Tôhoku Math. Journ.
22(1970), 240-248.

SOME REMARKS ON MINIMAL SUBMANIFOLDS

KATSUEI KENMOTSU

(Received Jan. 26, 1970)

This note consists of three topics for minimal submanifolds. M denotes an n-dimensional manifold which is minimally immersed in an $(n+p)$-dimensional Riemannian manifold $\bar{M}^{n+p}[c]$ of constant curvature c. In the section 1 we study a linear connection $\widehat{\nabla}$ on the normal vector bundle $N(M)$ which is naturally induced from the connection of the ambient space $\bar{M}^{n+p}[c]$. Let $\widehat{R}$ be the curvature tensor of $\widehat{\nabla}$ and let σ be the square of the length of the second fundamental form of this immersion. Then it is proved that if M is compact, orientable and $\widehat{R}=0$, then

$$\int_M \sigma(\sigma - nc)dv \geqq 0,$$

where dv denotes the volume element of M. It follows that if $\sigma \leqq nc$ everywhere on M, then either

$$(1) \qquad \sigma = 0 \quad (\text{i. e., } M \text{ is totally geodesic}),$$

or

$$(2) \qquad \sigma = nc.$$

The purpose of the section 1 is to determine all minimal submanifolds in a unit sphere $S^{n+p}[1]$ satisfying $\sigma = n$ and $\widehat{R} = 0$. The result can be found in Theorem 3.

In the section 2, we study a minimal hypersurface M in $S^{n+1}[1]$. R and R_1 denotes the curvature tensor and Ricci tensor of M, respectively. We will prove that if the Ricci tensor R_1 of M satisfies the condition $R(X,Y)\cdot R_1 = 0$, then, within rotations of $S^{n+1}[1]$, M is an open submanifold of one of the Clifford minimal hypersurfaces :

$$M_{k,n-k} = S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right), \text{ for } k = 0, 1, \cdots, \left[\frac{n}{2}\right].$$

If R_1 is parallel, then R_1 satisfies $R(X,Y)\cdot R_1 = 0$. Thus this result is a generalization of a result of [4].

In the last section we remark that a pseudo-Jacobi field which is defined by Y. Tomonaga [10] is identical with a Jacobi field which is defined by J. Simons [7].

1. Normal connection of minimal submanifolds. We choose a local field of orthonormal frames $\{e_1, \cdots, e_{n+p}\}$ in $\bar{M}^{n+p}[c]$ such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. The following ranges of indices will be used throughout this paper:

$$1 \leqq A, B, C, \cdots \leqq n + p,$$

$$1 \leqq i, j, k, \cdots \leqq n,$$

$$n + 1 \leqq \alpha, \beta, \gamma, \cdots \leqq n + p.$$

With respect to the frame field of $\bar{M}^{n+p}[c]$ chosen above, let $w^1, \cdots, w^{n+p}$ be the field of dual basis and let (w^A_B) be the connection form of $\bar{M}^{n+p}[c]$. Since $w^\alpha = 0$, we can put

$$(1) \qquad w^\alpha_i = \sum_j h^\alpha_{ij} w^j, \quad h^\alpha_{ij} = h^\alpha_{ji}.$$

Since (w^α_β) defines a linear connection on the normal vector bundle $N(M)$ in $\bar{M}^{n+p}[c]$, we call it the normal connection of M. When $R^\alpha_{\beta kl}$ denotes the curvature tensor of (w^α_β), by the structure equation of $\bar{M}^{n+p}[c]$:

$$dw^A_B = -\sum_C w^A_C \wedge w^C_B + cw^A \wedge w^B,$$

we have

$$(2) \qquad \hat{R}^\alpha_{\beta kl} = \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}).$$

Throughout this section, we assume that

(*) the normal connection $\hat{\nabla}$ is trivial, i. e., $\hat{R}^\alpha_{\beta kl} = 0$.

