Theorem 4 shows, these groups are indeed realizable as holonomy groups in this way.

In fact, it turns out (§5) that there is a very close connection between torsion-free $S^1 \cdot GL(n, \mathbb{R})$ -structures on $2n$ -manifolds and Finsler structures with constant flag curvature 1. When $n > 2$, a torsion-free $S^1 \cdot GL(n, \mathbb{R})$ -structure on a $2n$ -manifold Q that satisfies a mild positivity condition on its curvature arises from a canonical (generalized) Finsler structure of constant flag curvature 1 on an $(n+1)$ -manifold M . When $n = 2$, one must impose a further condition on the torsion-free

¹ Of course, this can always be arranged locally by restricting attention to a geodesically convex neighborhood in M .

structure, that of *integrability*, but the local generality of the integrable, torsion-free $S^1 \cdot \mathrm{GL}(2, \mathbb{R})$ -structures is also easily analyzable from this standpoint. Thus, the construction is reversible, so that Theorem 4 gives a method of describing the local generality of (generalized) Finsler structures of constant flag curvature.

The other main results deal with either special dimensions or more special Finsler structures:

First of all, an old result of Funk [18] describes the local Finsler metrics on the plane that have constant positive curvature and are rectilinear (i.e., the geodesic paths are straight lines) in terms of a holomorphic function of one variable. In [8], this construction was given a projectively invariant description in terms of certain holomorphic curves without real points in $\mathbb{C}\mathbb{P}^2$. This turns out to generalize in a natural way to higher dimensions: A Finsler metric on a domain in \mathbb{R}^{n+1} with constant flag curvature whose geodesics are straight lines gives rise to a holomorphic hypersurface $Q \subset \mathbb{C}\mathbb{P}^{n+1}$ satisfying certain open conditions and, conversely, such a hypersurface determines a (generalized) Finsler structure on a domain in \mathbb{R}^{n+1} in a projectively natural way. For a precise statement, see Theorem 2.

This result is used to derive two further results: First, the global Finsler metrics on $\mathbb{R}\mathbb{P}^{n+1}$ with constant flag curvature 1 and rectilinear geodesics are determined (Example 2). It turns out that these correspond naturally to the hyperquadrics in $\mathbb{C}\mathbb{P}^{n+1}$ that have no real points. Thus, up to isomorphism, these consist of an $(n+1)$ -parameter family of distinct global examples. Second, it is shown that for any closed, real analytic hypersurface $S \subset T_0\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$ that is strictly convex towards the origin, there exists a Finsler metric with constant flag curvature 1 and rectilinear geodesics on a neighborhood U of $0 \in \mathbb{R}^{n+1}$ that has S as its space of unit tangent vectors at $0 \in \mathbb{R}^{n+1}$.

Finally, in §5.1 the description in [6] of Finsler metrics on S^2 of constant positive curvature 1 in terms of a Riemannian metric $d\sigma^2$ and a (‘magnetic’) 1-form β on $Q \simeq S^2$, the space of oriented geodesics of the Finsler structure, is recalled and then combined with Guillemin’s classic result on the existence of Zoll metrics on the 2-sphere to prove the existence of a large family of global Finsler metrics on S^2 with constant positive curvature 1. This is still far from a complete description, of course, but it gives an indication that this family is much larger than previously believed.

1.2. Dedication. This article is dedicated to Professor Shiing-Shen Chern, whose research in and tireless efforts to promote the study of Finsler geometry for nearly

60 years have inspired much of the progress in the subject. The research in this article would not have been possible without his encouragement and interest.

The main results in this article (aside from those of §5.2) were announced at the 1998 Geometry Festival at Stony Brook, NY, but I had been unable to set aside time to write the article until this opportunity came along. Thus, I would like to thank the editorial board of the Houston Journal of Mathematics for inviting me to write this article for a volume dedicated to Professor Chern.²

2. THE STRUCTURE EQUATIONS

This first section is mainly to fix notation and to remind the reader of some basic facts about Finsler geometry that will be used in this article. It will also be necessary to generalize the notion of Finsler structure slightly since some of the constructions that will be made have to first be done in this slightly more general context.

2.1. Generalized Finsler structures. Let M be a manifold of dimension $n+1$. Classically, a Finsler structure on M is a non-negative function $F : TM \rightarrow \mathbb{R}$ that is smooth and positive away from the zero section of TM , homogeneous of degree 1 (i.e., $F(\lambda v) = |\lambda| F(v)$ for all $v \in TM$ and $\lambda \in \mathbb{R}$), and strictly convex on each tangent space $T_x M$ for $x \in M$. For background, the reader is referred to [2].

The function F determines and is determined by the set

$$(2.1.1) \quad \Sigma_F = \{v \in TM \mid F(v) = 1\},$$

which is known as the *tangent indicatrix* or *unit tangent bundle* of F . For each $x \in M$, the intersection $\Sigma_F(x) = \Sigma_F \cap T_x M$ is a smooth, compact hypersurface in the vector space $T_x M$ that is transverse to the radial vector field on TM and is strictly convex towards the origin.

Definition 1. A *generalized Finsler structure* on a manifold M^{n+1} is a pair (Σ, ι) where Σ is a connected, smooth manifold of dimension $2n+1$ together with a radially transverse immersion $\iota : \Sigma \rightarrow TM$ with the following two properties:

- (1) The composition $\pi \circ \iota : \Sigma \rightarrow M$ is a submersion with connected fibers.

²After this article was submitted, Z. Shen contacted me and informed me that he had recently found a different local representation (in terms of power series) of the projectively flat Finsler metrics of constant flag curvature. See his preprint [26] for further details. In the positive curvature case, his representation is related to, but not equivalent to, the description given in §4.

- (2) Setting $\Sigma_x = \iota^{-1}(T_x M)$ for each $x \in M$, the mapping $\iota_x : \Sigma_x \rightarrow T_x M$ immerses Σ_x as a hypersurface in $T_x M$ that is locally strictly convex towards the origin 0_x .

Remark 1 (Equivalence). Two generalized Finsler structures, say (Σ_1, ι_1) on M_1 and (Σ_2, ι_2) on M_2 , will be said to be *isometric* if there is a diffeomorphism $\psi : \Sigma_1 \rightarrow \Sigma_2$ and a diffeomorphism $\phi : M_1 \rightarrow M_2$ so that $\phi' \circ \iota_1 = \iota_2 \circ \psi$.

The reader might prefer to regard a generalized Finsler structure as an isometry class of generalized Finsler structures as they were defined in Definition 1. While this is natural, it can be cumbersome, so this course has not been adopted.

Of course, the canonical inclusion into TM of the tangent indicatrix of a classical Finsler structure on M is a generalized Finsler structure. Obviously, this is not the only kind of example. For instance, there is no requirement that any of the Σ_x be compact, or that ι be an embedding.

The reasons for considering generalized Finsler structures is two-fold. First, all of the classical constructions of canonical connections, bundles, and curvature will work just as well for generalized Finsler structures as for Finsler structures with no increase in difficulty. Second, as will be seen, imposing differential equations (such as curvature constraints) on Finsler structures often leads to problems where the best strategy is to first solve the problem in the more general class of generalized Finsler structures and then look among the solutions for Finsler structures in the classical sense.

2.2. The structure bundle. Let (Σ, ι) be a generalized Finsler structure on a manifold M^{n+1} . Following Cartan [11, 12] and Chern [13, 14], one can define a canonical $O(n)$ -structure with connection on Σ . This section will review their constructions via the method of equivalence and establish the notation to be used throughout this article.

2.2.1. *The Hilbert form.* One constructs a contact form ω_0 on Σ as follows: For each $u \in \Sigma$, the vector $\iota(u)$ lies in $T_x M$ where $x = \pi(\iota(u))$. Moreover, the image $(\pi \circ \iota)'(T_u \Sigma_x) \subset T_x M$ is a hyperplane not containing $\iota(u)$. Consequently, there exists a unique linear form $u^* \in T_x^* M$ whose kernel is $(\pi \circ \iota)'(T_u \Sigma_x)$ and so that $u^*(\iota(u)) = 1$. Define the 1-form ω_0 on Σ so that

$$(2.2.1) \quad (\omega_0)_u = (\pi \circ \iota)^*(u^*).$$

The assumption that $\iota : \Sigma \rightarrow TM$ is radially transverse (i.e., transverse to the orbits of scalar multiplication on TM) implies that ω_0 is a contact form, i.e.,

that $\omega_0 \wedge (d\omega_0)^n \neq 0$. This form is known in the calculus of variations as the *Hilbert form*.

2.2.2. *The Reeb field.* Since ω_0 is a contact form, there exists a unique vector field E on Σ that satisfies $\omega_0(E) = 1$ and $E \lrcorner (d\omega_0) = 0$. This vector field is known as the *Reeb vector field*. Its flow on Σ is simply the geodesic flow when (Σ, ι) is an actual Finsler structure, so it will be referred to as the geodesic flow or the Reeb flow in this more general context.

In particular, a Σ -geodesic will be a smooth curve $\gamma : (a, b) \rightarrow M$ such that $\gamma' : (a, b) \rightarrow TM$ lifts back to Σ as an integral curve of E .

The generalized Finsler structure Σ will be said to be *geodesically complete* if E is complete, i.e., if the flow of E is globally defined.

2.2.3. *The Legendrian foliation.* The foliation \mathcal{M} whose leaves are the fibers Σ_x for $x \in M$ is ω_0 -Legendrian. As a consequence, each point $u \in \Sigma$ has a neighborhood, say U , on which there exist n 1-forms $\omega_1, \dots, \omega_n$ with the properties that

- (1) $\omega_0 \wedge \dots \wedge \omega_n \neq 0$,
- (2) each of the ω_i vanishes when pulled back to any Σ_x ,
- (3) $\omega_i(E) = 0$ for $1 \leq i \leq n$, and
- (4) $d\omega_0 \wedge \omega_1 \wedge \dots \wedge \omega_n = 0$.

It follows that there exist 1-forms $\theta^1, \dots, \theta^n$ on U so that

$$(2.2.2) \quad d\omega_0 = -\theta^1 \wedge \omega_1 - \dots - \theta^n \wedge \omega_n.$$

The forms $\omega_0, \omega_1, \dots, \omega_n, \theta^1, \dots, \theta^n$ are linearly independent on U .

The conditions imposed on the n 1-forms $\omega_1, \dots, \omega_n$ so far determine them up to a change of basis (with variable coefficients). If one were to make a different choice subject to the same conditions, one would have 1-forms

$$(2.2.3) \quad {}^*\omega_i = A_i^j \omega_j$$

for some invertible matrix of functions $A = (A_i^j)$ defined on U . Any ${}^*\theta^1, \dots, {}^*\theta^n$ for which the equation $d\omega_0 = -{}^*\theta^i \wedge {}^*\omega_i$ holds must then be of the form

$$(2.2.4) \quad {}^*\theta^i = B_j^i (\theta^j + S^{jk} \omega_k)$$

where $(B_j^i) = B = {}^tA^{-1}$ and $S^{ij} = S^{ji}$ are functions on U .

In the language of G -structures, the local coframings $(\omega_0, \omega_i, \theta^i)$ that satisfy the above conditions are the local sections of a G_1 -bundle over Σ where $G_1 \subset$

$\mathrm{GL}(2n+1, \mathbb{R})$ is the group of matrices of the form

$$(2.2.5) \quad \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & {}^t A^{-1} S & {}^t A^{-1} \end{bmatrix} \mid A \in \mathrm{GL}(n, \mathbb{R}), S = {}^t S \in M_n(\mathbb{R}) \right\}.$$

Such coframings are said to be *1-adapted*.

2.2.4. *Convexity.* Since the system spanned by $\omega_0, \omega_1, \dots, \omega_n$ is Frobenius, there must be functions H_{ij} on U so that

$$(2.2.6) \quad d\omega_i \equiv H_{ij} \theta^j \wedge \omega_0 \pmod{\omega_1, \dots, \omega_n}.$$

Using (2.2.2) to expand the identity $d(d\omega_0) = 0$, reducing modulo $\omega_1, \dots, \omega_n$, and then using (2.2.6) shows that $H_{ij} = H_{ji}$.

The geometric significance of H is that the quantity $H_{ij} \theta^i \circ \theta^j$ pulls back to each Σ_x to be the centro-affine invariant metric induced on it by its radially transverse immersion into the vector space $T_x M$.

In particular, the strict local convexity hypothesis implies that the symmetric matrix $H = (H_{ij})$ is positive definite everywhere on U .

Moreover, relative to a coframing $(\omega_0, {}^* \omega_i, {}^* \theta^i)$ on U given by (2.2.3) and (2.2.4), the corresponding symmetric matrix ${}^* H$ satisfies ${}^* H = A H {}^t A$.

Thus, the coframing $\omega_0, \omega_1, \dots, \omega_n, \theta^1, \dots, \theta^n$ can be chosen so that it satisfies $H = I_n$, i.e., so that

$$(2.2.7) \quad d\omega_i \equiv \theta^j \wedge \omega_0 \pmod{\omega_1, \dots, \omega_n}.$$

Henceforth, assume that (2.2.2) and (2.2.7) hold. In the language of G -structures, the local coframings $(\omega_0, \omega_i, \theta^i)$ that satisfy these conditions are the local sections of a G_2 -bundle over Σ where $G_2 \subset \mathrm{GL}(2n+1, \mathbb{R})$ is the group of matrices of the form

$$(2.2.8) \quad \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & AS & A \end{bmatrix} \mid A \in \mathrm{O}(n), S = {}^t S \in M_n(\mathbb{R}) \right\}.$$

Such coframings are said to be *2-adapted*.

