SOME REMARKS ON PATH INDEPENDENCE IN THE SMALL IN PLASTICITY*

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Introduction. Drucker's [1] definition of work-hardening has had a significant influence on the development of stress-strain relations in the mathematical theory of plasticity. Also, recognition that this definition provides a condition which is sufficient to ensure uniqueness of solution, in problems involving small deformations, has lead to its consideration as a stability postulate with extensions to time-dependent materials [2, 3] and as a basis for idealized models of soil behavior. The stability postulate assists in defining the class of materials covered by the theory. Some materials are excluded and for these a different starting point must be used.

Frictional materials provide a number of examples of exceptions to Drucker's postulate. The postulate also excludes materials that soften. If these are to be brought within the scope of the theory of plasticity, a less restrictive postulate is required which allows softening but still provides the accepted forms of flow rule for hardening plasticity. With this in mind, Drucker has suggested an alternative postulate based on the concept of path independence in the small [4]. The object of this paper is to examine some of the implications of this idea. Inviscid plasticity is considered first. Following this, an example of a frictional material is taken to show that the new postulate is restrictive and that some forms of material are excluded. Finally, softening is considered in an application to an ideal material which fractures in a progressive manner.

Path independence in the large and small. For an elastic material, the work done in deforming an element of material from one state of strain to another is independent of the path in deformation space used in passing between the two strain states. It is in the spirit of the present enquiry to use this form of path independence as a basic postulate to define elasticity and to develop the theory. Thus, for an elastic material,

$$\int_{(1)}^{(2)} \sigma_{ij}\, d\epsilon_{ij} = W(\epsilon_{ij}{}^{(2)}) - W(\epsilon_{ij}{}^{(1)})$$

or

$$\int_{(1)}^{(2)} \epsilon_{ij}\, d\sigma_{ij} = W^*(\sigma_{ij}{}^{(2)}) - W^*(\sigma_{ij}{}^{(1)}). \tag{1}$$

Either statement can be used as the fundamental postulate, the other being a direct consequence of the first. Considering only the second statement, we note that the existence of the complementary potential energy function $W^*(\sigma_{ij})$ provides stress-strain relations in the form $\epsilon_{ij} = \partial W^*/\partial\sigma_{ij}$ or

$$\frac{\partial\epsilon_{ij}}{\partial\sigma_{km}} = \frac{\partial^2 W^*}{\partial\sigma_{ij}\,\partial\sigma_{km}} = c_{ijkm} = c_{kmij}\ . \tag{2}$$

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In this result, we note particularly the symmetry in the tensor of compliances and also that the compliances may themselves be functions of stress.

With inelastic materials, the processes associated with deformation are generally path-dependent in the sense that the prior history of loading influences the response to load at any particular instant. Because of this, there is a preference for using incremental stress-strain relations which give the change in strain $\delta\epsilon_{ij}$ due to a small increment in stress $\delta\sigma_{km}$, or *vice versa.* However, there is an infinite number of paths in stress space though which the stress increment $\delta\sigma_{km}$ could be effected. If no distinction is made between these alternative stress paths, there is the implication that the process of deformation is, in some way, path-independent over the interval $\delta\sigma_{km}$.

It appears that the total work done over the interval $\delta\sigma_{km}$ is always independent of the choice of path if terms involving products of small quantities are ignored. Thus, if path independence over the interval $\delta\sigma_{km}$ is to provide information concerning the relation between increments of stress and strain, a more restrictive condition than (1) is required. In effect, Drucker [5] proposes that we take this condition to be

$$\int_0^{\dot{\epsilon}_{ij}} \dot{\sigma}_{ij}\, d\dot{\epsilon}_{ij} = \phi(\dot{\epsilon}_{ij}, \epsilon_{ij})$$

or

$$\int_0^{\dot{\sigma}_{ij}} \dot{\epsilon}_{ij}\, d\dot{\sigma}_{ij} = \phi^*(\dot{\sigma}_{ij}, \sigma_{ij}), \tag{3}$$

where we follow the usual convention and use the superior dot to indicate differentiation with respect to parametric time.