By (3.1) of [2], we have

$$(3)\qquad -<h,\triangle h> = \sum_{\substack{\alpha,\beta,i\\ j,k,l}} (h^\alpha_{ik}h^\beta_{kj} - h^\beta_{ik}h^\alpha_{kj})(h^\alpha_{il}h^\beta_{lj} - h^\beta_{il}h^\alpha_{lj}) + \sum_{\substack{\alpha,\beta,i\\ j,k,l}} h^\alpha_{ij}h^\alpha_{kl}h^\beta_{ij}h^\beta_{kl} - nc\sum_{\alpha,i,j}(h^\alpha_{ij})^2.$$

By (2), (*) and (3), one obtains

$$(4)\qquad -<h,\triangle h> = \sum_{\alpha,\beta} S^2_{\alpha\beta} - nc\sigma,$$

where $S_{\alpha\beta} = \sum_{i,j} h^\alpha_{ij}h^\beta_{ij}$ and $\sigma = \sum_{i,j,\alpha}(h^\alpha_{ij})^2$. Since the $(p\times p)$-matrix $(S_{\alpha\beta})$ is symmetric, it can be assumed to be diagonal for a suitable choice of $e_{n+1},\cdots,e_{n+p}$. Setting $S_\alpha = S_{\alpha\alpha}$ $(\geqq 0)$, (4) may be rewritten as follows:

$$(5)\qquad \begin{aligned} -<h,\triangle h> &= \sum_\alpha S^2_\alpha - nc\sigma \\ &= \left(\sum_\alpha S_\alpha\right)^2 - \sum_{\alpha\neq\beta} S_\alpha S_\beta - nc\sigma \\ &\leqq \sigma^2 - nc\sigma. \end{aligned}$$

Thus we have

THEOREM 1. *Let M be an n-dimensional compact oriented manifold which is minimally immersed in an $(n+p)$-dimensional Riemannian manifold of constant curvature c. If the normal connection of M is trivial, then*

$$\int_M \sigma(\sigma - nc)dv \geqq 0.$$

PROOF. This follows immediately from (5) and the Lemma 2 of [2] or (6.18) of [1].

From the Theorem 1 we have easily the following Corollary 1.

COROLLARY 1. *Let M be a compact oriented manifold minimally immersed in a space $\bar{M}^{n+p}[c]$ of constant curvature c. If $\hat{R}^\alpha_{\beta kl}=0$, then either M is totally geodesic in $\bar{M}^{n+p}[c]$, or $\sigma = nc$ (>0) or at some point $x\in M$, $\sigma(x) > nc$.*

When we study minimal submanifolds with $\sigma = nc\ (>0)$ in $\bar{M}^{n+p}[c]$ we may assume that $c=1$ and $\sigma = n$. To state the proposition 1 we prepare the notion of M-index of a minimal submanifold which is defined by T. Ōtsuki: For any $x \in M$, we denote the normal space to M_x in $\bar{M}^{n+p}[c]_x$ by N_x. For a frame $b = (x, e_1, \cdots, e_n, \cdots, e_{n+p})$ we define a linear mapping ψ_b from N_x into the space of all $n \times n$ symmetric matrices by

$$\psi_b\left(\sum_\alpha \xi_\alpha e_\alpha\right) = \left(\sum_\alpha \xi_\alpha h^\alpha_{ij}\right).$$

Then we call dim $(\psi_b(N_x))$ M-index of a minimal submanifold M in $\bar{M}^{n+p}[c]$ at x.

PROPOSITION 1. *Let M be an n-dimensional minimal submanifold immersed in an $(n+p)$-dimensional Riemannian manifold $\bar{M}^{n+p}[1]$ of constant curvature* 1. *If M satisfies the condition* (*) *and* $\sigma = n$, *then M-index is* 1 *everywhere.*

PROOF. Since $\sigma =$ constant we have (see, p.42 of [1])

$$(6) \qquad h^\alpha_{ijk} = 0 \quad \text{and} \quad \langle h, \triangle h \rangle = 0,$$

where h^α_{ijk} is, by the definition,

$$(7) \qquad \sum_k h^\alpha_{ijk} w^k = dh^\alpha_{ij} - \sum_l h^\alpha_{il} w^l_j - \sum_l h^\alpha_{lj} w^l_i + \sum_\beta h^\beta_{ij} w^\alpha_\beta.$$