2.2.5. *The quadratic form γ .* The forms $\omega_1, \dots, \omega_n$ now are determined up to an orthogonal change of basis. Thus, the quadratic form

$$(2.2.9) \quad \gamma = \omega_1^2 + \dots + \omega_n^2$$

is globally well-defined on Σ . Here is the geometric meaning of γ :

For any $u \in \Sigma$ with basepoint $x = \pi(u(u))$ in M , there is a unique positive definite quadratic form g_u on $T_x M$ with the property that $\iota(u)$ is a unit vector

for g_u and that the unit sphere of g_u in $T_x M$ osculates to second order to $\iota(\Sigma_x)$ at $\iota(u)$. This family of quadratic forms can be shown to satisfy

$$(2.2.10) \quad (\pi \circ \iota)^*(g_u) = (\omega_0^2 + \gamma)|_u = (\omega_0^2 + \omega_1^2 + \cdots + \omega_n^2)|_u.$$

Since the ambiguity in the choice of the ω_i lies in the orthogonal group, there is no longer any reason to preserve a distinction between upper and lower indices. Henceforth, all indices will be written as subscripts. In particular, θ^i will now be written as θ_i .

2.2.6. *Further normalizations.* In view of (2.2.7), there must exist 1-forms ω_{ij} on U so that

$$(2.2.11) \quad d\omega_i = \theta_j \wedge \omega_0 - \omega_{ij} \wedge \omega_j.$$

The relations (2.2.11) do not determine the ω_{ij} uniquely. Evidently, one can keep the same relations while replacing each ω_{ij} by ${}^*\omega_{ij} = \omega_{ij} + P_{ijk} \omega_k$ where $P_{ijk} = P_{ikj}$ are arbitrary functions on U .

Write $\omega_{ij} = \theta_{ij} + \sigma_{ij}$ where $\theta_{ij} = -\theta_{ji}$ and $\sigma_{ij} = \sigma_{ji}$. Expand σ_{ij} in the coframing as follows:

$$(2.2.12) \quad \sigma_{ij} = S_{ij} \omega_0 + B_{ijk} \omega_k + I_{ijk} \theta_k,$$

where $S_{ij} = S_{ji}$, $B_{ijk} = B_{jik}$, and $I_{ijk} = I_{jik}$ are functions on U .

The ambiguities in the choices so far can be exploited to eliminate the quantities S_{ij} and B_{ijk} by the following normalizations:

A Lagrangian splitting. First, note that by replacing θ_i by ${}^*\theta_i = \theta_i - S_{ij} \omega_j$ and ω_{ij} by ${}^*\omega_{ij} = \omega_{ij} - S_{ij} \omega_0$, one preserves the formulae (2.2.2) and (2.2.7), but the S_{ij} are replaced by ${}^*S_{ij} = 0$.

Thus, it will be assumed from now on that $S_{ij} = 0$.

Remark 2 (Geometric S elimination). This normalization has the following intrinsic description, which, in slightly different form, can essentially be found in the work of Foulon [15]:

Since $\omega_0(E) = 1$ and $\omega_i(E) = \theta_i(E) = 0$, the Lie derivative $\dot{\gamma}$ of γ with respect to E can be computed in the form

$$(2.2.13) \quad \begin{aligned} \dot{\gamma} &= 2\omega_i \circ (E \lrcorner d\omega_i) = 2\omega_i \circ (-\theta_i - (\theta_{ij}(E) + \sigma_{ij}(E)) \omega_j) \\ &= -2\omega_i \circ (\theta_i - S_{ij} \omega_j) \end{aligned}$$

(remember that $\theta_{ij}(E) = -\theta_{ji}(E)$).

It follows that the $(n+1)$ -plane field on U defined by $\theta_i - S_{ij} \omega_j = 0$ is the unique one that is transverse to the fibers of $\pi \circ \iota$ and is both Lagrangian with

respect to $d\omega_0$ and null with respect to the quadratic form γ' . Consequently, this plane field is globally defined on Σ .

In the language of G -structures, the 2-adapted coframings $(\omega_0, \omega_i, \theta^i)$ that satisfy equations of the form

$$(2.2.14) \quad d\omega_i = \theta_i \wedge \omega_0 - \theta_{ij} \wedge \omega_j - B_{ijk} \omega_k \wedge \omega_j - I_{ijk} \theta_k \wedge \omega_j,$$

where $\theta_{ij} = -\theta_{ji}$ and $B_{ijk} = B_{jik}$ while $I_{ijk} = I_{jik}$, are the local sections of a G_3 -bundle over Σ where $G_3 \subset \text{GL}(2n+1, \mathbb{R})$ is the group of matrices of the form

$$(2.2.15) \quad \left\{ \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & A & 0 & \\ 0 & 0 & A & \end{array} \right] \mid A \in \text{O}(n) \right\}.$$

Such coframings are said to be 3-*adapted*.

A connection adaptation. By making a replacement $\theta_{ij} \mapsto * \theta_{ij} = \theta_{ij} + P_{ijk} \omega_k$ where $P_{ijk} = -P_{jik}$, one can arrange $*B_{ijk} = 0$ and this uniquely determines the P_{ijk} .³ Thus, it will be assumed from now on that $B_{ijk} = 0$. This normalization determines the θ_{ij} uniquely.

2.2.7. *An $\text{O}(n)$ -structure.* The analysis so far has produced a coframing

$$(2.2.16) \quad (\omega_0, \omega_1, \dots, \omega_n, \theta_1, \dots, \theta_n)$$

on U that satisfies the equations

$$(2.2.17) \quad \begin{aligned} d\omega_0 &= -\theta_j \wedge \omega_j \\ d\omega_i &= \theta_i \wedge \omega_0 - \theta_{ij} \wedge \omega_j - I_{ijk} \theta_k \wedge \omega_j \end{aligned}$$

for some functions $I_{ijk} = I_{jik}$ and 1-forms $\theta_{ij} = -\theta_{ji}$. Moreover, coframings satisfying these equations are unique up to an orthogonal change of coframing of the form $*\omega_i = A_{ij} \omega_j$ and $*\theta_i = A_{ij} \theta_j$ where $A = (A_{ij})$ is a smooth mapping of U into $\text{O}(n)$.

Thus, such coframings are the local sections of an $\text{O}(n)$ -structure $u : F \rightarrow \Sigma$, where $\text{O}(n)$ is embedded into $\text{GL}(2n+1, \mathbb{R})$ as the subgroup G_3 of matrices of the form (2.2.15).

³ This is precisely the algebraic lemma that is used to prove the Fundamental Lemma of Riemannian geometry.

The $\theta_{ij} = -\theta_{ji}$ are simply connection forms for this $O(n)$ -structure relative to the given coframing.⁴

To keep the notation simple, the symbols $\omega_0, \omega_i, \theta_i$ will also be used to stand for the corresponding tautological forms on F while the symbols θ_{ij} will also be used to stand for the connection forms on F . Context will be used to determine whether these forms are to be understood as defined globally on F or locally on Σ . In most cases, this will make no practical difference.

For example, the quadratic form γ is defined globally on Σ while the expression $\omega_1^2 + \cdots + \omega_n^2$ is defined globally on F . Logically, one should write $u^*(\gamma) = \omega_1^2 + \cdots + \omega_n^2$ as a global equation on F , but, as is common practice in moving frame computations, one simply writes $\gamma = \omega_1^2 + \cdots + \omega_n^2$ and either the u^* is understood or else the equation is meant locally on Σ , relative to a 2-adapted coframing.

2.2.8. *The symmetry of I.* Differentiating the first equation of (2.2.17) yields the relation

$$(2.2.18) \quad 0 = (d\theta_i + \theta_j \wedge \theta_{ji} + I_{ijk} \theta_j \wedge \theta_k) \wedge \omega_i,$$

which, in particular, implies

$$(2.2.19) \quad d\theta_i + \theta_j \wedge \theta_{ji} + I_{ijk} \theta_j \wedge \theta_k \equiv 0 \pmod{\omega_1, \dots, \omega_n}.$$

Differentiating the second equation of (2.2.17) and reducing modulo $\omega_1, \dots, \omega_n$ yields

$$(2.2.20) \quad 0 \equiv (d\theta_i + \theta_j \wedge \theta_{ji} + I_{ijk} \theta_k \wedge \theta_j) \wedge \omega_0 \pmod{\omega_1, \dots, \omega_n},$$

which, in particular, implies

$$(2.2.21) \quad d\theta_i + \theta_j \wedge \theta_{ji} - I_{ijk} \theta_j \wedge \theta_k \equiv 0 \pmod{\omega_0, \omega_1, \dots, \omega_n}.$$

However, comparing (2.2.19) with (2.2.21) yields $I_{ijk} \theta_j \wedge \theta_k = 0$, i.e., $I_{ijk} = I_{ikj}$. In particular, I_{ijk} is fully symmetric in its indices.

⁴ This is essentially Cartan's connection, but, as was observed explicitly by Chern [14], there are other natural connections one could conceivably attach to F . For example, one could take the Levi-Civita connection of the Riemannian metric $ds^2 = \omega_0^2 + \omega_i^2 + \theta_i^2$ on Σ and project it to F (which is a subbundle of the ds^2 -orthonormal frame bundle) in the usual way. Indeed, several different connections have been attached to Finsler geometry in the literature; having to sort through all of them while learning the subject is something of a chore. See [2] for an account.

2.2.9. *The Cartan torsion.* The symmetric cubic form

$$(2.2.22) \quad I = I_{ijk} \theta_i \theta_j \theta_k$$

is well-defined globally on Σ . Its geometric interpretation at a point u is that it measures the failure of the unit sphere of g_u in $T_x M$ to osculate to $\iota(\Sigma_x)$ to third order at $\iota(u)$. In fact, I pulls back to each Σ_x to be the classical centro-affine cubic form induced on Σ_x by its radially transverse, strictly locally convex immersion into the vector space $T_x M$.

It is a standard result of Cartan that the equation $I \equiv 0$ is the necessary and sufficient condition that the image $\iota(\Sigma)$ should be an open subset of the unit sphere bundle of a Riemannian metric g (necessarily unique) defined on the open set $\pi \circ \iota(\Sigma) \subset M$.

In fact, note that, when $I \equiv 0$, if one writes $\theta_i = \theta_{0i}$ and sets $\theta_{i0} = -\theta_{0i}$ and $\theta_{00} = 0$, then (2.2.17) can be written in the simple form

$$(2.2.23) \quad d\omega_a = -\theta_{ab} \wedge \omega_b, \quad \theta_{ab} + \theta_{ba} = 0,$$

where the indices a and b lie in the range $0, \leq a, b \leq n$. It follows that the quadratic form $\omega_0^2 + \omega_1^2 + \dots + \omega_n^2$ is simply the $\pi \circ \iota$ -pullback of the desired metric g .

2.2.10. *Realizations.* Because the notational change in the Riemannian case is so suggestive, it will be adopted here for the general case. Thus, from now on, θ_i will be written as θ_{0i} while θ_{i0} (to be used on rare occasions) will mean $-\theta_{0i} = -\theta_i$. With this notational change, the structure equations so far take the form

$$(2.2.24) \quad \begin{aligned} d\omega_0 &= -\theta_{0j} \wedge \omega_j, \\ d\omega_i &= -\theta_{i0} \wedge \omega_0 - \theta_{ij} \wedge \omega_j - I_{ijk} \theta_{0k} \wedge \omega_j. \end{aligned}$$

These equations have been derived starting with a generalized Finsler structure (Σ, ι) . It will be important in what follows to know that there is the following sort of converse. The proposition below may seem somewhat strange, but it is the fundamental tool for ensuring that, once one has found differential forms satisfying the appropriate structure equations, they come from a (generalized) Finsler structure in a natural way.

Proposition 1. *Suppose that X is a manifold of dimension at least $2n+1$ and that there exist linearly independent 1-forms $\omega_0, \omega_1, \dots, \omega_n, \theta_{01}, \dots, \theta_{0n}$ on X for which the equations (2.2.24) hold for some 1-forms $\theta_{ij} = -\theta_{ji}$ and some functions $I_{ijk} = I_{jik} = I_{ikj}$ on X .*

Suppose that there exist submersions $\tau : X \rightarrow M^{n+1}$ and $\psi : X \rightarrow \Sigma^{2n+1}$, respectively, with connected fibers such that the fibers of each are, respectively, the

leaves of the Frobenius system generated by $\{\omega_0, \dots, \omega_n\}$ and $\{\omega_0, \dots, \omega_n, \theta_{01}, \dots, \theta_{0n}\}$.

Then there exists an immersion $\iota : \Sigma \rightarrow TM$ that defines a generalized Finsler structure and a mapping $\phi : X \rightarrow F$, where F is the canonical $O(n)$ -structure associated to (Σ, ι) , so that ϕ pulls back the tautological forms and connection forms on F to be the given forms on X .

PROOF. The key point is to explain how ι is defined: Take any vector field E on X (locally defined, if necessary) for which $\omega_0(E) = 1$ and $\omega_i(E) = \theta_{0i}(E) = 0$. Now define a mapping $\tilde{\iota} : X \rightarrow TM$ by setting $\tilde{\iota}(x) = \tau'(E(x))$. It is not difficult to show that $\tilde{\iota}$ is constant on the fibers of $\psi : X \rightarrow \Sigma$ and therefore that there exists a mapping $\iota : \Sigma \rightarrow TM$ such that $\tilde{\iota} = \iota \circ \psi$. The mapping ϕ is defined by a similar abstract diagram chase.

The remainder of the proof is a matter of checking details and can be left to the reader. □

2.2.11. *A even wider sense of Finsler structure.* At some point during this subsection (if not earlier), the reader may have realized that even the generalization of Finsler structure proposed in Definition 1 is unnecessarily restrictive.

The only ingredients used in the construction are

- (1) a $(2n+1)$ -manifold Σ ,
- (2) a contact form ω_0 , and
- (3) a ω_0 -Legendrian foliation \mathcal{M} of Σ that satisfies the local convexity property needed to ensure that the matrix H that shows up in (2.2.6) is positive definite. (This H represents a tensor that is globally defined on Σ using only the data of ω_0 and \mathcal{M} .)

Thus, one could define a generalized Finsler structure to be a triple $(\Sigma, \omega_0, \mathcal{M})$ as above, subject to the appropriate local convexity condition. The construction of the canonical $O(n)$ -structure $u : F \rightarrow \Sigma$ then proceeds just as before.