Eqs. (3) provide the condition for what Drucker terms "path independence in the small". This is to be contrasted with the condition given by Eqs. (1) for "path independence in the large". It will be noted that as

$$\phi(\dot{\epsilon}_{ij}, \epsilon_{ij}) = \tfrac{1}{2}\dot{\epsilon}_{ij}\dot{\epsilon}_{km}\frac{\partial^2 W}{\partial\epsilon_{ij}\,\partial\epsilon_{km}},$$

the function ϕ exists if W exists. Thus a material that is path-independent in the large is also path-independent in the small. The converse statement is not necessarily true.

The condition for path independence in the small appears to be a natural extension of Drucker's original postulate with attention now being focussed on the manner of applying the external agency $\delta\epsilon_{ij}$ or $\delta\sigma_{ij}$ rather than on the scalar product $\delta\sigma_{ij}\delta\epsilon_{ij}$. The latter approach has already yielded useful results for work-hardening plasticity, However, similar results can be obtained without the restriction to work-hardening by taking the property of path independence in the small as the initial postulate of material behavior.

Plasticity. At any particular instant, the stress at a point in a body is σ_{ij} with corresponding strain ϵ_{ij}. The rates of change of stress and strain, with respect to some arbitrary time scale, are $\dot{\sigma}_{ij}$ and $\dot{\epsilon}_{ij}$ respectively, where the strain rate $\dot{\epsilon}_{ij}$ may be considered to be the sum of two components: an elastic component $\dot{\epsilon}_{ij}'$ and a plastic component $\dot{\epsilon}_{ij}''$.

We assume that a yield surface $F(\sigma_{ij}, \epsilon_{ij}'', h) = 0$ can be defined in stress space to separate points representing states of stress that can be reached without change in the plastic strain ($\dot{\epsilon}_{ij}'' = 0$) from those that can only be achieved if there is some additional plastic deformation. The yield surface is described in terms of the current stress σ_{ij}, the total plastic strain ϵ_{ij}'' and possibly parameters h describing the history of plastic deformation. The sign of F is chosen so that, if the point σ_{ij} lies on the yield surface, a stress increment $\delta\sigma_{ij}$ requires a stress path terminating outside the current yield surface if

$$\frac{\partial F}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0. \tag{4}$$

If this occurs, the process is termed loading. Loading is accompanied by both elastic and plastic deformation.

The elastic behavior is described by Eq. (1). We shall assume that plastic deformation is path-independent in the small so that

$$\dot{\epsilon}_{ij}'' = \frac{\partial \psi}{\partial \dot{\sigma}_{ij}}, \tag{5}$$

where $\psi = \psi(\dot{\sigma}_{ij}, \sigma_{ij})$. This implies that, during loading,

$$\dot{\epsilon}_{ij} = \frac{\partial}{\partial \dot{\sigma}_{ij}} \left(\psi + \tfrac{1}{2}\dot{\sigma}_{pq}\dot{\sigma}_{rs} \frac{\partial^2 W^*}{\partial \sigma_{pq}\, \partial \sigma_{rs}} \right),$$

so that the total strain is also path-independent in the small. Thus, the yield surface can be considered as a boundary separating stress paths that are path-independent in the large (i.e. elastic behavior) from those that are path-independent only in the small.

In order to derive the flow rule, we consider loading paths in which

$$\dot{\sigma}_{ij} = \dot{\sigma}_{ij}{}^T + \alpha(\partial F/\partial \sigma_{ij}). \tag{6}$$

Here $\dot{\sigma}_{ij}{}^T$ represents loading in a particular direction tangential to the yield surface. Different values of the scalar α thus provide a range of loading paths which lie in the hyperplane containing $\dot{\sigma}_{ij}{}^T$ and the normal to the yield surface at the point σ_{ij}. For loading from this particular point,

$$\dot{\epsilon}_{ij}'' = \dot{\epsilon}_{ij}''(\dot{\sigma}_{ij}) = \dot{\epsilon}_{ij}''(\dot{\sigma}_{ij}{}^T + \alpha(\partial F/\partial \sigma_{ij}))\cdot$$

Thus, using Taylor's series,

$$\dot{\epsilon}_{ij}'' = \dot{\epsilon}_{ij}''(\dot{\sigma}_{ij}{}^T) + \alpha \left(\frac{\partial \dot{\epsilon}_{ij}''}{\partial \dot{\sigma}_{pq}} \right)_{\alpha=0} \frac{\partial F}{\partial \sigma_{pq}} + \frac{\alpha^2}{2!} \left(\frac{\partial^2 \dot{\epsilon}_{ij}''}{\partial \dot{\sigma}_{pq}\, \partial \dot{\sigma}_{rs}} \right)_{\alpha=0} \frac{\partial F}{\partial \sigma_{pq}} \frac{\partial F}{\partial \sigma_{rs}} + \cdots \tag{7}$$

in which the first term must be zero in order to satisfy Prager's condition for continuity [6], i.e. a stress increment tangential to the yield surface cannot cause a change in plastic deformation.