By (5), (6), $c=1$ and $\sigma = n$ we have

$$(8) \qquad \sum_{\alpha \neq \beta} S_\alpha \cdot S_\beta = 0.$$

From (8), $\sigma = n$ and $S_\alpha \geqq 0$, we may assume that $S_{n+1} = n$ and $S_\alpha = 0$ for $\alpha > n+1$. By the definition of S_α, one obtains

$$(9) \qquad \begin{cases} \sum_{i,j} (h^{n+1}_{ij})^2 = n, \\ h^\alpha_{ij} = 0 \text{ for any } \alpha > n+1 \text{ and any } i, j. \end{cases}$$

Taking account of (9) and the definition of M-index, Proposition 1 follows. Q. E. D.

Using the Proposition 1, Theorem 1 of [6] and Theorem 2 of [2] we have

THEOREM 2. *Under the same assumption as the Proposition* 1,

(i) *there exists an* $(n+1)$-*dimensional totally geodesic submanifold* N^{n+1} *in* $\bar{M}^{n+p}[1]$ *containing* M *as a minimal hypersurface*

and

(ii) M *is locally a Riemannian direct product* $M \supset U = V_1 \times V_2$ *of spaces* V_1 *and* V_2 *of constant curvature*, $\dim V_1 = m \geqq 1$ *and* $\dim V_2 = n - m \geqq 1$.

Now we have easily the following global version of Theorem 2.

THEOREM 3. *Let* M *be an* n-*dimensional compact connected minimal submanifold in an* $(n+p)$-*dimensional unit sphere* $S^{n+p}[1]$. *If* M *satisfies the conditions that the normal connection of* M *is trivial and* $\sigma = n$, *then there exists an* $(n+1)$-*dimensional unit sphere* $S^{n+1}[1]$ *containing* M *as a Clifford minimal hypersurface* $M_{k,n-k}$ *for* $k = 1, 2, \cdots, \left[\frac{n}{2}\right]$.

PROOF. By Theorem 2 there exists a totally geodesic submanifold N^{n+1} which is of constant curvature 1 in $S^{n+p}[1]$. Since it is well-known [5] that the totally geodesic maximal integral submanifold of an involutive distribution on a complete Riemannian manifold is also complete for the induced metric, N^{n+1} is complete for the induced metric. Therefore we have $N^{n+1} = S^{n+1}[1]$. The latter half of the Theorem 3 follows from the following Theorem C ([2], [4]):

THEOREM C. *Let* M *be an* n-*dimensional hypersurface immersed in* $S^{n+1}[1]$. *If* $\sigma = n$, *then* M *is an open submanifold of one of the* $M_{k,n-k}$ *for* $k = 1, 2, \cdots, \left[\frac{n}{2}\right]$.

REMARK 1. By (2) a hypersurface in a Riemannian manifold of constant curvature have always $\hat{R}^{\alpha}_{\beta kl} = 0$. It follows that Theorem 3 is a generalization of Theorem 1 in [4].

2. Classification of minimal hypersurfaces with $R(X,Y)\cdot R_1 = 0$ in $S^{n+1}[1]$. For any tangent vectors X and Y, $R(X,Y)$ is an endomorphsm of the tangent space at each point. $R(X,Y)$ acts on R_1 as a derivation of the tensor algebra at each point of M. Hypersurfaces with $R(X,Y)\cdot R_1 = 0$ is studied by S. Tanno [7] and S. Tanno and T. Takahashi [8]. The following Theorem 4 is essentially a Corollary of Theorem 1 in [8].