It will be useful to speak of generalized Finsler structures $(\Sigma, \omega_0, \mathcal{M})$ in this wider sense, so the reader should be alert for this usage in the rest of this article. The expression “generalized Finsler structure on M ” will still be reserved for a pair (Σ, ι) as in Definition 1.

By Proposition 1, any generalized Finsler structure in the wider sense is locally realizable as a generalized Finsler structure on a manifold M and uniquely up to isometry to boot. Thus, the extra generality is only relevant when one does not have a manifold structure for the leaves for \mathcal{M} explicitly in hand.

2.3. The flag curvature. Differentiating the equations (2.2.24) (and reducing the second one modulo $\omega_1, \dots, \omega_n$ to remove the derivatives of the functions I_{ijk}) shows that there exist functions $R_{0i0j} = R_{0j0i}$, $R_{0ijk} = -R_{0ikj}$, and $J_{ijk} = J_{jik}$ so that

$$(2.3.1) \quad d\theta_{0i} = -\theta_{0j} \wedge \theta_{ji} + R_{0i0j} \omega_0 \wedge \omega_j + \frac{1}{2} R_{0ijk} \omega_j \wedge \omega_k + J_{ijk} \theta_{0k} \wedge \omega_j.$$

Moreover, $R_{0ijk} + R_{0jki} + R_{0kij} = 0$, just as in the Riemannian case.

Equation (2.3.1) completes the first level of structure equations for the $O(n)$ -structure F .

It will not be necessary to carry out a full development of the structure equations here. The interested reader is referred to [2] for a thorough treatment.

The most important aspect of these equations for the present article is the so-called *flag curvature*, represented by the symmetric tensor

$$(2.3.2) \quad \mathcal{R} = R_{0i0j} \omega_i \circ \omega_j.$$

The geometric significance of the tensor \mathcal{R} is that it furnishes the lowest order term for the Jacobi equation that governs the second variation of geodesics of the Finsler structure. (This should not be surprising since, as the structure equations show, \mathcal{R} can be recovered from the *second* Lie derivative of γ with respect to the Reeb vector field E .)

Definition 2 (Constant flag curvature). A generalized Finsler structure (Σ, ι) is said to have *constant flag curvature* if there exists a constant c such that $\mathcal{R} = c\gamma$.

2.3.1. A PDE system. While a local hypersurface Σ in TM depends essentially on one arbitrary function of $2n+1$ variables, the equation $\mathcal{R} = c\gamma$ is $\frac{1}{2}n(n+1)$ non-linear partial differential equations for Σ , thought of as such a hypersurface. Thus, the condition of constant flag curvature is an overdetermined system of PDE for generalized Finsler structures as soon as $n > 1$.

The generality of the solutions of this system up to local isometry is easily understood when $n = 1$ (see §5.1 for a description in the case $c = 1$), but for $n > 1$ this system is overdetermined and it is not at all clear how many solutions there are, even locally. In §5, this question will be addressed for $n \geq 2$.

2.3.2. The Jacobi equation. When a generalized Finsler structure (Σ, ι) has constant flag curvature c , the Jacobi equation along any of its geodesics is metrically conjugate to the Jacobi equation of a geodesic in a Riemannian manifold of constant sectional curvature c . Indeed, a Riemannian metric has constant flag curvature when regarded as a Finsler structure if and only if it has constant sectional curvature.

2.3.3. *Examples.* Many (non-Riemannian) Finsler metrics that have constant flag curvature are now known. Here are a few examples:

- (1) Hilbert showed that there is a canonically defined, complete Finsler metric of constant negative flag curvature on any bounded convex domain M in \mathbb{R}^{n+1} with strictly convex smooth boundary. The geodesics of Hilbert's metric are the (open) line segments in M . Hence, Hilbert's examples are projectively flat.⁵ Hilbert's Finsler metric is Riemannian if and only if the boundary of M is an ellipsoid.
- (2) When $n = 2$, it is known [6] that the generalized Finsler metrics of constant flag curvature c depend essentially on two arbitrary functions of two variables, up to local isometry. Moreover, for $c = 1$, there are many examples defined globally on S^2 , at least as many (in some sense) as there are Zoll metrics on S^2 (see §5.1.2). There is even a two-parameter family of mutually inequivalent, projectively flat Finsler metrics of constant flag curvature $c = 1$ on the 2-sphere [6].
- (3) Recently, Bao and Shen [3] have constructed a one-parameter family of mutually inequivalent, left-invariant Finsler metrics on S^3 that have constant flag curvature $c = 1$ that are not projectively flat.

However, the present knowledge of existence and/or classification (in either the local or global case) of (generalized) Finsler structures with constant flag curvature is still very preliminary. The remaining sections of this article discuss various different aspects of this problem.

3. A KÄHLER STRUCTURE

This section will be concerned with properties of the space of geodesics of a generalized Finsler structure.

Definition 3 (Geodesic simplicity). A generalized Finsler structure (Σ, ι) on M^{n+1} is said to be *geodesically simple* if the set Q of integral curves of its Reeb vector field E can be given the structure of a smooth, Hausdorff manifold of dimension $2n$ in such a way that the natural mapping $\ell : \Sigma \rightarrow Q$ is a smooth submersion.

⁵ A Finsler metric $\Sigma \subset TM$ is projectively flat if every point of M has a neighborhood U that can be embedded into \mathbb{R}^{n+1} in such a way that it carries the Σ -geodesics in U to straight line segments. In contrast to the Riemannian case, where, by a theorem of Bonnet, projective flatness is equivalent to having constant sectional curvature, there are projectively flat Finsler metrics that do not have constant flag curvature and Finsler metrics with constant flag curvature that are not projectively flat.

Any generalized Finsler structure is locally geodesically simple, so for local calculations, one can always assume that Σ is geodesically simple. Thus, throughout this section, unless it is explicitly stated otherwise, it will be assumed that Σ is geodesically simple.

Now, it is a classical fact drawn from the calculus of variations that, for any generalized Finsler structure Σ , the space Q inherits a canonical symplectic structure. In fact, the 2-form $d\omega_0$ is manifestly invariant under the flow of E and satisfies $E \lrcorner d\omega_0 = 0$, so there exists a symplectic form Ω on Q so that

$$(3.0.3) \quad \ell^*\Omega = -d\omega_0 = \theta_{0i} \wedge \omega_i.$$

(The minus sign is for later convenience.)

In general, there is no natural metric on Q . However, when Σ has constant flag curvature, such a ‘metric’ does exist:

Proposition 2 (Quotient quadratic form). *Suppose that (Σ, ι) is a generalized Finsler structure with constant flag curvature c that is geodesically simple. Then there exists a quadratic form $d\sigma^2$ on Q for which*

$$(3.0.4) \quad \ell^*(d\sigma^2) = \theta_{01}^2 + \cdots + \theta_{0n}^2 + c\omega_1^2 + \cdots + c\omega_n^2.$$

PROOF. It suffices to show that the quadratic form on the right hand side of (3.0.4) is invariant under the flow of E . However, by hypothesis $R_{0i0j} = c\delta_{ij}$. Substituting this into the structure equations allows one to compute the Lie derivative with respect to E of the right hand side of (3.0.4) and see that it is equal to zero. \square

Remark 3. Proposition 2 and Proposition 3 below are necessary parts of the main result of this section, Theorem 1 below, which shows that Q actually inherits a Kähler structure when Σ has constant positive flag curvature.

These two propositions are essentially equivalent to the result of Bejancu and Farran [4] that (in the present article’s notation) asserts that a Finsler structure $\Sigma \subset TM$ has constant flag curvature 1 if and only if the Reeb field E is a Killing field for the Riemannian metric $ds^2 = \omega_0^2 + \omega_i^2 + \theta_{0i}^2$ on Σ .

While Theorem 1 was announced both at a June 1997 meeting at Oberwolfach, Germany and at the 1998 Geometry Festival at Stony Brook, NY, it appears that Bejancu and Farran arrived at their results independently.

3.1. The geodesic flow. Assume that the generalized Finsler structure (Σ, ι) has constant flag curvature c . Then the structure equations derived so far take

the form⁶

$$\begin{aligned}
 (3.1.1) \quad d\omega_0 &= -\theta_{0j} \wedge \omega_j \\
 d\omega_i &= \theta_{0i} \wedge \omega_0 - \theta_{ij} \wedge \omega_j - I_{ijk} \theta_{0k} \wedge \omega_j, \\
 d\theta_{0i} &= c\omega_0 \wedge \omega_i - \theta_{ij} \wedge \theta_{0j} + \frac{1}{2}R_{0ijk} \omega_j \wedge \omega_k + J_{ijk} \theta_{0k} \wedge \omega_j,
 \end{aligned}$$

where the forms θ_{ij} and functions R_{0ijk} , I_{ijk} , and J_{ijk} have the symmetries already discussed.

In what follows, the Lie derivative of a form or function with respect to the Reeb vector field E will be denoted by an overdot.⁷

Proposition 3. *The structure forms on the $O(n)$ -bundle of a generalized Finsler structure of constant flag curvature c satisfy the following identities:*

- (1) $\dot{\omega}_i = -\theta_{0i}$ and $\dot{\theta}_{0i} = c\omega_i$.
- (2) $\dot{I}_{ijk} = J_{ijk}$ and $\dot{J}_{ijk} = -cI_{ijk}$
- (3) $R_{0ijk} = 0$
- (4) $\dot{\theta}_{ij} = 0$.

(In particular, J is fully symmetric in its indices.)

PROOF. The formulae in Item (1) follow immediately from the definition of Lie derivative, the defining properties of E , and the second and third equations of (3.1.1).

Now compute the Lie derivative with respect to E of the second line of (3.1.1), using the fact that this operation is a derivation that commutes with exterior derivative, and add the result to the third equation. The result is

$$(3.1.2) \quad 0 = -\dot{\theta}_{ij} \wedge \omega_j + \frac{1}{2}R_{0ijk} \omega_j \wedge \omega_k + (J_{ijk} - \dot{I}_{ijk}) \theta_{0k} \wedge \omega_j.$$

(The reader who performs this calculation will note that it uses the fact that I_{ijk} is fully symmetric in its indices.) Since $\theta_{ij} = -\theta_{ji}$ while $I_{ijk} = I_{jik}$ and $J_{ijk} = J_{jik}$, it follows that $J_{ijk} - \dot{I}_{ijk} = 0$ and thus that

$$(3.1.3) \quad 0 = -\dot{\theta}_{ij} \wedge \omega_j + \frac{1}{2}R_{0ijk} \omega_j \wedge \omega_k.$$

⁶ The reader may wonder about the choice of notation, which is not entirely standard. The letters I and J and their signs were chosen to conform, insofar as possible, to Cartan's article [11], in which he treated the case $n = 1$. This notation has turned out to be very felicitous for computations in the constant flag curvature case.

⁷ Bear in mind, too, that E (which was originally defined on Σ) has a canonical lifting to the $O(n)$ -bundle F so that it satisfies $\omega_0(E) = 1$ while $\omega_i(E) = \theta_{0i}(E) = \theta_{ij}(E) = 0$. No separate notation will be introduced for the lifted vector field on F . The reader should have no difficulty determining which is meant from context.

In particular, the first equation of Item (2) is verified, which shows that J is indeed fully symmetric in all its indices.

Equation (3.1.3) also implies that there must exist (unique) functions $T_{ijk} = -T_{jik}$ that satisfy

$$(3.1.4) \quad \dot{\theta}_{ij} = -T_{ijk} \omega_k, \quad \text{and} \quad R_{0ijk} = T_{ijk} - T_{ikj}.$$

Now compute the Lie derivative with respect to E of the third line of (3.1.1), using the fact that this operation is a derivation that commutes with exterior derivative, and subtract c times the second equation from the result. This yields the relation

$$(3.1.5) \quad 0 = (2T_{ijk} - T_{ikj} - \dot{J}_{ijk} - cI_{ijk}) \omega_k \wedge \theta_{0j} + \frac{1}{2} \dot{R}_{0ijk} \omega_j \wedge \omega_k.$$

Of course, this implies both $\dot{R}_{0ijk} = 0$ and

$$(3.1.6) \quad 0 = 2T_{ijk} - T_{ikj} - \dot{J}_{ijk} - cI_{ijk}.$$

However, the symmetry of I and J and the skewsymmetry $T_{ijk} = -T_{jik}$ now combine to show that

$$(3.1.7) \quad T_{ijk} = 0 \quad \text{and} \quad \dot{J}_{ijk} = -cI_{ijk}.$$

This gives the second equation of Item (2) and, in view of (3.1.4), Items (3) and (4) as well. \square

By Proposition 3, the structure equations for a generalized Finsler structure of constant flag curvature c simplify to:

$$(3.1.8) \quad \begin{aligned} d\omega_0 &= -\theta_{0j} \wedge \omega_j \\ d\omega_i &= -\omega_0 \wedge \theta_{0i} - \theta_{ij} \wedge \omega_j - I_{ijk} \theta_{0k} \wedge \omega_j, \\ d\theta_{0i} &= c\omega_0 \wedge \omega_i - \theta_{ij} \wedge \theta_{0j} + J_{ijk} \theta_{0k} \wedge \omega_j, \end{aligned}$$

where I and J are fully symmetric in their indices. The similarity of the second and third lines is very suggestive and will be exploited in the next subsection.

3.2. The Kähler structure. The main concern of this article is the case of constant positive flag curvature. To treat this case, it suffices (by homothety) to treat the case $c = 1$, so assume this from now on.

The pieces are now in place for the main result of this section:

Theorem 1. *Let (Σ, ι) be a geodesically simple generalized Finsler structure with constant flag curvature $+1$. Then the metric $d\sigma^2$ induced on Q is a Kähler metric,*

with Kähler form Ω . Moreover, Σ has a natural immersion into the unit canonical bundle of Q that commutes with the submersion $\ell : \Sigma \rightarrow Q$.