Now we consider a particular loading path corresponding to a stress rate $\dot{\sigma}_{ij}$. For this path, there will be a particular value of $\dot{\sigma}_{ij}{}^T$ for which

$$\alpha = \frac{\partial F}{\partial \sigma_{pq}} \dot{\sigma}_{pq} \bigg/ \frac{\partial F}{\partial \sigma_{rs}} \frac{\partial F}{\partial \sigma_{rs}}. \tag{8}$$

Thus, $\dot{\sigma}_{ij}{}^T$ is a function of $\dot{\sigma}_{ij}$ for any one path and

$$\dot{\epsilon}_{ij}''(\dot{\sigma}_{ij}) = g_{ij} \frac{\partial F}{\partial \sigma_{pq}} \dot{\sigma}_{pq}, \tag{9}$$

where $g_{ij} = g_{ij}(\dot{\sigma}_{pq})$. We now note the way symmetry occurs in the tensor of compliances for elastic behavior (Eq. (2)) and observe that the assumption of path independence in the small requires that

$$\frac{\partial \dot{\epsilon}_{ij}''}{\partial \dot{\sigma}_{km}} = g_{ij} \frac{\partial F}{\partial \sigma_{km}} + \frac{\partial g_{ij}}{\partial \dot{\sigma}_{km}} \frac{\partial F}{\partial \sigma_{pq}} \dot{\sigma}_{pq}$$

be symmetric in the sense that the suffixes ij and km can be exchanged. For this to be possible

$$g_{ij} = \lambda \frac{\partial F}{\partial \sigma_{ij}}, \qquad \frac{\partial \lambda}{\partial \dot{\sigma}_{km}} \frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial \lambda}{\partial \dot{\sigma}_{ij}} \frac{\partial F}{\partial \sigma_{km}}. \tag{10}$$

Eqs. (10) imply that,

$$\lambda = G + H \frac{\partial F}{\partial \sigma_{pq}} \dot{\sigma}_{pq} \tag{11}$$

where the scalar G is not a function of the $\dot{\sigma}_{ij}$. On substituting the value for λ in Eqs. (9, 10) we obtain the flow rule,

$$\dot{\epsilon}_{ij}'' = \left(G + H \frac{\partial F}{\partial \sigma_{pq}} \dot{\sigma}_{pq}\right) \frac{\partial F}{\partial \sigma_{ij}} \frac{\partial F}{\partial \sigma_{rs}} \dot{\sigma}_{rs}\ . \tag{12}$$

This form of flow rule has been obtained using Eq. (5) but with no further restriction on the relation between $\dot{\sigma}_{ij}$ and $\dot{\epsilon}_{ij}$. If a linear relationship is assumed, ψ must be taken to be a quadratic function of the stress rates, which implies that g_{ij} is not a function of $\dot{\sigma}_{ij}$ and H must be zero. On putting $H = 0$, Eq. (12) reduces to the form of flow rule which is already available for the special case of linearity in the small [1]. Clearly, the flow rule based on the assumption of linearity can be regarded as a satisfactory approximation if $H(\partial F/\partial \sigma_{pq})\dot{\sigma}_{pq}$ is sufficiently small compared with G.