THEOREM 4. *Let M be a connected minimal hypersurface with $R(X,Y)\cdot R_1 = 0$ in $S^{n+1}[1]$, $(n \geqq 3)$. Then, within rotations of $S^{n+1}[1]$, M is an open submanifold of one of the $M_{k,n-k}$ for $k = 0, 1, \cdots, \left[\frac{n}{2}\right]$.*

PROOF. we set $h_{ij} = h_{ij}^{n+1}$. We choose our frame field in such a way that

$$h_{ij} = 0 \quad \text{for} \quad i \neq j. \tag{10}$$

and we set $h_i = h_{ii}$. Then the condition $R(X,Y)\cdot R_1 = 0$ is written as

$$(1 + h_i h_j)(R_{ii} - R_{jj}) = 0, \tag{11}$$

where $R_{ih} = R_1(e_i, e_h)$, (see 1.3 of [9]). Taking account of the Gauss equation of M, since M is a minimal hypersurface, one obtains (cf. see 1.4 of [9])

$$R_{ij} = (n-1)\,\delta_{ij} - h_i h_j \delta_{ij}. \tag{12}$$

By (11) and (12), one obtains

$$(1 + h_i h_j)(h_i^2 - h_j^2) = 0 \text{ for any } i \neq j. \tag{13}$$

By virtue of (13), (h_{ij}) has at most two eigenvalues and we define h and k as $h = \max\{h_i\}$ (with multiplicity s) and $k = \min\{h_i\}$ (with multiplicity $(n-s)$), respectively. Taking account of Lemma 5 in [9] and the minimality of M, if M is not totally geodesic at a point x_0, then M is not totally geodesic at any point of M. If $h^2 = k^2$ $(\neq 0)$ holds at any point of M, then, by (12), M is an Einstein space. Thus M is an open submanifold of $M_{n/2,n/2}$ (see Corollary 2 of [4]). If $h^2 \neq k^2$ holds at some point $x_0 \in M$, then we have $1 + hk = 0$ at x_0 where the type number, $t(x_0)$, at x_0 is n. In [9] Tanno and Takahashi proved that if $1 + hk = 0$ at x_0 where $t(x_0)$ is n, then $1 + hk = 0$ and $t(x) = n$ hold on M.

Thus we have

$$0 = \sum_i h_i = sh + (n-s)k \text{ at any point}. \tag{14}$$

By virtue of $hk = -1$ and (14) we have $h^2 = (n-s)/s$. Therefore the square of the length of the second fundamental form (h_{ij}) is equal to

$$\sum_i h_i^2 = sh^2 + (n-s)\frac{1}{h^2} = n.$$

Theorem 4 follows immediately from the Theorem C. Q. E. D.

3. Jacobi field on a minimal submanifold. Let $\bar{M}^{n+p}$ be an $(n+p)$-dimensional Riemannian manifold and M an n-dimensional minimal submanifold in $\bar{M}^{n+p}$. $\bar{\nabla}$ (resp. ∇) denotes the linear connection for the Riemannian metric $\bar{g}$ of $\bar{M}^{n+p}$ (resp. the induced metric g of M). In the paper [7], J.Simons defined the Laplace operator on the Riemannian vector bundle. The purpose of this section is to give a decomposition formula of the Laplace operator, $\hat{\nabla}^2$, on the cross-sections in the normal vector bundle $N(M)$. The last statement in the Introduction follows easily from the decomposition formula.

$B(X,Y)$ denotes the second fundamental form, i. e., $B(X,Y)=(\bar{\nabla}_X Y)^N$. Let $\hat{\nabla}$ be the connection induced by $\bar{\nabla}$ in $N(M)$: Let V be a cross-section in $N(M)$ and $X\in M_x$. Then we can set

$$\bar{\nabla}_X V = -A^V(X) + \hat{\nabla}_X V, \tag{15}$$

where $g(A^V(X),Y)=\bar{g}(B(X,Y),V)$. We define $\triangle V$ by

$$(\triangle V)^C = V^C_{;i;j}g^{ij},\ C=1,\ 2,\cdots,n+p, \tag{16}$$

where the semicolon denotes the covariant differentiation along M. And we define $\tilde{A}(V)\in N(M)$ by (2. 2. 5) in [7], i. e.,

$$\bar{g}(\tilde{A}(V),W) = g_{ij}g^{st}(A^W)^i_s(A^V)^j_t \text{ for any } W\in N(M). \tag{17}$$