PROOF. Define complex valued 1-forms

$$(3.2.1) \quad \zeta_i = \omega_i - i\theta_{0i}.$$

Using this notation and the condition $c = 1$, one finds that the pullbacks of Ω and $d\sigma^2$ can be written in the form

$$(3.2.2) \quad \begin{aligned} \ell^*(\Omega) &= \frac{1}{2} \zeta_i \wedge \bar{\zeta}_i, \\ \ell^*(d\sigma^2) &= \zeta_1 \circ \bar{\zeta}_1 + \dots + \zeta_n \circ \bar{\zeta}_n. \end{aligned}$$

Thus, the metric and 2-form are algebraically compatible and the 2-form is closed. The only condition remaining to verify in order to show that this metric is Kähler with the given 2-form as Kähler form is whether or not the almost complex structure defined by this pair is integrable.

Now, the almost complex structure induced on Q is the one for which the ℓ -pullback of a $(1, 0)$ -form is a linear combination of ζ_1, \dots, ζ_n . By the Newlander-Nirenberg Theorem, the integrability of the almost complex structure is equivalent to the condition that these forms define a differentially closed ideal.

To see this, note that the second and third structure equations (3.1.8) (when $c = 1$) can be combined in complex form as

$$(3.2.3) \quad d\zeta_i = -i\omega_0 \wedge \zeta_i - \theta_{ij} \wedge \zeta_j + \frac{1}{2}(I_{ijk} + iJ_{ijk})\bar{\zeta}_j \wedge \zeta_k.$$

Thus, the complex Pfaffian system spanned by the ζ_i is Frobenius, as desired.

For the final statement, consider the complex-valued n -form defined on Σ by

$$(3.2.4) \quad \zeta = \zeta_1 \wedge \zeta_2 \wedge \dots \wedge \zeta_n.$$

Let K be the canonical bundle of Q (regarded as a complex manifold), i.e., K is the top exterior power of the complex cotangent bundle of Q . Let Υ be the tautological holomorphic n -form on K and let $\Sigma(K) \subset K$ denote the circle bundle of unit complex volume forms on Q with respect to the Kähler structure constructed in the first part of the proof.

Evidently, there is a unique smooth mapping $\hat{\ell} : \Sigma \rightarrow \Sigma(K)$ that lifts ℓ and satisfies $\hat{\ell}^*(\Upsilon) = \zeta$. Because ζ satisfies

$$(3.2.5) \quad d\zeta = -i\rho \wedge \zeta, \quad \text{where} \quad \rho = n\omega_0 - \text{Re}((I_{ij} - iJ_{ij})\zeta_j),$$

it follows that $\hat{\ell}$ is an immersion. □

3.3. A canonical flow. Proposition 3 has another very interesting consequence: Let Φ_t denote the (locally defined) flow of the Reeb vector field E , lifted to F . Then Proposition 3 implies (bearing in mind that $c = 1$ throughout this discussion):

$$(3.3.1) \quad \begin{aligned} \Phi_t^* \omega_0 &= \omega_0, \\ \Phi_t^* \omega_i &= \cos t \omega_i - \sin t \theta_{0i}, \\ \Phi_t^* \theta_{0i} &= \sin t \omega_i + \cos t \theta_{0i}, \\ \Phi_t^* \theta_{ij} &= \theta_{ij}. \end{aligned}$$

In particular, this flow commutes with the $O(n)$ -action that defines the bundle structures

Of course, Φ_t is only locally defined unless Σ is geodesically complete. However, this does not prevent one from defining a circle of coframings $(\omega_0^t, \omega_i^t, \theta_{0i}^t, \theta_{ij}^t)$ on the $O(n)$ -bundle F by the formulae

$$(3.3.2) \quad \begin{aligned} \omega_0^t &= \omega_0, \\ \omega_i^t &= \cos t \omega_i - \sin t \theta_{0i}, \\ \theta_{0i}^t &= \sin t \omega_i + \cos t \theta_{0i}, \\ \theta_{ij}^t &= \theta_{ij}. \end{aligned}$$

These forms (together with the appropriately rotated functions I_{ijk} and J_{ijk}) evidently satisfy the structure equations (3.1.8) with $c = 1$ for any value of t and so make F into the $O(n)$ -bundle of a circle of generalized Finsler structures with constant flag curvature 1.

The members of this circle of generalized Finsler structures are generally not isometric among themselves. Thus, this constructs a nontrivial flow on the space of generalized Finsler structures with constant flag curvature 1. In the projectively flat case (see the next section), however, the resulting generalized Finsler structure is a fixed point of this flow.

3.4. Related G -structures. In the language of G -structures, a Kähler structure on a $2n$ -manifold Q is the same thing as a torsion-free $U(n)$ -structure on Q where $U(n)$ is embedded into $GL(2n, \mathbb{R})$ in the usual way. Theorem 1 describes how a generalized Finsler structure of constant flag curvature 1 determines a natural Kähler structure on Q .

3.4.1. An $S^1 \cdot O(n)$ -structure. Now, Proposition 3 actually implies that the structure on Σ determines a canonical $S^1 \cdot O(n)$ -structure on Q , where $S^1 \cdot O(n)$ is

the subgroup of $U(n)$ generated by $O(n)$ and the central subgroup $S^1 \subset U(n)$ consisting of scalar multiplication by unit complex numbers.

This $S^1 \cdot O(n)$ -structure is defined as follows: Let $\tau : F \rightarrow Q$ be the composition of the submersions $u : F \rightarrow \Sigma$ and $\ell : \Sigma \rightarrow Q$. For $f \in F$, the kernel of $\tau'(f) : T_f F \rightarrow T_{\tau(f)} Q$ is defined by the equations $\omega_i = \theta_{0i} = 0$. It follows that there is a unique isomorphism $v(f) : T_{\tau(f)} Q \rightarrow \mathbb{C}^n$ for which $\zeta = (\omega_i - i\theta_{0i})$ satisfies

$$(3.4.1) \quad \zeta(w) = v(f)(\tau'(f)(w))$$

for all $w \in T_f F$. Thus, $v(f)$ is a \mathbb{C}^n -valued coframe at $\tau(f)$.

Proposition 3 implies that if $\tau(f_1) = \tau(f_2)$, then $v(f_1)$ and $v(f_2)$ are related by $v(f_1) = e^{is} A v(f_2)$ for some $s \in \mathbb{R}$ and $A \in O(n)$. Thus, the fibers of τ are mapped into $S^1 \cdot O(n)$ -orbits in the bundle of \mathbb{C}^n -valued coframes on Q and dimension count plus the structure equations on F imply that v is actually a local diffeomorphism on each fiber. Thus, there is a well-defined $S^1 \cdot O(n)$ -structure $\hat{u} : \hat{F} \rightarrow Q$ so that $v : F \rightarrow \hat{F}$ immerses F as an open set in \hat{F} .

Equation (3.2.3) can now be interpreted as the structure equation on \hat{F} , where ω_0 and the θ_{ij} are the connection forms for \hat{F} as a $S^1 \cdot O(n)$ -structure over Q . Note that, when $n > 1$, this $S^1 \cdot O(n)$ -structure has torsion unless $I = J = 0$. For the situation when $n = 1$, see §5.1.

3.4.2. *Extending Σ .* Note that the quotient $\hat{\Sigma} = \hat{F}/O(n)$ is naturally a circle bundle over Q and there is a canonical map $\Sigma \rightarrow \hat{\Sigma}$ that is a local diffeomorphism, carrying the vector field E into the infinitesimal generator of the S^1 -action. Moreover, $\hat{\Sigma}$ carries a natural generalized Finsler structure in the wider sense of §2.2.11. The needed data are as follows: First ω_0 is well-defined on $\hat{\Sigma}$ as the connection form associated to the S^1 -action. Second, there is a \mathcal{M} of $\hat{\Sigma}$ defined as the image of the leaves of $\omega_0 = \text{Re}(\zeta) = 0$ on \hat{F} . One checks directly that this foliation extends the foliation \mathcal{M} on Σ and satisfies the convexity assumptions as discussed in §2.2.11. In particular, it follows that, any geodesically simple generalized Finsler structure Σ of constant flag curvature 1 can be canonically immersed in a generalized Finsler structure $\hat{\Sigma}$ that is geodesically complete, with all geodesics closed of period 2π .

The leaves of \mathcal{M} on $\hat{\Sigma}$ project via ℓ to become Lagrangian submanifolds of Q . In fact, the (circle) fiber in $\hat{\Sigma}$ over a point $q \in Q$ represents a circle of Lagrangian n -planes in $T_q Q$ and the images of the leaves of \mathcal{M} are the Lagrangian submanifolds of Q whose tangent planes belong to $\hat{\Sigma}$ when regarded as a subset of the space of Lagrangian planes of Q .

Unfortunately, it can happen that the foliation \mathcal{M} on $\hat{\Sigma}$ is not simple.

3.4.3. *An $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure.* Finally, it is important to note that, although \hat{F} has torsion, it underlies a canonical $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure on Q that does not. Going back to (3.2.3), and setting

$$(3.4.2) \quad \sigma_{ij} = \frac{i}{2}(I_{ijk} - iJ_{ijk})\zeta_k - \frac{i}{2}(I_{ijk} + iJ_{ijk})\overline{\zeta_k} = \overline{\sigma_{ij}},$$

one notes that, because I and J are symmetric in all their indices, (3.2.3) can be written in the form

$$(3.4.3) \quad d\zeta = -(i\omega_0 I_n + \theta + \sigma) \wedge \zeta$$

where $\theta = (\theta_{ij})$ is real-valued and skewsymmetric while $\sigma = (\sigma_{ij})$ is real-valued and symmetric. Since $(i\omega_0 I_n + \theta + \sigma)$ takes values in the Lie algebra of $S^1 \cdot \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(2n, \mathbb{R})$, it follows that, as promised, the $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure underlying \hat{F} has vanishing torsion.⁸

This is surprising because the subgroup $S^1 \cdot \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(2n, \mathbb{R})$ acts irreducibly on \mathbb{R}^{2n} and yet does not appear on the accepted list [22, 23] of irreducibly-acting holonomies of torsion-free connections. More will be said about this point in §5.2, where the structure equations will be investigated more thoroughly.

4. COMPLEX HYPERSURFACES IN $\mathbb{C}\mathbb{P}^{n+1}$

The goal of this section is to explain how certain generalized Finsler structures on $\mathbb{R}\mathbb{P}^{n+1}$ of constant flag curvature 1 can be constructed from complex hypersurfaces in $\mathbb{C}\mathbb{P}^{n+1}$ that satisfy some genericity assumptions.

As usual, regard $\mathbb{R}\mathbb{P}^{n+1} = \mathbb{P}^{n+1}$ as the space of real lines in \mathbb{R}^{n+2} through the origin. The notation $\tilde{\mathbb{P}}^{n+1}$ will denote its nontrivial double cover, i.e., the space of real rays in \mathbb{R}^{n+2} emanating from the origin, or, equivalently, the set of oriented real lines in \mathbb{R}^{n+2} through the origin. If $v \in \mathbb{R}^{n+2}$ is nonzero, then $[v] \in \mathbb{P}^{n+1}$ denotes the line spanned by v and $[v]_+ \in \tilde{\mathbb{P}}^{n+1}$ denotes the ray spanned by v .

Each oriented 2-plane $E \subset \mathbb{R}^{n+2}$ through the origin determines an oriented line in $\tilde{\mathbb{P}}^{n+1}$ that is a closed circle, as follows: If (v, w) is an oriented basis of E , then the curve $\gamma_{(v,w)}(s) = [(\cos s)v + (\sin s)w]_+$ is an oriented embedding of the circle into $\tilde{\mathbb{P}}^{n+1}$. It is easy to see that this oriented line (as an image) depends only on the oriented 2-plane E and not on the specific oriented basis (more will be said about this below). Thus, $\mathrm{Gr}_2^+(\mathbb{R}^{n+2})$ parametrizes a family of oriented lines in $\tilde{\mathbb{P}}^{n+1}$ that has the property that there is a unique such line passing through a given point and having a given oriented tangent direction there.

⁸ The reader will note that, when $n > 1$, a $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure has at most one torsion-free connection. Again, the case $n = 1$ is special, but is treated by other means anyway.

A generalized Finsler structure (Σ, ι) on $\tilde{\mathbb{P}}^{n+1}$ will be said to be *rectilinear* if each of its (oriented) geodesics is a line (up to reparametrization). (Note that this is stronger than requiring saying that the generalized Finsler structure be projectively flat. It requires that the geodesics actually *be* lines in $\tilde{\mathbb{P}}^{n+1}$, not just that one can transform them into lines by local reparametrizations in $\tilde{\mathbb{P}}^{n+1}$.) In such a case, there is a canonical submersion $\lambda : \Sigma \rightarrow \text{Gr}_2^+(\mathbb{R}^{n+2})$ whose fibers are unions of integral curves of E . (It need not be true, in general, that the λ -fibers are connected.)

It was first observed by Funk [16] that if $D \subset \mathbb{R}^2 \subset \tilde{\mathbb{P}}^2$ is a domain with a rectilinear Finsler structure $\Sigma \subset TD$ whose flag curvature is identically 1, then every Σ -geodesic in D has a *unit speed* parametrization of the form

$$(4.0.4) \quad \gamma_{(v,w)}(s) = [(\cos s)v + (\sin s)w]_+$$

for some linearly independent $v, w \in \mathbb{R}^3$.

Now, it is immediate that $\gamma_{(v_1, w_1)}$ and $\gamma_{(v_2, w_2)}$ as defined by (4.0.4) parametrize the same oriented geodesic with the same speed if and only if $[[v_1 + iw_1]] = [[v_2 + iw_2]]$ as points in $\mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{P}^{n+1}$.

Consequently, for a rectilinear Finsler structure $\Sigma \subset TD$ with flag curvature identically 1, there is a refined ‘geodesic map’ $\ell : \Sigma \rightarrow \mathbb{C}\mathbb{P}^2 \setminus \mathbb{P}^2$ such that Σ is essentially recoverable from its image under ℓ . (One gets not only the geodesics, but their unit speed parametrizations as well; hence one can recover their unit tangent vectors, i.e., $\Sigma \subset TD$.)

Busemann [10] later pointed out that Funk’s observation applies just as well to a rectilinear Finsler structure of constant flag curvature 1 on a projective space of any dimension. In fact, the argument is purely local, so that it applies to any rectilinear generalized Finsler structure (Σ, ι) on $\tilde{\mathbb{P}}^{n+1}$ with constant flag curvature 1. The result is that one has a natural mapping $\ell : \Sigma \rightarrow \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{P}^{n+1}$ for such a (Σ, ι) whose differential has constant rank $2n$ and whose fibers are discrete unions of integral curves of E .

Back in the case $n = 1$, Funk [17] eventually observed that, when Σ is a rectilinear Finsler structure on $D \subset \tilde{\mathbb{P}}^2$ with constant flag curvature 1, the 2-dimensional image $\ell(\Sigma) \subset \mathbb{C}\mathbb{P}^2 \setminus \mathbb{P}^2$ is actually a holomorphic curve. He then showed how, conversely, starting with a holomorphic curve in $\mathbb{C}\mathbb{P}^2 \setminus \mathbb{P}^2$ satisfying some open conditions, one could construct a (generalized) rectilinear Finsler structure on a domain in $\tilde{\mathbb{P}}^2$ with constant flag curvature 1.

In [8], it was shown how Funk’s construction could be globalized so as to classify the rectilinear Finsler structures on $\tilde{\mathbb{P}}^2$ with constant flag curvature 1.

It was shown there that, up to projective equivalence, these structures form a non-compact, 2-parameter family.

The goal of this section is to explain how these constructions generalize to higher dimensions. This turns out to be straightforward. However, it will be useful to examine the proofs directly via the moving frame for use in the next section.

4.1. Some notation. It will be necessary to consider projective spaces of vector spaces with real or complex coefficients. It will also sometimes be necessary to consider, for a real or complex vector space V , the set $\tilde{\mathbb{P}}(V)$ of *real rays* in V , i.e., the equivalence classes in $V \setminus \{0\}$ generated by scalar multiplication by positive real numbers. For a nonzero vector $v \in V$, the notation $[v]_+$ will be used for the real ray containing v . The standard notation $[v]$ will be used for the real line containing v and, in case V is a complex vector space, the notation $[[v]]$ will be used for the complex line containing v .

When V is a real vector space, $\tilde{\mathbb{P}}(V)$ is naturally the (non-trivial) double cover of $\mathbb{P}(V)$ and is diffeomorphic to a sphere. The space $\tilde{\mathbb{P}}(\mathbb{R}^m)$ will be denoted $\tilde{\mathbb{P}}^{m-1}$.

When V is a complex vector space, $\tilde{\mathbb{P}}(V)$ is naturally an S^1 -bundle over $\mathbb{P}(V)$ (which, as usual, denotes the complex projectivization), and the natural mapping will be denoted by $\ell : \tilde{\mathbb{P}}(V) \rightarrow \mathbb{P}(V)$.

The constructions will be designed so as to be equivariant under the action of $SL(n+2, \mathbb{R})$, so it will be useful to consider some of the spaces on which this group acts.

First of all, there is $\mathbb{S} = \tilde{\mathbb{P}}(\mathbb{C}^{n+2}) \setminus \tilde{\mathbb{P}}(\mathbb{R}^{n+2})$. A typical element of \mathbb{S} is of the form $[v + iw]_+$ where v and w are linearly independent in \mathbb{R}^{n+2} .

There is a natural mapping $\iota : \mathbb{S} \rightarrow T\tilde{\mathbb{P}}^{n+1}$ that sends $[v + iw]_+ \in \mathbb{S}$ to the velocity at $t = 0$ of the curve $\gamma_{(v,w)} : \mathbb{R} \rightarrow \tilde{\mathbb{P}}^{n+1}$ defined by $\gamma_{(v,w)}(t) = [v + tw]_+$.

The image of $\ell : \mathbb{S} \rightarrow \mathbb{C}\mathbb{P}^{n+1}$ is the open set $X_{n+1} = \mathbb{C}\mathbb{P}^{n+1} \setminus \mathbb{R}\mathbb{P}^{n+1}$.

The following diagrams may help to fix these homogeneous spaces of $SL(n+2, \mathbb{R})$ and $SL(n+2, \mathbb{R})$ -equivariant mappings in their proper perspective:

$$\begin{array}{ccccc}
 \mathbb{S} & \xrightarrow{\iota} & T\tilde{\mathbb{P}}^{n+1} & & [v + iw]_+ \xrightarrow{\iota} \frac{d}{dt}\Big|_{t=0}([v + tw]_+) \\
 \ell \downarrow & & \downarrow \pi & & \ell \downarrow & & \downarrow \pi \\
 (4.1.1) \quad X_{n+1} & & \tilde{\mathbb{P}}^{n+1} & & [[v + iw]] & & [v]_+ \\
 \lambda \downarrow & & & & \lambda \downarrow & & \\
 \mathrm{Gr}_2^+(\mathbb{R}^{n+2}) & & & & [v \wedge w]_+ & &
 \end{array}$$

The map λ is a surjective submersion and its fibers in X_{n+1} are Poincaré disks. In fact, the fiber over $[v \wedge w]_+$ is one of the two disks cut out of the $\mathbb{C}\mathbb{P}^1$ that is the projectivization of the (complex) span of v and w by the removal of the real points. The points of the fiber can be thought of as metric structures on the oriented $\mathbb{R}\mathbb{P}^1$ that was removed.⁹

For further details about these and similar constructions, see [8, §4.1].

4.2. The moving frame. A moving frame for the homogeneous space X_{n+1} can be constructed by starting with the standard moving frame

$$(4.2.1) \quad \mathbf{e} = (\mathbf{e}_0 \ \mathbf{e}_1 \ \cdots \ \mathbf{e}_{n+1}) : \mathrm{SL}(n+2, \mathbb{R}) \longrightarrow M_{n+2}(\mathbb{R})$$

with its canonical matrix-valued 1-form $\omega = (\omega_\alpha^\beta)$ satisfying the usual structure equations

$$(4.2.2) \quad d\mathbf{e} = \mathbf{e}\omega, \quad d\omega = -\omega \wedge \omega, \quad \text{and} \quad \mathrm{tr}(\omega) = 0,$$

and making a complex change of basis

$$(4.2.3) \quad (\mathbf{f}_0 \ \mathbf{f}_1 \ \cdots \ \mathbf{f}_n) = (\mathbf{e}_0 \ \mathbf{e}_1 \ \cdots \ \mathbf{e}_n) \begin{bmatrix} 1 & 0 & 1 \\ 0 & I_n & 0 \\ i & 0 & -i \end{bmatrix}.$$

Note that $\overline{\mathbf{f}_0} = \mathbf{f}_{n+1}$, $\overline{\mathbf{f}_{n+1}} = \mathbf{f}_0$, and $\overline{\mathbf{f}_i} = \mathbf{f}_i$ for $1 \leq i \leq n$. The structure equation $d\mathbf{f} = \mathbf{f}\varphi$ can now be expanded in the following form (which establishes notation¹⁰ for the entries of φ)

$$(4.2.4) \quad d(\mathbf{f}_0 \ \mathbf{f}_i \ \mathbf{f}_{n+1}) = (\mathbf{f}_0 \ \mathbf{f}_j \ \mathbf{f}_{n+1}) \begin{bmatrix} \alpha & -\frac{1}{2} \overline{\pi}_i & \overline{\zeta}^{n+1} \\ \zeta^j & \omega_i^j & \overline{\zeta}^j \\ \zeta^{n+1} & -\frac{1}{2} \pi_i & \overline{\alpha} \end{bmatrix}.$$

Note the trace relation

$$(4.2.5) \quad \mathrm{tr}(\varphi) = \alpha + \overline{\alpha} + \omega_i^i = 0$$

and the second structure equation $d\varphi = -\varphi \wedge \varphi$, which expands in the obvious way to provide formulae for $d\alpha$, etc.

By construction, the map $[\mathbf{f}_0] : \mathrm{SL}(n+2, \mathbb{R}) \rightarrow X_{n+1}$ is a surjective submersion and pulls back $(1, 0)$ -forms on X_{n+1} to be linear combinations of $\{\zeta^1, \dots, \zeta^{n+1}\}$.

⁹ The reader is reminded that the foliation of X_{n+1} by the fibers of λ is *not* holomorphic. If it were, then $\mathrm{Gr}(2, n+2)$ would have a $\mathrm{SL}(n+2, \mathbb{R})$ -invariant holomorphic structure, which it does not.

¹⁰ The introduction of the $\frac{1}{2}$ coefficients simplifies later normalizations.

One more fact about this moving frame construction will be important. The mapping

$$(4.2.6) \quad ([\mathbf{f}_0], [\mathbf{f}_0 \wedge \mathbf{f}_1 \wedge \dots \wedge \mathbf{f}_n]) : \mathrm{SL}(n+2, \mathbb{R}) \longrightarrow X_{n+1} \times (\mathbb{C}\mathbb{P}^{n+1})^*$$

has, as its image, the set of pairs (p, H) where $p \in X_{n+1}$ is any point and $H \subset \mathbb{C}\mathbb{P}^{n+1}$ is any complex hyperplane through p that is transverse to the λ -fiber through p . The easy verification of this fact is left to the reader.

4.3. Transverse, convex hypersurfaces. Now let $Q \subset \mathbb{C}\mathbb{P}^{n+1}$ be a (not necessarily compact) nonsingular complex hypersurface that is transverse to the fibers of λ .

Let Σ_Q denote the preimage $\ell^{-1}(Q) \subset \mathbb{S}$, and let $\iota_Q : \Sigma_Q \rightarrow T\tilde{\mathbb{P}}^{n+1}$ be the restriction of ι to Σ_Q . Then Σ_Q is a smooth manifold of dimension $2n+1$ and it is not hard to see that the assumption that Q be transverse to the fibers of λ implies that the map $\iota_Q : \Sigma_Q \rightarrow T\tilde{\mathbb{P}}^{n+1}$ is a radially transverse immersion.

Define the *first order frame bundle* $F_Q^1 \subset \mathrm{SL}(n+2, \mathbb{R})$ of Q to be the set of \mathbf{f} in $\mathrm{SL}(n+2, \mathbb{R})$ so that $[\mathbf{f}_0]$ lies in Q and the hyperplane $[\mathbf{f}_0 \wedge \mathbf{f}_1 \wedge \dots \wedge \mathbf{f}_n]$ is tangent to Q at \mathbf{f}_0 . The projection $[\mathbf{f}_0] : F_Q^1 \rightarrow Q$ is then a bundle over Q whose fibers are left cosets of the subgroup $G_1 \subset \mathrm{SL}(n+2, \mathbb{R})$ consisting of the set of matrices of the form

$$(4.3.1) \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & I_n & 0 \\ i & 0 & -i \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \bar{a} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & I_n & 0 \\ i & 0 & -i \end{bmatrix}^{-1} .$$

where $A \in \mathrm{GL}(n, \mathbb{R})$ and $a \in \mathbb{C}^*$ satisfy $a\bar{a} \det(A) = 1$.

Pulling the forms on $\mathrm{SL}(n+2, \mathbb{R})$ back to F_Q^1 , one has the relations

$$(4.3.2) \quad \zeta^{n+1} = 0, \quad \zeta^1 \wedge \dots \wedge \zeta^n \neq 0.$$

Since the structure equations entail

$$(4.3.3) \quad d\zeta^{n+1} = (\alpha - \bar{\alpha}) \wedge \zeta^{n+1} + \frac{1}{2} \pi_j \wedge \zeta^j ,$$

it follows that $\pi_j \wedge \zeta^j = 0$. By Cartan's Lemma, there exist functions $H_{ij} = H_{ji}$ on F_Q^1 so that

$$(4.3.4) \quad \pi_i = H_{ij} \zeta^j .$$

The structure equations entail

$$(4.3.5) \quad d\alpha = \frac{1}{2} \bar{\pi}_i \wedge \zeta^i = -\frac{1}{2} \overline{H_{ij}} \zeta^i \wedge \bar{\zeta}^j$$

and

$$(4.3.6) \quad d\zeta^i = (\delta_j^i \alpha - \omega_j^i) \wedge \zeta^j .$$

After an application of Cartan’s Lemma, these relations yield

$$(4.3.7) \quad dH_{ij} - (\alpha + \bar{\alpha}) H_{ij} + H_{ik} \omega_j^k + H_{kj} \omega_i^k = H_{ijk} \zeta^k$$

for some functions $H_{ijk} = H_{jik} = H_{ikj}$ on F_Q^1 .

In particular, the complex-valued quadratic form $\mathcal{H} = H_{ij} \zeta^i \bar{\zeta}^j$ is well-defined on Q , as are its real and imaginary parts.

Example 1 (Standard null quadric). When restricted to $SO(n+2) \subset SL(n+2, \mathbb{R})$, the map $[f_0]$ has image equal to the standard null quadric, whose homogeneous equation is

$$(4.3.8) \quad (z^0)^2 + (z^1)^2 + \dots + (z^{n+1})^2 = 0.$$

In this case, the structure matrix φ reduces to

$$(4.3.9) \quad \begin{bmatrix} i\omega_{n+1}^0 & -\frac{1}{2}\bar{\zeta}_i & \zeta^{n+1} \\ \zeta^j & \omega_i^j & \bar{\zeta}^j \\ \zeta^{n+1} & -\frac{1}{2}\zeta_i & -i\omega_{n+1}^0 \end{bmatrix}, \quad \text{where} \quad \omega_j^i + \omega_i^j = 0.$$

In particular, $H = I_n$ in this case.

With this example in mind, the following definition will be adopted:

Definition 4. A smooth embedded complex hypersurface $Q \subset X_{n+1}$ that is transverse to the λ -fibers will be said to be *convex* if the function $\text{Re}(H)$ on F_Q^1 takes values in positive definite matrices, or, equivalently, if the quadratic form $\text{Re}(\mathcal{H})$ is positive definite on Q .

It is an elementary exercise to check that the condition that Q be convex is equivalent to the condition that (Σ_Q, ι_Q) satisfies the local convexity condition given in Definition 1 that is needed to ensure that it be a generalized Finsler structure on $\tilde{\mathbb{P}}^{n+1}$. Unfortunately, it need not satisfy the fiber-connectedness hypotheses given in Definition 1, so this is only a generalized Finsler structure in the wider sense.

The way is now prepared for stating the main result of this section, which generalizes the construction in [8] for $n = 1$ that was based on an idea of Funk [17].

Theorem 2. *Suppose that the complex hypersurface $Q \subset X_{n+1}$ is transverse and convex. Then the generalized Finsler structure (Σ_Q, ι_Q) on $\tilde{\mathbb{P}}^{n+1}$ is rectilinear and has constant flag curvature $+1$.*

Conversely, for any rectilinear generalized Finsler structure (Σ, ι) on $\tilde{\mathbb{P}}^{n+1}$ with constant flag curvature $+1$, the canonical geodesic map $\ell : \Sigma \rightarrow X_{n+1}$ has, as its image, a complex (immersed) hypersurface Q that is transverse and convex.

The proof of the first half of the theorem will be given in the following subsection. (Afterwards, the proof of the converse statement can safely be left to the reader.) Note that the example of the standard null quadric, Example 1, must correspond to the Riemannian metric of constant curvature 1 on $\tilde{\mathbb{P}}^{n+1} \simeq S^{n+1}$. A further discussion of examples will be taken up after the proof.

4.4. Structure reduction. Assume for the rest of this section that Q is convex.

The equation $\operatorname{Re}(H) = I_n$ defines a sub-bundle $F_Q^2 \subset F_Q^1$ whose structure group is the group $G_2 \subset G_1$ consisting of the matrices of the form (4.3.1) with $a\bar{a} = 1$ and $A \in \operatorname{SO}(n)$. Thus $G_2 \simeq S^1 \times \operatorname{SO}(n)$. Henceforth, all functions and forms will be regarded as pulled back to F_Q^2 , though, as is customary in moving frame calculations, this pullback will not be notated.

It will be useful to separate ζ^i into its real and imaginary parts, so introduce real-valued forms ω_i and θ_{0i} for $1 \leq i \leq n$ by the equations¹¹

$$(4.4.1) \quad \zeta^i = \omega_i - i\theta_{0i}.$$

Now the equation

$$(4.4.2) \quad H = I_n + iY$$

holds, where Y is symmetric and real-valued. Define 1-forms ρ and ω_0 so as to separate α into its real and imaginary parts as

$$(4.4.3) \quad \alpha = \rho - i\omega_0.$$

Then (4.3.5) becomes

$$(4.4.4) \quad d(\rho - i\omega_0) = -\frac{1}{2}(\delta_{ij} - iY_{ij})\zeta^i \wedge \bar{\zeta}^j.$$

Separating this equation into its real and imaginary parts yields

$$(4.4.5) \quad d\rho = \frac{i}{2}Y_{ij}\zeta^i \wedge \bar{\zeta}^j = Y_{ij}\theta_{0i} \wedge \omega_j$$

and

$$(4.4.6) \quad d\omega_0 = -\frac{i}{2}\zeta^i \wedge \bar{\zeta}^i = -\theta_{0i} \wedge \omega_i.$$

Now (4.3.7) can be written in the form

$$(4.4.7) \quad dH_{ij} = (\delta_{ik}\rho - \omega_i^k)H_{kj} + (\delta_{kj}\rho - \omega_j^k)H_{ik} + H_{ijk}\zeta^k.$$

It will be useful to separate this into its real and imaginary parts. First, set

$$(4.4.8) \quad H_{ijk} = 2(J_{ijk} + iI_{ijk}),$$

¹¹ This notation is chosen so as to agree with the notation in earlier sections.

where I_{ijk} and J_{ijk} are real-valued and then define new 1-forms $\theta_{ij} = -\theta_{ji}$ and $\sigma_{ij} = \sigma_{ji}$ by the relations

$$(4.4.9) \quad \delta_{ij} \rho - \omega_j^i = -\theta_{ij} - \sigma_{ij} .$$

The real part of (4.4.7) can now be written in the form

$$(4.4.10) \quad \sigma_{ij} = \frac{1}{2} \operatorname{Re}(H_{ijk} \zeta^k) = J_{ijk} \omega_k + I_{ijk} \theta_{0k} .$$

Moreover, the structure equation $d\zeta = (\delta_j^i \alpha - \omega_j^i) \wedge \zeta^j$ separates into real and imaginary parts as

$$(4.4.11) \quad \begin{aligned} d\omega_i &= \theta_{0i} \wedge \omega_0 - \theta_{ij} \wedge \omega_j - I_{ijk} \theta_{0k} \wedge \omega_j , \\ d\theta_{0i} &= \omega_0 \wedge \omega_i - \theta_{ij} \wedge \theta_{0j} + J_{ijk} \theta_{0k} \wedge \omega_j , \end{aligned}$$

The reader will recognize equations (4.4.6) and (4.4.11) as the structure equations of the canonical $SO(n)$ -bundle of a generalized Finsler structure of constant flag curvature 1.

Of course, there needs to be a base manifold of dimension $n+1$, but this is easily constructed: Note that, by the structure equations and definitions so far

$$(4.4.12) \quad d\mathbf{e}_0 = d(\operatorname{Re}(\mathbf{f}_0)) = \mathbf{e}_0 \rho + \mathbf{e}_i \omega_i + \mathbf{e}_{n+1} \omega_0 .$$

Thus, the mapping $[\mathbf{e}_0]_+ : F_Q^2 \rightarrow \tilde{\mathbb{P}}^{n+1}$ is a submersion and its fibers are (unions of) leaves of the Frobenius system $\omega_0 = \omega_1 = \dots = \omega_n = 0$. Moreover, the fibers of the map $[\mathbf{f}_0]_+ : F_Q^2 \rightarrow \mathbb{S}$ are the $SO(n)$ -orbits on F_Q^2 and the image of this map is Σ_Q , by definition.

It now follows from the structure equations that (Σ_Q, ι_Q) is a rectilinear generalized Finsler structure on $\tilde{\mathbb{P}}^{n+1}$ with constant flag curvature $+1$, as desired.

4.5. Examples. It is now time to consider some examples of transverse, convex hypersurfaces.

Example 2 (Non-real Hyperquadrics). Let $Q \subset X_{n+1}$ be a hypersurface so that the induced generalized Finsler structure is actually a Finsler structure on $\tilde{\mathbb{P}}^{n+1}$. By construction, this means that Q is compact and hence algebraic. Moreover, since each geodesic in $\tilde{\mathbb{P}}^{n+1}$ occurs with two orientations, it follows that Q must meet each λ -fiber transversely in two points. It follows that $Q \subset \mathbb{C}\mathbb{P}^{n+1}$ has degree two, i.e., is a hyperquadric and has no real points.

Now, a hyperquadric Q with no real points is $SL(n+2, \mathbb{R})$ -equivalent to a unique hyperquadric of the form

$$(4.5.1) \quad (z^0)^2 + e^{ip_1} (z^1)^2 + \dots + e^{ip_{n+1}} (z^{n+1})^2 = 0,$$

where $p = (p_1, \dots, p_{n+1})$ is a real vector satisfying

$$(4.5.2) \quad 0 = p_0 \leq p_1 \leq \dots \leq p_{n+1} < \pi.$$

Conversely, it is not difficult to see that the quadric Q_p defined by (4.5.1) where the p_i are subject to (4.5.2) is both transverse and convex. Moreover, it is easy to see that distinct values of p give rise to non-isometric Finsler structures.

Thus, this provides an $(n+1)$ -parameter family of distinct, rectilinear Finsler structures with constant flag curvature 1 on $S^{n+1} = \tilde{\mathbb{P}}^{n+1}$.

Only the case $p = (0, \dots, 0)$ is Riemannian. When the p_i (including $p_0 = 0$) are distinct, the group of isometries of the corresponding Finsler metric is discrete, but it has positive dimension when two or more of the p_i are equal.

Example 3 (Prescribed Indicatrix). Theorem 2 can be used to construct rectilinear Finsler structures with constant flag curvature 1 and a prescribed tangent indicatrix at one point. In fact, one has the following result:

Proposition 4. *Let $[v]_+ \in \tilde{\mathbb{P}}^{n+1}$ be any point and let $S \subset T_{[v]_+} \tilde{\mathbb{P}}^{n+1}$ be a compact, real-analytic hypersurface that is strictly convex towards the origin in $T_{[v]_+} \tilde{\mathbb{P}}^{n+1}$. Then there is an open neighborhood U of $[v]_+$ in $\tilde{\mathbb{P}}^{n+1}$, together with a Finsler structure (Σ, ι) on U , so that $\iota(\Sigma) \subset TU$ contains S and so that (Σ, ι) is rectilinear and has constant flag curvature 1.*

PROOF. Choose a hyperplane $W \subset \mathbb{R}^{n+2}$ that is transverse to the line $[v]$ and set $\hat{S} = \{[v + iw]_+ \in \mathbb{S} \mid w \in W, \iota([v + iw]_+) \in S\}$. Of course, \hat{S} is diffeomorphic to $S \simeq S^n$. The image $\ell(\hat{S}) \subset X_{n+1}$ is a totally real, real analytic n -dimensional submanifold of X_{n+1} whose complexified tangent space is transverse to the fibers of λ . Thus, there exists a unique complex hypersurface $Q \subset X_{n+1}$ that contains $\ell(\hat{S})$. By restricting Q to a sufficiently small tubular neighborhood of $\ell(\hat{S})$ (in some metric on X_{n+1}), one can assume that Q is embedded and everywhere transverse to the fibers of λ (since it is along $\ell(\hat{S})$). Moreover, the hypothesis that S is strictly convex towards the origin in $T_{[v]_+} \tilde{\mathbb{P}}^{n+1}$ implies that Q is convex (in the sense of Definition 4) on a neighborhood of $\ell(\hat{S})$, so by shrinking Q again if necessary, one can assume that Q is convex everywhere.

Consider the corresponding (Σ_Q, ι_Q) , which is a rectilinear generalized Finsler structure on $\tilde{\mathbb{P}}^{n+1}$ with constant flag curvature 1. By construction, the fiber $\Sigma_{[v]_+} = \hat{S}$ is compact and convex. It is now not difficult to see that there must be an open neighborhood U of $[v]_+$ in $\tilde{\mathbb{P}}^{n+1}$ with the property that, for all $[v']_+ \in U$, the fiber $\Sigma_{[v']_+}$ is also compact and nonempty. This U is the desired neighborhood. \square

Remark 4 (A more general construction). Note that the argument in the proof does not construct all of the possible rectilinear Finsler structures on a neighborhood of $[v]_+$ with constant flag curvature 1 and with the given tangent indicatrix at the point $[v]_+$.

In fact, if $\lambda : \hat{S} \rightarrow \mathbb{R}$ is any real analytic function, set

$$(4.5.3) \quad \hat{S}_\lambda = \left\{ [v + i(w + \lambda v)]_+ \in \mathbb{S} \mid [v + iw]_+ \in \hat{S} \right\}.$$

Then one can use $\ell(\hat{S}_\lambda)$ instead of $\ell(\hat{S})$ to generate a complex hypersurface and the construction proceeds as before. This more general construction does give all of the the possible rectilinear Finsler structures on a neighborhood of $[v]_+$ with constant flag curvature 1 with the given tangent indicatrix at the point $[v]_+$.

Of course, these methods do not give any easy method to estimate how large the domain U will be.

In some sense, this construction is the positive curvature analog of Hilbert's construction of rectilinear Finsler metrics with constant flag curvature -1 on convex domains in \mathbb{R}^{n+1} .

5. GENERALITY

5.1. The case of dimension 2. For comparison, the local description of generalized Finsler metrics on surfaces with constant flag curvature 1 will be recalled from [6].

The structure equations in case $n = 1$ take the form

$$(5.1.1) \quad \begin{aligned} d\omega_0 &= -\theta_{01} \wedge \omega_1, \\ d\omega_1 &= -\omega_0 \wedge \theta_{01} - I \theta_{01} \wedge \omega_1 = -(\omega_0 - I \omega_1 + J \theta_{01}) \wedge \theta_{01}, \\ d\theta_{01} &= \omega_0 \wedge \omega_1 + J \theta_{01} \wedge \omega_1 = (\omega_0 - I \omega_1 + J \theta_{01}) \wedge \omega_1, \end{aligned}$$

where, throughout this subsection, I_{111} and J_{111} will be written as I and J , respectively. These are the structure equations on the $O(1)$ -structure F over Σ . By passing to a double cover if necessary, it will be assumed that these equations hold on Σ itself.

Assuming that Σ is geodesically simple with geodesic projection $\ell : \Sigma \rightarrow Q$, Proposition 3 implies that, not only do there exist a metric $d\sigma^2$ and area form Ω on Q satisfying

$$(5.1.2) \quad \ell^*(\Omega) = \theta_{01} \wedge \omega_1, \quad \ell^*(d\sigma^2) = \theta_{01}^2 + \omega_1^2,$$

but there also exists a 1-form β on Q satisfying

$$(5.1.3) \quad \ell^*(\beta) = -I \omega_1 + J \theta_{01}.$$

A glance at (5.1.1) coupled with knowledge of the structure equations of a Riemannian metric shows that

$$(5.1.4) \quad d\beta = (1 - K)\Omega$$

where K is the Gauss curvature of the metric $d\sigma^2$.

Conversely, suppose that one has a surface Q endowed with a metric $d\sigma^2$ with Gauss curvature K , an area form Ω , and a 1-form β that satisfies $d\beta = (1 - K)\Omega$. Let $\ell : \Sigma \rightarrow Q$ be the oriented orthonormal frame bundle of Q endowed with the metric $d\sigma^2$ and orientation Ω . Then the usual tautological and connection forms $\eta_1, \eta_2, \eta_{12}$ defined on Σ satisfy

$$(5.1.5) \quad \ell^*(d\sigma^2) = \eta_1^2 + \eta_2^2, \quad \ell^*(\Omega) = \eta_1 \wedge \eta_2, \quad \ell^*(\beta) = -I\eta_2 + J\eta_1,$$

for some functions I and J on Σ , the structure equations

$$(5.1.6) \quad d\eta_1 = -\eta_{12} \wedge \eta_2, \quad d\eta_2 = \eta_{12} \wedge \eta_1, \quad d\eta_{12} = \ell^*(K)\eta_1 \wedge \eta_2,$$

and the equation

$$(5.1.7) \quad d(-I\eta_2 + J\eta_1) = (1 - \ell^*(K))\eta_1 \wedge \eta_2.$$

Consequently, setting

$$(5.1.8) \quad \omega_0 = -\eta_{12} + I\eta_2 - J\eta_1, \quad \omega_1 = \eta_2, \quad \theta_{01} = \eta_1,$$

yields a coframing on Σ that satisfies the structure equations for a generalized Finsler structure with constant flag curvature 1.

Thus, the local prescription for generalized Finsler surfaces with constant flag curvature 1 is equivalent to prescribing data on a surface: a metric $d\sigma^2$, its area form Ω , and a 1-form β that satisfies the equation $d\beta = (1 - K)\Omega$. Up to local isometry, a metric $d\sigma^2$ on a surface depends on one arbitrary function of two variables and the 1-form β is determined up to the addition of an exact 1-form df , which is also one function of two variables.

Thus, (local) generalized Finsler structures for surfaces with constant flag curvature 1 depend on two arbitrary functions of two variables.

5.1.1. *β -geodesics.* Generally, given a metric $d\sigma^2$ with area form Ω on a surface Q and a 1-form β , a curve $\gamma \subset Q$ that satisfies $\kappa_\gamma ds = \beta|_\gamma$ will be called a *β -geodesic* with respect to $(d\sigma^2, \Omega)$. Here, κ_γ represents the geodesic curvature of γ when one fixes an orientation of γ . Of course, reversing the orientation of γ reverses the sign of both its arc length ds and its geodesic curvature κ_γ , so the expression $\kappa_\gamma ds$ is unchanged.

The orientation of the surface is significant: A curve γ is a *β -geodesic* with respect to $(d\sigma^2, \Omega)$ if and only if it is a *$(-\beta)$ -geodesic* with respect to $(d\sigma^2, -\Omega)$.

Just as in the case of ordinary geodesics (i.e., the 0-geodesics), there is a unique β -geodesic with respect to $(d\sigma^2, \Omega)$ with any given initial point and direction on the surface Q .

The 1-form β is sometimes called the “magnetic field” for particles moving on Q .

5.1.2. *CFC 2-spheres.* Now return to the case of a geodesically simple generalized Finsler structure $\ell : \Sigma \rightarrow Q$ endowed with a coframing $(\omega_0, \omega_1, \theta_{01})$ satisfying (5.1.1). Define $d\sigma^2$, Ω , and β on Q by (5.1.5).

The leaves of the system $\omega_0 = \omega_1 = 0$ on Σ , i.e., the fibers of a realization $\pi \circ \iota : \Sigma \rightarrow M$ as a generalized Finsler structure on a surface M^2 , are then mapped to the β -geodesics with respect to $(d\sigma^2, \Omega)$

For example, when $\beta = 0$, these curves are geodesics. Of course, the condition $\beta = 0$ implies that $K = 1$, so that these are just the geodesics on a standard 2-sphere Q of constant Gauss curvature 1. The corresponding M is just the 2-sphere of oriented geodesics on the standard 2-sphere. More interesting examples will be constructed below.

In general, if the data $(d\sigma^2, \Omega, \beta)$ on Q has the property that the β -geodesics with respect to $(d\sigma^2, \Omega)$ are all closed, then they lift to closed curves in Σ regarded as the unit tangent bundle of Q and the quotient surface M will exist globally.

There are now two elementary results to note. Each is a calculation that can be left to the reader.

Proposition 5. *Let Q be a surface endowed with a metric $d\sigma^2$, an area form Ω , and a 1-form β . For any function $L > 0$ on Q define*

$$(5.1.9) \quad d\bar{\sigma}^2 = L d\sigma^2, \quad \bar{\Omega} = L \Omega, \quad \bar{\beta} = \beta + *d(\log \sqrt{L}).$$

Then the $\bar{\beta}$ -geodesics with respect to $(d\bar{\sigma}^2, \bar{\Omega})$ are the same as the β -geodesics with respect to $(d\sigma^2, \Omega)$. □

Proposition 6. *Let Q be a surface endowed with a metric $d\sigma^2$ with Gauss curvature $K > 0$ and area form Ω . Then the data*

$$(5.1.10) \quad d\bar{\sigma}^2 = K d\sigma^2, \quad \bar{\Omega} = K \Omega, \quad \bar{\beta} = *d(\log \sqrt{K}),$$

satisfy $d\bar{\beta} = (1 - \bar{K})\bar{\Omega}$, where \bar{K} is the Gauss curvature of $d\bar{\sigma}^2$. □

Recall that a metric $d\sigma^2$ on the 2-sphere is said to be a *Zoll metric* (see [5, Chapter 4]) if all of its geodesics are closed and of length 2π . It is elementary to show that, in this case, the space of oriented $d\sigma^2$ -geodesics is itself a 2-sphere M .

In resolving a question of Funk, Guillemin [19] has shown that there exist many Zoll metrics near the metric of constant Gauss curvature 1 on S^2 . See [5, Chapter 4] for another account and further discussion of related problems.

Theorem 3. *Let $d\sigma_0^2$ be a Zoll metric on $Q = S^2$ with positive Gauss curvature K_0 . Let Ω_0 be the area form of $d\sigma_0^2$ and let $M \simeq S^2$ be the space of oriented $d\sigma_0^2$ -geodesics on Q . Then there exists a unique Finsler structure $\Sigma \subset TM$ on M with constant flag curvature 1 whose geodesic projection $\ell : \Sigma \rightarrow Q$ has the induced data*

$$(5.1.11) \quad d\sigma^2 = K_0 d\sigma_0^2, \quad \Omega = K_0 \Omega_0, \quad \beta = *d(\log \sqrt{K_0}).$$

PROOF. By hypothesis, the 0-geodesics of $(d\sigma_0^2, \Omega_0)$ are all closed, so, by Proposition 5, the β -geodesics of $(d\sigma^2, \Omega)$ (which are the same) are also closed. Moreover, by Proposition 6, the data $(d\sigma^2, \Omega, \beta)$ satisfy $d\beta = (1 - K)\Omega$ where K is the Gauss curvature of $d\sigma^2$. By the discussion at the beginning of this subsection, there is a canonically constructed coframing $(\omega_0, \omega_1, \theta_{01})$ on $\ell : \Sigma \rightarrow Q$, the unit tangent bundle of $d\sigma^2$ over Q , that satisfies the structure equations (5.1.1) of a generalized Finsler structure of constant flag curvature 1 and that induces the given data $(d\sigma^2, \Omega, \beta)$ on Q , its space of geodesics. Because its foliation \mathcal{M} given by $\omega_0 = \omega_1 = 0$ has closed leaves and, in fact, has M as its leaf space, Proposition 1 shows that there is an immersion $\iota : \Sigma \rightarrow TM$ that realizes Σ as a generalized Finsler structure on M . The reader can easily check that Σ is, in fact, an embedding and defines a genuine Finsler structure on M , as desired. \square

Remark 5 (Other global possibilities). Theorem 3 provides one way to construct data $(d\sigma^2, \Omega, \beta)$ on S^2 satisfying $d\beta = (1 - K)\Omega$ and the condition that the β -geodesics with respect to $(d\sigma^2, \Omega)$ be closed.

Note that this Zoll construction only produces data $(d\sigma^2, \Omega, \beta)$ with $d(*\beta) = 0$. In fact, by writing $*\beta = du$ for some function u (uniquely determined up to an additive constant), one can recover the original Zoll metric from this data by dividing $d\sigma^2$ by e^{2u} . Thus, the Finsler structure $\Sigma \subset TM$ determines the original Zoll metric. Consequently, Theorem 3 provides an injection of the set of isometry classes of Zoll metrics with positive Gauss curvature into the set of isometry classes of Finsler metrics on S^2 with constant flag curvature 1.

The Zoll method is far from the only method of constructing global examples, though it is the most general found so far. For example, one can find other examples by assuming rotational symmetry in the data. Also, the projectively flat examples constructed in Example 2 (with $n = 1$) do not arise from the Zoll construction (except for the Riemannian one).

None of these examples (other than the Riemannian one) are reversible, i.e., $\Sigma \neq -\Sigma \subset TM$. In fact, the data $(d\sigma^2, \Omega, \beta)$ on Q give rise to a reversible Finsler structure on M if and only if there exists a fixed-point free involution $\iota : Q \rightarrow Q$ that fixes $d\sigma^2$ and reverses Ω and β . No such example with $\beta \neq 0$ is known at present (nor has it been ruled out).

5.2. The structure equations in higher dimensions. As was already mentioned in §3.4.3, a generalized Finsler structure (Σ, ι) with constant flag curvature 1 that is geodesically simple induces a torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure on the space Q of geodesics. It turns out that this construction is essentially reversible, as will now be explained. Then, in later subsections, this reversibility will be used to investigate the generality of generalized Finsler structures with constant flag curvature 1.

For the rest of this section, the assumption $n > 1$ will be in force.

5.2.1. A circle of totally real n -planes. Since $S^1 \cdot \text{GL}(n, \mathbb{R})$ is a subgroup of $\text{GL}(n, \mathbb{C})$ (assuming their standard embeddings into $\text{GL}(2n, \mathbb{R})$), a torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure on a $2n$ -manifold Q underlies an integrable almost complex structure. Geometrically, the reduction from an integrable almost complex structure to an $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure is represented by the choice of a totally real n -plane in each tangent space, defined up to multiplication by $e^{i\theta}$. Equivalently, one has a subbundle $R \subset \text{Gr}(n, TQ)$ of totally real tangent n -planes $E \subset T_q Q$ (i.e., $E \cap iE = \{0_q\}$) for which the fiber over each point $R_q \subset R$ consists of the complex multiples of single totally real n -plane.

Conversely, the choice of such a circle bundle $R \subset \text{Gr}(n, TQ)$ over a complex n -manifold Q defines a $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure $q : F \rightarrow Q$: A coframing $u : T_q Q \rightarrow \mathbb{C}^n$ belongs to the structure F if and only if u carries the elements of the fiber R_q to the n -planes $e^{i\theta} \mathbb{R}^n$.

Given such a circle bundle $R \subset \text{Gr}(n, TQ)$, a n -dimensional submanifold $P \subset Q$ will be said to *belong* to R if its tangent plane at every point is an element of R . Belonging to R is an overdetermined system of first order partial differential equations for submanifolds $P \subset Q$. If $P \subset Q$ belongs to R , then it has a canonical lifting $\tau : P \rightarrow R$ defined by $\tau(q) = T_q P$ for $q \in P$. This will be called the *tangential lifting* of P .

It is easy to see that, for every n -plane $E \in R$, there is at most one connected n -dimensional submanifold $P \subset Q$ that belongs to R and has E as its tangent plane. (This uses the hypothesis $n > 1$.) The bundle R and, by association, the corresponding $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure $q : F \rightarrow Q$ will be said to be *integrable* if every element of R actually is tangent to an n -manifold that belongs to R . The

condition of being integrable is equivalent to the condition that R be foliated by the tangential lifts of the n -manifolds that belong to R .

Example 4 (Generalized Finsler structures). If (Σ, ι) is a generalized Finsler structure on M^{n+1} with constant flag curvature 1 that is geodesically simple, with geodesic projection $q : \Sigma \rightarrow Q$, then the images $q(\Sigma_x) \subset Q$ for $x \in M$ belong to the canonical torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure constructed in §3.4.3. Their liftings foliate an open set in the associated circle bundle R and, in fact, R is integrable, as will be seen below.

5.2.2. *Torsion-free structures.* An $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure $q : F \rightarrow Q$ will be said to be *torsion-free* if it admits a connection without torsion.

Denote the Lie algebra of $S^1 \cdot \text{GL}(n, \mathbb{R}) \subset \text{GL}(2n, \mathbb{R})$ by $\mathfrak{t} \oplus \mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{gl}(2n, \mathbb{R})$. It is straightforward to compute that the first prolongation¹² of this subalgebra of $\mathfrak{gl}(2n, \mathbb{R})$ vanishes (this uses the assumption $n > 1$). Consequently, if $q : F \rightarrow Q$ does admit a torsion-free connection, it admits only one.

It will be necessary to examine the structure equations of F in the torsion-free case, in particular, to compute the space of curvature tensors of torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structures.

Let $\zeta = (\zeta^i)$ be the tautological \mathbb{C}^n -valued 1-form on F . The assumption that F be torsion-free is equivalent to assuming that there exist on F a 1-form ω_0 and a $\mathfrak{gl}(n, \mathbb{R})$ -valued 1-form $\phi = (\phi_j^i)$ so that the *first structure equation*

$$(5.2.1) \quad d\zeta = -(\mathfrak{i}\omega_0 \mathbf{I}_n + \phi) \wedge \zeta$$

holds. These forms ω_0 and ϕ are the connection forms of the structure.

The *second structure equation* will give expressions for the curvature forms

$$(5.2.2) \quad \Omega_0 = d\omega_0, \quad \Phi = d\phi + \phi \wedge \phi,$$

that are based on the *first Bianchi identity*

$$(5.2.3) \quad 0 = -(\mathfrak{i}\Omega_0 \mathbf{I}_n + \Phi) \wedge \zeta,$$

which is derived by computing the exterior derivative of (5.2.1). This computation, which is left to the reader, has the following result.

Proposition 7 (Second structure equations). *If $n > 2$, there exist on F real-valued functions $b_{ij} = b_{ji}$ and $r_{jkl}^i = r_{kjl}^i = r_{jlk}^i$ so that*

$$(5.2.4) \quad \begin{aligned} \Omega_0 &= -\mathfrak{i} b_{kl} \zeta^k \wedge \bar{\zeta}^l, \\ \Phi_j^i &= b_{jl} (\zeta^i \wedge \bar{\zeta}^l + \bar{\zeta}^i \wedge \zeta^l) + \mathfrak{i} r_{jkl}^i \zeta^k \wedge \bar{\zeta}^l. \end{aligned}$$

¹² See [7] for information related to prolongation.

If $n = 2$, there exist on F , in addition to the real-valued functions $b_{ij} = b_{ji}$ and $r_{jkl}^i = r_{kjl}^i = r_{jlk}^i$, a complex-valued function A and a real-valued function a so that

(5.2.5)

$$\begin{aligned} \Omega_0 &= \text{Im} (A \zeta^1 \wedge \zeta^2) + 3a (\zeta^1 \wedge \bar{\zeta}^2 - \zeta^2 \wedge \bar{\zeta}^1) - i b_{kl} \zeta^k \wedge \bar{\zeta}^l, \\ \Phi_j^i &= \delta_j^i \text{Re} (A \zeta^1 \wedge \zeta^2) + b_{jl} (\zeta^i \wedge \bar{\zeta}^l + \bar{\zeta}^i \wedge \zeta^l) + i (r_{jkl}^i + a(\epsilon_{jk} \delta_l^i + \epsilon_{jl} \delta_k^i)) \zeta^k \wedge \bar{\zeta}^l. \end{aligned}$$

where $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{12} = 1$. □

Remark 6 (Prolongation algebra). Let V be an abstract real vector space of dimension n with complexification $V^{\mathbb{C}}$. The algebra $\mathfrak{gl}(V)$ is naturally included into $\mathfrak{gl}(V^{\mathbb{C}})$ and one can consider the Lie algebra $\mathfrak{g} = \mathbb{C} \cdot \text{I}_{V^{\mathbb{C}}} + \mathfrak{gl}(V)$ as a (real) sub-algebra of $\mathfrak{gl}(V^{\mathbb{C}})$. This is a proper subalgebra as long as $n > 1$.

It has already been remarked that, when $n > 1$, the first prolongation vanishes: $\mathfrak{g}^{(1)} = 0$. Proposition 7 computes $\mathcal{K}(\mathfrak{g})$, the space of curvature tensors of a torsion-free \mathfrak{g} -connection. The result is

$$(5.2.6) \quad \mathcal{K}(\mathfrak{g}) = \begin{cases} S^2(V^*) \oplus V \otimes S^3(V^*), & \text{when } n > 2, \\ S^2(V^*) \oplus V \otimes S^3(V^*) \oplus \mathbb{C} \oplus \mathbb{R}, & \text{when } n = 2. \end{cases}$$

Note that the generic element in $S^2(V^*) \oplus V \otimes S^3(V^*) \subset \mathcal{K}(\mathfrak{g})$ does not lie in $\mathcal{K}(\mathfrak{h})$ for any proper sub-algebra $\mathfrak{h} \subset \mathfrak{g}$, so a \mathfrak{g} -connection whose curvature assumes such a generic value will have holonomy equal to the full group $S^1 \cdot \text{GL}(n, \mathbb{R})$. Thus, Berger’s first criterion for $S^1 \cdot \text{GL}(n, \mathbb{R})$ to exist as the holonomy of a torsion-free connection is satisfied.

Corollary 1 (Integrability). *When $n > 2$, a torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure $q : F \rightarrow Q$ is integrable. When $n = 2$, such a structure is integrable if and only if the functions A and a vanish identically on F .*

PROOF. The integrability condition is equivalent to the condition that the Pfaffian system on F generated by ω_0 and the components of $\text{Im}(\zeta)$ be Frobenius. By Proposition 7, this condition is satisfied when $n > 3$ and is satisfied when $n = 2$ if and only if $A = a = 0$. □

Since the only $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structures that arise in the study of generalized Finsler structures with constant flag curvature 1 are integrable and torsion-free, only the integrable, torsion-free case will be considered further in this article. In order to have a uniform notation, let $\mathcal{K}_o(\mathfrak{g}) \subset \mathcal{K}(\mathfrak{g})$ denote the subspace consisting of the tensors of integrable, torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structures. Thus $\mathcal{K}_o(\mathfrak{g}) \simeq S^2(V^*) \oplus V \otimes S^3(V^*)$ for all $n \geq 2$.

For an integrable, torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure $q : F \rightarrow Q$, the structure equations derived so far can be written in the form

$$\begin{aligned}
 (5.2.7) \quad & d\zeta^i = -(\mathfrak{i} \delta_j^i \omega_0 + \phi_j^i) \wedge \zeta^j \\
 & d\omega_0 = -\mathfrak{i} b_{kl} \zeta^k \wedge \bar{\zeta}^l, \\
 & d\phi_j^i + \phi_k^i \wedge \phi_j^k = b_{jl} (\zeta^i \wedge \bar{\zeta}^l + \bar{\zeta}^i \wedge \zeta^l) + \mathfrak{i} r_{jkl}^i \zeta^k \wedge \bar{\zeta}^l.
 \end{aligned}$$

where $b_{ij} = b_{ji}$ and $r_{jkl}^i = r_{kjl}^i = r_{ljk}^i$ are real-valued functions on F .

For later purposes, it will be necessary to understand the second Bianchi identity as well. This is computed by applying the exterior derivative to the second and third equations of (5.2.7) and working out the consequences.

The result of the computation is that there exist *complex*-valued functions $B_{ijk} = B_{jik} = B_{ikj}$ and $R_{jklm}^i = R_{kjl m}^i = R_{jlk m}^i = R_{jkm l}^i$ on F so that

$$\begin{aligned}
 (5.2.8) \quad & db_{ij} = b_{kj} \phi_i^k + b_{ik} \phi_j^k + \text{Re} (B_{ijk} \zeta^k), \\
 & dr_{jkl}^i = -r_{jkl}^m \phi_m^i + r_{mkl}^i \phi_j^m + r_{jml}^i \phi_k^m + r_{jkm}^i \phi_l^m \\
 & \quad + \text{Re} ((R_{jklm}^i - \mathfrak{i} (\delta_j^i B_{klm} + \delta_k^i B_{ljm} + \delta_l^i B_{kjm})) \zeta^m).
 \end{aligned}$$

Remark 7 (Prolongation algebra continued). In the notation of Remark 6, this calculation has the following interpretation: When $n > 2$, this second Bianchi identity calculation determines the space $\mathcal{K}^1(\mathfrak{g})$, i.e., the space of first covariant derivatives of curvature tensors of torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structures. Then formula (5.2.8) implies the isomorphism

$$(5.2.9) \quad \mathcal{K}^1(\mathfrak{g}) = S^3(V^*)^{\mathbb{C}} \oplus (V \otimes S^4(V^*))^{\mathbb{C}}.$$

When $n = 2$, this is not the calculation of $\mathcal{K}^1(\mathfrak{g})$ since the integrability condition $A = a = 0$ has been imposed. However, in this case, the formula above does describe the space of covariant derivatives of curvature tensors of *integrable* torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structures, which, it turns out, is the space that needed to be computed for applications in this article anyway, since this space is the prolongation of $\mathcal{K}_\circ(\mathfrak{g})$ (regarded as a second-level tableau) in either case.

In particular, it follows that $\mathcal{K}^1(\mathfrak{g}) \neq 0$ for all $n \geq 2$, so Berger’s second criterion for $S^1 \cdot \text{GL}(n, \mathbb{R})$ to be the holonomy of a torsion-free connection that is not locally symmetric is also satisfied.

Proposition 8 (Involutivity). *Regard $\mathcal{K}_\circ(\mathfrak{g})$ as a subspace of $\mathfrak{g} \otimes \Lambda^2((V^\mathbb{C})^*)$, i.e., as a second-level tableau. This subspace is involutive, with Cartan characters*

$$(5.2.10) \quad s_k = \begin{cases} 0, & k = 0, 1, \\ k - 1 + n(n + (n+1-k)(k-2)), & 2 \leq k \leq n+1, \\ 0, & n+1 < k \leq 2n. \end{cases}$$

The characteristic variety of this tableau consists of the covectors $\xi \in \mathbb{P}((\mathbb{C} \otimes V)^*) \simeq \mathbb{P}^{2n-1}$ of the form $\lambda \otimes \xi'$ for $\xi' \in \mathbb{P}(V^*)$ and is of degree $n+1$ in $\mathbb{P}((\mathbb{C} \otimes V)^*)$.

PROOF. This is a straightforward calculation: Choose a flag that is non-characteristic with respect to the claimed characteristic variety. One then finds that the characters of this flag are as given in (5.2.10). However, by combinatorics, one sees that, not only does one have the identity

$$(5.2.11) \quad s_2 + \cdots + s_{n+1} = \binom{n+1}{2} + n \binom{n+2}{3} = \dim \mathcal{K}_\circ(\mathfrak{g}),$$

but also that Cartan’s test is satisfied, i.e.,

$$(5.2.12) \quad 2s_2 + \cdots + (n+1)s_{n+1} = 2 \binom{n+2}{3} + 2n \binom{n+3}{4} = \dim (\mathcal{K}_\circ(\mathfrak{g}))^{(1)},$$

as was verified in the computation of the second Bianchi identity for integrable, torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structures. □

This has an immediate consequence:

Theorem 4. *Up to diffeomorphism, the local, integrable, torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structures in dimension $2n$ depend on $n(n+1)$ functions of $n+1$ variables. Moreover, for any curvature tensor in $\mathcal{K}_\circ(\mathfrak{g})$, there exists an integrable, torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure on a neighborhood of $0 \in \mathbb{R}^{2n}$ that assumes this value at 0.*

PROOF. These results follow from the usual Cartan-style construction of an exterior differential system whose integral manifolds are the the local, integrable, torsion-free $S^1 \cdot \text{GL}(n, \mathbb{R})$ -structure plus the algebraic result of Proposition 8. The proof is similar in all details to those executed in [7], to which the reader is referred if more detail is needed. □

Remark 8 (Exotic Holonomies). In [7], the existence of torsion-free $S^1 \cdot \text{GL}(2, \mathbb{R})$ -structures on 4-manifolds whose canonical connections have holonomy equal to $S^1 \cdot \text{GL}(2, \mathbb{R})$ was established and in [9], the existence of torsion-free $H_\lambda \cdot \text{SL}(2, \mathbb{R})$ -structures on 4-manifolds whose canonical connections have holonomy $H_\lambda \cdot \text{SL}(2, \mathbb{R})$ for any 1-parameter subgroup $H_\lambda \subset \mathbb{C}^*$ (other than \mathbb{R}^*) was established.

Using Proposition 8, one can similarly demonstrate the existence of torsion-free $H_\lambda \cdot \mathrm{SL}(n, \mathbb{R})$ -structures on $2n$ -manifolds whose canonical connections have holonomy equal to $H_\lambda \cdot \mathrm{SL}(n, \mathbb{R})$ for any 1-parameter subgroup $H_\lambda \subset \mathbb{C}^*$ (other than \mathbb{R}^*).

This is interesting because these holonomy groups are not on Berger's original lists of holonomies of irreducible holonomy torsion-free connections (and hence fall into the category of 'exotic' holonomies) and also were apparently overlooked in the recent classification of such holonomies by Merkulov and Schwachhöfer [22, 23].

All that remains is to tie the geometry of these structures to that of generalized Finsler structures with constant flag curvature 1. The key to doing this is structure reduction.

Note that, if $q : F \rightarrow Q$ is a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure, then the curvature 2-form Ω_0 is actually the q -pullback of a 2-form that is well-defined on Q . By abuse of notation, the symbol Ω_0 will be used to denote this 2-form on Q as well. If the structure is also integrable, then, by Proposition 7 and Corollary 1, the form Ω_0 is of type $(1, 1)$ on Q .

Say that the structure F is *positive* if the symmetric matrix $b = (b_{ij})$ takes values in positive definite matrices or, equivalently, if $-\Omega_0$ is a positive $(1, 1)$ -form on Q , i.e., it defines a Kähler structure on Q . In this case, there is a canonical $S^1 \cdot \mathrm{O}(n)$ -substructure $F_\circ \subset F$ defined by the equations $b_{ij} = \frac{1}{2}\delta_{ij}$. This will be called the *Kähler reduction* of the torsion-free, integrable $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure F .

Now the preparations have been made for the statement of the final main result of this article:

Theorem 5. *Let $q : F \rightarrow Q$ be a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ -structure, assumed integrable if $n = 2$. If $-\Omega_0$ is a positive $(1, 1)$ -form on Q , then the Kähler reduction of F defines a generalized Finsler structure with constant flag curvature 1.*

PROOF. This is a matter of computation and expansion of the definitions. The point is that if one reduces to the locus in F where $b_{ij} = \frac{1}{2}\delta_{ij}$, this clearly defines an $S^1 \cdot \mathrm{O}(n)$ -substructure $F_\circ \subset F$ as mentioned above. One can then write $\zeta_i = \omega_i - i\theta_{0i}$ and write $\phi_j^i = \theta_{ij} + \sigma_{ij}$, just as in the previous section. Then the first equation of (5.2.8) shows how one can define I_{ijk} and J_{ijk} in terms of the real and imaginary parts of B_{ijk} so that equations (3.1.8) hold. Finally, applying Proposition 1 generates the desired Finsler structure. \square

Finally, combining Theorems 4 and 5 leads to the following description of the generality of the space of Finsler metrics of constant positive flag curvature:

Corollary 2. *The space of local equivalence classes of Finsler metrics in dimension $n+1$ that have constant flag curvature 1 depends on $n(n+1)$ functions of $n+1$ variables.*

Of course, this “dependency” must be understood in the sense of Cartan-Kähler.

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