Looking again at Eq. (12), we note that the flow rule implies that the plastic strain increment vector is normal to the yield surface and that this property is independent of any assumption of linearity. Following this, convexity of the yield surface can be demonstrated if the work done in a cycle of deformation is required to be positive, as proposed by Il'iushin [7]. Drucker's work-hardening postulate [1] can be used additionally to restrict G and H to positive values. This is not a requirement of the theory, however, and the flow rule is applicable to both hardening and softening situations. If softening occurs, the yield surface contracts locally during plastic deformation so that an alternative criterion of loading must be adopted in place of condition (4). One approach is to combine (4) with Drucker's definition of work-hardening to give a condition for loading in the form

$$\dot{\sigma}_{ij}\dot{\epsilon}_{ij}'' \frac{\partial F}{\partial \sigma_{rs}} \dot{\sigma}_{rs} > 0. \tag{13}$$

Now, using the flow rule with this condition, we see that loading occurs if

$$\left(G + H \frac{\partial F}{\partial \sigma_{pq}} \dot{\sigma}_{pq}\right) \frac{\partial F}{\partial \sigma_{rs}} \dot{\sigma}_{rs} > 0. \tag{14}$$

During loading, $G = G(\sigma_{ij}, \epsilon_{ij}'', h)$ must be positive for hardening and negative for softening. For the special case of $G = 0$, Eq. (12) provides the associated flow rule for perfect plasticity if H approaches infinity. This is particularly satisfying as it seems quite reasonable that a theory which describes both work-hardening and softening should contain perfect plasticity as a special case.

A model frictional material. In order to consider behavior that is path-dependent in the small, we consider a two-dimensional model of a frictional material. The model consists of a block which slides on a rough surface in the xy plane. It is loaded by forces X and Y and the displacement of the block has components u and v in the directions x and y respectively. This motion is opposed by forces ku and kv induced in two long springs, which themselves lie in the x and y directions, and a frictional force which is developed between the block and the surface on which it slides. The frictional force acts in a direction opposed to that of the motion of the block and has magnitude $N + \beta X$. Here N can be thought of as the frictional force developed by the weight of the block alone and the coefficient β provides for some form of coupling between the force X and the normal force between the block and the plane on which it slides. If a physical model were to be constructed, this type of coupling could be arranged by applying the load X through a device similar to a pair of spring-loaded sugar tongs so arranged that the spring provides the force N and with the coefficient β related to the geometry of the tongs. We regard the model as an analogue of a rigid plastic material in which the displacements u and v correspond to strains (all plastic) and the forces X and Y are analogous to stresses.

The forces acting on the block are shown in Fig. 1. During motion, or at the instant of incipient motion,

$$(X - ku) \cos \theta + (Y - kv) \sin \theta = N + \beta X, \tag{15}$$

$$(X - ku) \sin \theta - (Y - kv) \cos \theta = 0, \tag{16}$$

where θ is the angle between the x-direction and the instantaneous direction of motion. Thus $\dot{v} \cos \theta = \dot{u} \sin \theta$ or, using Eq. (16),

$$(X - ku)\dot{v} = (Y - kv)\dot{u}. \tag{17}$$

On eliminating θ from Eqs. (15) and (16), we obtain the following equation for the yield surface of the material in X, Y (stress) space:

$$(X - ku)^2 + (Y - kv)^2 - (N + \beta X)^2 = 0. \tag{18}$$

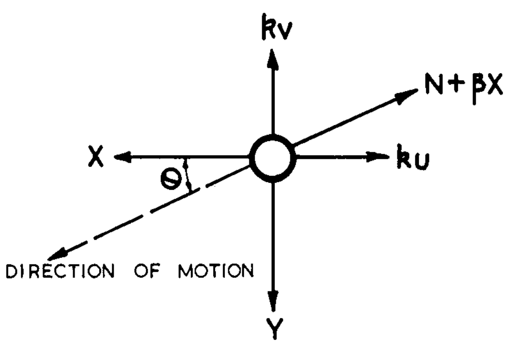

FIG. 1. Force diagram for the model frictional material.

The special case obtained when $\beta = 0$ is of interest. This occurs if no coupling occurs between X and the frictional force. The frictional force then has constant magnitude N and the yield surface is a circle of radius N with its centre at the point (ku, kv). The model thus behaves as a kinematically hardening solid of the form proposed by Prager [8]. The yield surface is regular and Eq. (17) implies that the analogue to the strain increment vector $(\dot{u}, \dot{v})$ is normal to the yield surface. Because of this, we can expect the model to be path-independent in the small if $\beta = 0$.

We return to the more general model and differentiate Eq. (18) with respect to time to obtain an expression connecting $\dot{u}$ and $\dot{v}$. This expression is taken together with Eq. (17) to give the displacement rates in the form

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \frac{\frac{1}{k}}{\left(\frac{X - ku}{Y - kv} + \frac{Y - kv}{X - ku}\right)} \left[\begin{array}{c|c} \frac{1}{A} & 1 \\ \hline 1 - \beta\left(\frac{\beta X + N}{X - ku}\right) & B \end{array}\right] \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} \tag{19}$$

where $A = Y - kv/((X - ku) - \beta(\beta X + N))$ is the slope of the normal to the yield surface at the point (X, Y) and $B = (Y - kv)/(X - ku)$ is the slope of the corresponding plastic strain increment vector $(\dot{u}, \dot{v})$. We note that A and B are not in general equal and that the plastic strain increment vector is normal to the yield surface only if $\beta = 0$ or if the loading is uniaxial in the x-direction (i.e. with $Y = 0, v = 0$). In general, normality is not obtained and the direction of the plastic strain increment vector is determined by the history of loading as indicated by Eq. (17).

The matrix connecting the stress and strain rates would be symmetric if the material were path-independent in the small. As anticipated, this occurs if $\beta = 0$. However, for all other values of β the matrix is non-symmetric and the material is seen to be path-dependent in the small. This produces the departures from normality noted in the previous paragraph.

A typical yield surface for $\beta > 0$ is shown in Fig. 2. The yield surface is in the form of an ellipse which increases in size, with displacement of the block in the x-direction, and also moves in stress space in a manner resembling the kinematically hardening solid. Let us now examine the behavior of the material for alternative load increments from a particular point on the yield surface.

Fig. 3 shows a portion of the yield surface in the vicinity of the point P. The line ab is the tangent to the yield surface at P and the line $P - P'$ represents the direction of the plastic strain increment vector $(\dot{u}, \dot{v})$. The line PQ points outwards from, the yield surface and is at right angles to PP'. If PQ coincides with the direction of the load increment vector, Drucker's second-order work term $\dot{\sigma}_{ij}\dot{\epsilon}_{ij}$ (or $\dot{X}\dot{u} + \dot{Y}\dot{v}$ for the model) is zero. In Drucker's terminology [2] the material is stable if $\dot{\sigma}_{ij}\dot{\epsilon}_{ij} \geq 0$. This condition is satisfied for any load increment vector directed into the unshaded region between PQ and the tangent ab. However, the material is apparently unstable if the load path is within the shaded region. If this occurs, work is extracted from the material and $\dot{\sigma}_{ij}\dot{\epsilon}_{ij} \leq 0$. We note that it is possible to have $\dot{\sigma}_{ij}\dot{\epsilon}_{ij} \geq 0$ for all load paths only if the neutral line PQ is tangential to the yield surface. This requires normality of the plastic strain increment vector and therefore path independence in the small.

Path dependence in the small can be demonstrated for the model material. Fig. 4a shows a point A on the yield surface, corresponding to loads X and Y, and a neighboring

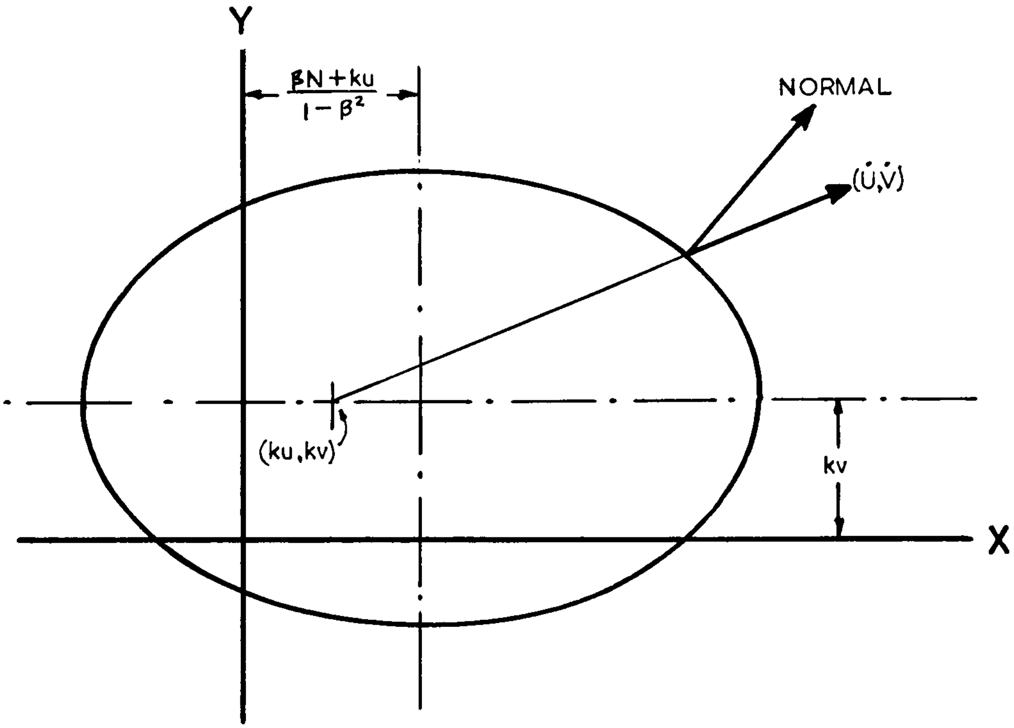

FIG. 2. Typical yield surface for the model frictional material with $\beta > 0$.

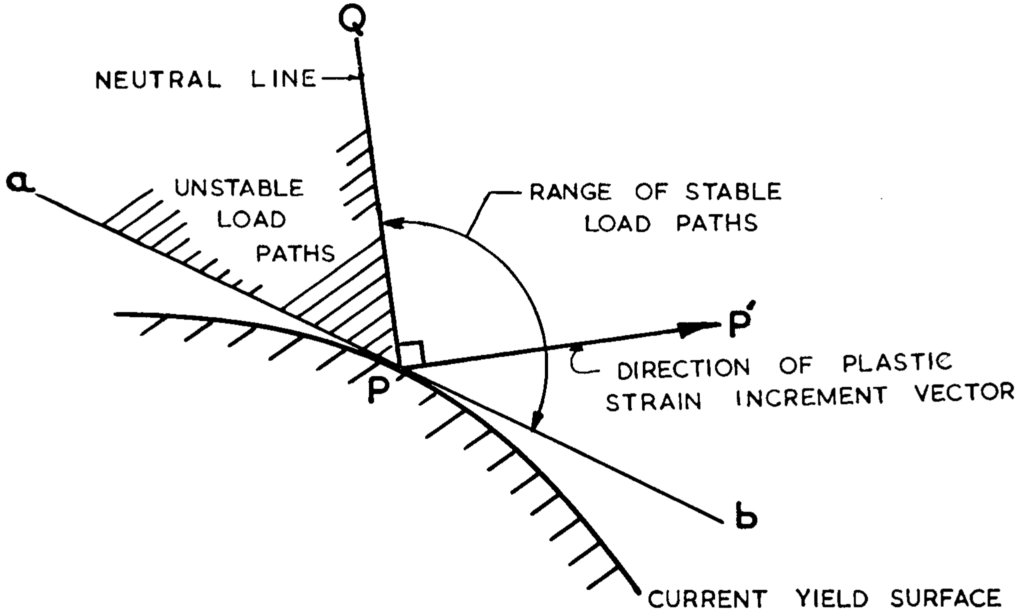

FIG. 3. Consequences of a departure from normality.

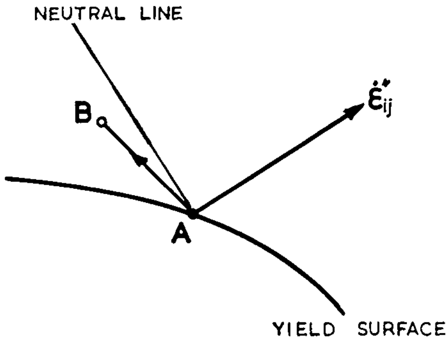

FIG. 4a. An unstable load path, A to B.

point B where the loads are $X + \delta X$ and $Y + \delta Y$. The direct path AB is an unstable load path. Thus, if the forces X and Y are provided by dead weights, it is not possible to add increments of loading δX and δY by passing along the load path AB. However, because the material is path-dependent in the small, it is possible to add the increments of load δX and δY, in a stable manner with the dead load system, by using an alternative load path. One of a variety of possible loading paths is shown in Fig. 4b. Here the model is subjected to a small amount of unloading from A before loading along a stable path to the point B. This possibility of alternative modes of behavior corresponding to the same total increment of loading is a characteristic of materials that are path-dependent in the small.

Before looking briefly at the application of these ideas to real materials we consider another example of an idealized material.

The progressively fracturing solid. Consider a material that is made up of a large number of linear elastic fibers having various strengths and stiffnesses. Suppose that a fiber fractures with its stiffness being reduced to zero when the stress in the element becomes equal to its strength. If such a material is subjected to a proportional increase in strain ϵ_{ij}, the stress σ_{ij} first increases, with $\dot{\sigma}_{ij}\dot{\epsilon}_{ij} > 0$, until a maximum value of the stress is reached and then decreases with $\dot{\sigma}_{ij}\dot{\epsilon}_{ij} < 0$. The material thus exhibits the property of softening with increasing strain. In addition, if the sense of deformation is

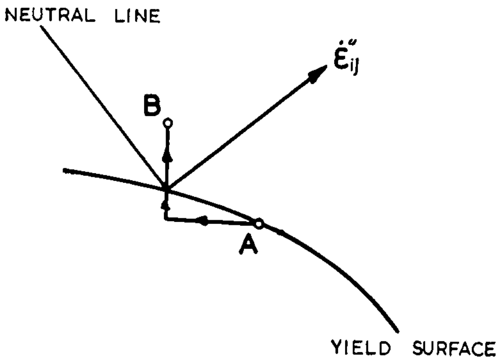

FIG. 4b. An alternative stable load path from A to B.

reversed, the material behaves in a linear elastic fashion and is therefore unstressed when its original dimensions are recovered.

We attempt to describe the behavior of this material in terms of the theory of plasticity. In order to do this, we ignore the detailed structure of the material and assume that it is permissible to consider stress and strain as averages taken throughout a representative volume which is itself taken to correspond to a point in a continuum. It will be noted that even for proportional loading strain is a multi-valued function of stress. Because of this, it seems convenient to describe the history of loading by paths in deformation space ϵ_{ij} rather than stress space σ_{ij}.

Consider any deformation path which starts with the unstrained state $\epsilon_{ij} = 0$. For all such paths, the material behaves in a linear elastic manner until fracture occurs. Thus, there is a surface in strain space $F(\epsilon_{ij}) = 0$ which encloses all combinations of strain that can be obtained without increasing the amount of fractured material. We call this the fracture surface, in deference to the conceptual model, although there is no essential difference between it and a yield surface in hardening plasticity.

During progressive fracture, the fracture surface expands in order to accommodate the additional strain states that can be reached by linear elastic behavior alone. Clearly, any fracture surface must surround the origin and be star-shaped in the sense that any ray from the origin cuts the surface in only one point. More particularly, the form of the fracture surface will depend on the nature of the material being described and on the history of loading.

The fracture surface enables a distinction to be made between strain increments which are accompanied by progressive fracture and those that correspond to elastic unloading and reloading. During elastic unloading and reloading, the stress σ_{ij} and strain ϵ_{km} are related by the generalized Hooke's Law:

$$\sigma_{ij} = S_{ijkm}\epsilon_{km} \tag{20}$$

where S_{ijkm} is a fourth-order tensor of moduli which are dependent on the same factors that determine the current yield surface. Thus, the tensor S_{ijkm} determines the relationship between stress and strain for all possible deformation paths that are entirely within the current fracture surface.

Progressive fracture changes the elastic response of the material which is described by the tensor S_{ijkm}. Thus, during progressive fracture

$$\dot{\sigma}_{ij} = S_{ijkm}\dot{\epsilon}_{km} + \dot{S}_{ijkm}\epsilon_{km}. \tag{21}$$

This equation suggests that it is possible to consider the stress rate $\dot{\sigma}_{ij}$ as the sum of an elastic component

$$\dot{\sigma}_{ij}' = S_{ijkm}\dot{\epsilon}_{km} \tag{22}$$

and the fracture stress decrement

$$\dot{\sigma}_{ij}'' = \dot{S}_{ijkm}\epsilon_{km}. \tag{23}$$

Arguments similar to those used in developing the theory of plasticity [1, 6] then lead to the result

$$\dot{\sigma}_{ij}'' = -g_{ij}\frac{\partial F}{\partial \epsilon_{km}}\dot{\epsilon}_{km} \tag{24}$$

where g_{ij} is a symmetric tensor. We now use the condition for the material to be path-independent in the small and choose g_{ij} so that the tensor connecting $\dot{\sigma}_{ij}''$ and $\dot{\epsilon}_{km}$ is symmetric. In this way,

$$\dot{\sigma}_{ij}'' = -K \frac{\partial F}{\partial \epsilon_{ij}} \frac{\partial F}{\partial \epsilon_{km}} \dot{\epsilon}_{km} ,$$

$$\dot{\sigma}_{ij} = \left(S_{ijkm} - K \frac{\partial F}{\partial \epsilon_{ij}} \frac{\partial F}{\partial \epsilon_{km}}\right)\dot{\epsilon}_{km} \tag{25}$$

where K is a scalar function of the strain and possibly the strain history.

It will be noted that the condition for the material to be path-independent in the small has led to normality in the sense that the fracture stress decrement vector is orthogonal to the fracture surface in strain space. This result was to be expected. However, it does not necessarily imply that the fracture surface must be convex. Il'iushin [7] has observed that a proof of convexity requires the elastic response to be unaffected by the plastic deformation. This has been emphasized by Palmer *et al.* [9] who obtained a yield surface with convex regions for a model composite material made of stable elements arranged in such a way that the elastic response was affected by plastic deformation. The idealized fracturing solid is similar to this but has the peculiarity that, although its elastic response is determined by the current position of the fracture surface, it retains a memory of an initial state ($\sigma_{ij} = 0$, $\epsilon_{ij} = 0$) which can always be regained on unloading. It is of interest that, for some particular forms of fracture surface, this property enables us to determine the rate of change of the moduli with progressive fracture.

Applications. So far no attempt has been made to relate the theoretical developments to applications. We recognize that the theory applies to materials that are path-independent in the small and for which a yield surface provides a sharp distinction between elastic behavior and more complex behavior in which energy is dissipated by inviscid flow, frictional sliding or fracture. These conditions are very restrictive and it is unlikely that they are satisfied precisely by any real material. In practice, the yield surface is itself dependent on the definition used for yield stress, and path independence in the small can only be recognized through the property of normality with respect to this somewhat arbitrarily defined surface. There are thus difficulties in assessing whether the theory can be applied in any given situation. Accordingly, a rather subjective viewpoint has to be adopted which is based more on convenience and utility than on a knowledge of the approximations involved.

In considering dense metallic materials, Drucker's hardening postulate provides a form of flow rule which is both accepted and used. The assumption of path independence in the small provides the same results, so that the use of the new postulate can be justified, for most metals, on the basis of past experience.

The use of path independence in the small as an initial postulate removes a restriction in the theory and enables situations involving softening to be examined in an idealized manner. However, it must be recognized that the postulate is restrictive. Some materials, including the model frictional material discussed earlier, do not behave in this way and it is by no means certain whether softening of real materials occurs in a way compatible with the postulate.

On a number of occasions the view has been expressed [1, 6, 9] that there is a fundamental difference between dry friction and plasticity and that this difference can only be resolved when behavior is examined at the microstructural level. One wonders whether this view is entirely correct. The model frictional material discussed here provides an exact analogue to the behavior of a kinematically hardening solid, if there is no coupling between the applied force and the friction that can be developed. Clearly, it is not friction itself that causes departures from normality and behavior that is expected in metals. Rather, it is the coupling between friction and the direct loads. The present model is anisotropic, but if an isotropic model were adopted the frictional force would depend on $X + Y$ and the yield function would include terms involving the sum of the direct stresses σ_{kk} . A more cautious view is that there may be difficulties in applying the theory of plasticity to any material for which the yield surface is very dependent on the first invariant of the stress tensor. Our experience with metals does not conflict with this view. However, both frictional and fracturing materials are likely to be very much influenced by a state of superimposed triaxial compression. Some work of a physical nature remains to be done to investigate what restrictions must be placed on the use of the theory with these types of material.

Concluding remarks. Drucker's [5] suggested postulate of path independence in the small has been formulated and shown to provide a less restrictive theory of plasticity than that following from a definition of work-hardening. The extended theory includes perfect plasticity as a special case and is not restricted in its application to stable materials. In addition, the assumption of linearity in the small is not required or, alternatively, its use can be justified in particular circumstances even for unstable materials.

An example using a model frictional material has indicated that some materials are excluded by the postulate. Thus, although the formulation can be used for a range of materials that dissipate energy in different ways, some caution is necessary in applications.

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