Then we have

PROPOSITION 2. *Let V be a cross-section in $N(M)$. Then $\hat{\nabla}^2 V$ for the Laplace operator $\hat{\nabla}^2$ on $N(M)$ can be decomposed in the following way,*

$$\hat{\nabla}^2 V = (\triangle V)^N + \tilde{A}(V). \tag{18}$$

PROOF. Let $\{e_1,\cdots,e_n\}$ be a basis in M_x at any point $x\in M$. Extend them to vector fields $E_1,\cdots,E_n$ in a neighborhood of x such that $g(E_i,E_j)=\delta_{ij}$ and $(\nabla_{E_i}E_j)_x=0$ at x. By Proposition 1. 2. 1 in [7] we have

$$(\hat{\nabla}^2 V)_x = \sum_{i=1}^n(\hat{\nabla}_{E_i}\hat{\nabla}_{E_i}V)_x. \tag{19}$$

For any cross-section W in $N(M)$ we have, using (15)$\sim$(19),

$$
\begin{aligned}
g((\hat{\nabla}^2 V)_x, W_x) &= \sum_{i=1}^{n} g((\hat{\nabla}_{E_i} \hat{\nabla}_{E_i} V)_x, W_x) \\
&= \sum_{i=1}^{n} \bar{g}((\bar{\nabla}_{E_i}(\bar{\nabla}_{E_i} V)^N)_x^N, W_x) \\
&= \sum_{i=1}^{n} g(\bar{\nabla}_{E_i}(\bar{\nabla}_{E_i} V + A^V(E_i))_x, W_x) \\
&= \sum_{i=1}^{n} \bar{g}((\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V)_x, W_x) \\
&\quad + \sum_{i=1}^{n} \bar{g}(B(E_i, A^V(E_i))_x, W_x) \\
&= g((\triangle V)_x, W_x) + \sum_{i=1}^{n} g((A^V(E_i))_x, (A^W(E_i))_x) \\
&= g((\triangle V + \tilde{A}(V))_x, W_x) .
\end{aligned}
$$

Thus one obtains Proposition 2. Q. E. D.

A cross-section V in $N(M)$ is called a Jacobi field [7] if it satisfies

$$\hat{\nabla}^2 V = \bar{R}(V) - \tilde{A}(V), \tag{20}$$

where $\bar{R}(V) = \sum_{i=1}^{n} (\bar{R}(E_i, V)E_i)^N$.

A cross-section V in $N(M)$ is called a pseudo-Jacobi field [10] if it satisfies, in our terminology,

$$(\triangle V)^N = \bar{R}(V) - 2\tilde{A}(V). \tag{21}$$

By (18), (20) and (21) a pseudo-Jacobi field is identical with a Jacobi field.

REMARK 2. The formula similar to (18) is seen in [3] (see (17) and (18) of [3]).

REFERENCES

[1] S. S. CHERN, Minimal submanifolds in a Riemannian manifold, Lecture note, 1968.
[2] S. S. CHERN, M. P. Do Carmo, S. KOBAYASHI, Minimal submanifolds of a sphere with second fundamental form of constant length, to appear.
[3] E. T. DAVIES, On the second and third fundamental forms of a subspace, J. London Math. Soc., 12(1937), 290-295.
[4] H. B. LAWSON, JR, Local rigidity theorems for minimal hypersurfaces, Ann. of Math., 89 (1969), 187-197.
[5] K. NOMIZU, On hypersurfaces satisfying a certain condition on the curvature tensor, Tôhoku Math. J., 20(1968), 46-59.
[6] T. ŌTSUKI, Minimal hypersurfaces in a Riemannian manifold of constant curvature, to appear.
[7] J. SIMONS, Minimal varieties in riemannian manifolds, Ann. of Math., 88(1968), 62-105.
[8] S. TANNO, Hypersurfaces satisfying a certain condition on the Ricci tensor, Tôhoku Math. J., 21(1969), 297-303.
[9] S. TANNO, T. TAKAHASHI, Some hypersurfaces of a sphere, Tôhoku Math. J., 22(1970), 212-219.
[10] Y. TOMONAGA, Pseudo-Jacobi fields on minimal varieties, Tôhoku Math. J., 21(1969), 539-547.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN