12-2017

Some Remarks on Ricci Solitons

Ramesh Sharma
University of New Haven, rsharma@newhaven.edu

S Balasubramanian
Sri Sathya Sai Institute of Higher Learning

N. Uday Kiran
Sri Sathya Sai Institute of Higher Learning

Publisher Citation

Sharma, R., Balasubramanian, S., & Kiran, N. U. (2017). Some remarks on Ricci solitons. Journal of Geometry, 108(3), 1031-1037. June 2017 doi: 10.1007/s00022-017-0392-0

Comments

This is a post-peer-review, pre-copyedit version of an article published in *Journal of Geometry*. The final authenticated version is available online at: http://dx.doi.org/10.1007/s00022-017-0392-0.
Mathematics Subject Classification 53C55 53C44 53C21

Some Remarks On Ricci Solitons

Ramesh Sharma[a], **S. Balasubramanian**[b], **and N. Uday Kiran**[c]

[a] *Department of Mathematics, University of New Haven, West Haven, CT 06516, U.S.A.* E-mail: rsharma@newhaven.edu

[b] *Department of Mathematics and Computer Science, Sri Sathya Sai Institute Of Higher Learning, Prasanthinilayam-515134, India*, E-mail: sbalasubramanian@sssihl.edu.in

[c] *Department of Mathematics and Computer Science, Sri Sathya Sai Institute Of Higher Learning, Prasanthinilayam-515134, India*, E-mail: nudaykiran@sssihl.edu.in

Abstract: We obtain an intrinsic formula of a Ricci soliton vector field and a differential condition for the non-steady case to be gradient. Next we provide a condition for a Ricci soliton on a Kaehler manifold to be a Kaehler-Ricci soliton. Finally we give an example supporting the first result.

2010 MSC:53C55,53C44,53C21

Keywords: Ricci soliton, Gradient, Intrinsic formula, Harmonic form, Kaehler-Ricci soliton.

1. Introduction

A Ricci Soliton is a generalization of a Einstein manifold, and defined as a complete Riemannian manifold (M, g) with a vector field V, satisfying the equation

$$\mathcal{L}_V g + 2Ric = 2\lambda g \tag{1}$$

where $\mathcal{L}_V$ denotes Lie derivative along V, Ric denotes the Ricci tensor of g and λ a real constant. The Ricci soliton is a special self similar solution of the Hamilton's Ricci flow: $\frac{\partial}{\partial t}g(t) = -2Ric(t)$ with initial condition $g(0) = g$; and is said to be shrinking, steady, or expanding accordingly as $\lambda > 0$, $= 0$ or < 0 respectively. In particular, if V is the gradient of a smooth function f on M, i.e., $V = grad\, f$, up to the addition of a Killing vector field, then we say that the Ricci soliton is gradient and f is the potential function. For a gradient Ricci soliton, equation (1) becomes

$$Hessf + 2Ric = 2\lambda g \tag{2}$$

where $Hess$ denotes the Hessian operator $\nabla\nabla$ (∇ denoting the covariant derivative operator with respect to the Riemannian connection of

g). The following formulas are well known (see Chow et.al [1] and Petersen and Wylie: [4]) for a gradient Ricci soliton:

$$Q(grad\, f) = \frac{1}{2} grad\, S \tag{3}$$

$$|grad\, f|^2 + S - 2\lambda f = \text{a constant} \tag{4}$$

$$\Delta f - |grad\, f|^2 + 2\lambda f = \text{a constant} \tag{5}$$

where Δf is the g-trace of $Hessf$, Q is the Ricci operator defined by $g(QX, Y) = Ric(X, Y)$ for arbitrary vector fields X, Y on M, and S denotes the scalar curvature of g.

A seminal result of Perelman [3] says that a compact Ricci soliton is necessarily gradient. In this article, we first provide a geometric operator-theoretic condition on V so that it may become gradient for the non-steady case. The 1-form metrically equivalent to V is denoted by v and is given by $v(X) = g(V, X)$ for an arbitrary vector field X on M. For a p-form ω, we denote the co-differential operator by δ, i.e., $\delta\omega$ is a $(p-1)$-form such that $(\delta\omega)_{i_2 \cdots i_p} = -\nabla^{i_1}\omega_{i_1 \cdots i_p}$. The interior product operator of ω by V is denoted by i_V such that $(i_V\omega)_{i_2 \cdots i_p} = V^{i_1}\omega_{i_1 \cdots i_p}$. We now state our result as follows.

Theorem 1.1. *The Ricci soliton vector field V and its metric dual 1-form v satisfy the following intrinsic formula:*

$$2\lambda v = d(|V|^2 + \delta v) + 2(\delta + i_V)dv \tag{6}$$

So, a non-steady Ricci soliton is gradient (i.e. v is exact) if and only if $(\delta + i_V)dv$ is exact.

Corollary 1.2. *A non-steady Ricci soliton (M, g, V, λ) with v closed is gradient.*

Remark 1. In general, v closed need not imply v exact (i.e., V gradient) unless M is simply connected.

Next, we consider a Kaehler-Ricci soliton (see [1]) which is defined as a Kaehler manifold (M, g, J) satisfying the Ricci soliton equation (1) for a vector field V which is an infinitesimal automorphism of the complex structure J, i.e.

$$\mathcal{L}_V J = 0. \tag{7}$$

A vector field V satisfying (7) is also known as a real holomorphic vector field or a contravariant analytic vector field (see Yano [5]). It is known (see Feldman, Ilmanen and Knopf [2]) that a Ricci soliton as a Kaehler metric is a Kaehler-Ricci soliton if it is gradient. We provide a generalization of this result as follows.

Theorem 1.3. *A Ricci soliton which is also a Kaehler metric is Kaehler-Ricci soliton if and only if dv is J-invariant.*

2. Proofs of The Results

In the following X, Y, Z will denote arbitrary vector fields on M.

Proof of Theorem 1.1 Equation (1) can be written as

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2Ric(X, Y) = 2\lambda g(X, Y) \tag{8}$$

The exterior derivative dv of the 1-form v is given by

$$g(\nabla_X V, Y) - g(\nabla_Y V, X) = 2(dv)(X, Y) \tag{9}$$

As dv is skew-symmetric, we define a tensor field F of type $(1, 1)$ by

$$(dv)(X, Y) = g(X, FY) \tag{10}$$

Obviously, F is skew self-adjoint, i.e. $g(X, FY) = -g(Y, FX)$. Thus equation (9) assumes the form.$g(\nabla_X V, Y) - g(\nabla_Y V, X) = 2g(X, FY)$. Adding it to equation (8) side by side, and factoring Y out gives

$$\nabla_X V = -QX + \lambda X - FX \tag{11}$$

Using this equation we compute $R(Y, X)V = \nabla_Y \nabla_X V - \nabla_X \nabla_Y V - \nabla_{[Y,X]} V$ and obtain

$$R(Y, X)V = (\nabla_X Q)Y - (\nabla_Y Q)X + (\nabla_X F)Y - (\nabla_Y F)X \tag{12}$$

We note that $(dv)(X, Y) = g(X, FY)$ and dv is closed. Hence

$$g(X, (\nabla_Y F)Z) + g(Y, (\nabla_Z F)X) + g(Z, (\nabla_X F)Y) = 0 \tag{13}$$

Taking inner product of (12) with Z we have

$$\begin{aligned} g(R(Y, X)V, Z) &= g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) \\ &+ g(Z, (\nabla_X F)Y) - g(Z, (\nabla_Y F)X) \end{aligned} \tag{14}$$

The skew self-adjointness of F implies skew self-adjointness of $\nabla_Y F$ and so the last term of (14) including the minus sign equals $g(X, (\nabla_Y F)Z)$. Using (13) in (14) gives

$$\begin{aligned} g(R(Y, X)V, Z) &= (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z) \\ &- g(Y, (\nabla_Z F)X) \end{aligned} \tag{15}$$

Let (e_i) be a local orthonormal frame on M. Setting $Y = Z = e_i$ in (15) and summing over $i = 1, \cdots, n$ provides

$$Ric(X, V) = \frac{1}{2}X(S) - (divF)X \tag{16}$$

Next, we compute the covariant derivative of the squared g-norm of V using (11) as follows.

$$\nabla_X |V|^2 = 2g(\nabla_X V, V) = -2Ric(X, V) + 2\lambda g(X, V) - 2g(FX, V) \tag{17}$$

Eliminating $Ric(X, V)$ between (16) and (17) shows

$$\nabla_X |V|^2 + X(S) = 2\lambda g(X, V) + 2((divF)X + g(FV, X)) \tag{18}$$

In view of (10) we note that the last term in (18) is equivalent to $-2(\delta + i_V)(dv)(X)$. Hence (18) can be expressed as

$$d(|V|^2 + S) = 2\lambda v - 2(\delta + i_V)dv \tag{19}$$

Now, taking the g-trace of equation (1) gives $\delta v = S - n\lambda$ and hence we get $d\delta v = dS$. Using this consequence in (19) we obtain the formula (6). The second part of the theorem follows from this formula, because $\lambda \neq 0$ by hypothesis. This completes the proof.

Remark 2. Contracting the Ricci soliton equation (1) in local coordinates and then differentiating gives

$$\nabla_j \nabla_i V^i = -\nabla_j S. \tag{20}$$

Differentiating the Ricci soliton equation (1) gives $\nabla_i \nabla_j V^i = -\nabla_i \nabla^i V_j - \nabla_j S$. Using this and (20) we obtain

$$R_j{}^k V_k + \nabla^i \nabla_i V_j = 0. \tag{21}$$

A vector field V on a Riemannian manifold (M, g) satisfying equation (21) was studied by K. Yano and T. Nagano in [7] and was termed a geodesic vector field (not to be confused with vector field whose integral curves are geodesics). Actually, (21) is equivalent to the condition $(\mathcal{L}_V \nabla)(e_i, e_i) = 0$ (i summed over $1, ..., n$), where e_i is a local orthonormal frame on M. Obvious examples of a geodesic vector field are Killing vector fields ($\mathcal{L}_V g = 0$) and affine Killing vector fields ($\mathcal{L}_V \nabla = 0$). For a compact Riemannian manifold we know that a divergence-free geodesic vector field is Killing (see Yano [6]). We noted earlier that a Ricci soliton vector field V on a Riemannian manifold (not necessarily compact) satisfies (21), and hence we conclude that a Ricci soliton vector field V is a new example of a geodesic vector field in the sense of [7].

Remark 3. Equations (16), (19) and (6) are generalizations of the corresponding formulas (3), (4) and (5) for a gradient Ricci soliton respectively, because in the gradient case $v = df$ which implies $dv = 0$ and hence $F = 0$.

Proof Of Theorem 1.3. Operating J on (11) we have

$$J\nabla_X V = -JQX + \lambda JX - JFX.$$

Next, substituting JX for X in (11) we get

$$\nabla_{JX} V = -QJX + \lambda JX - FJX.$$

Taking the difference between the above two equations and noting that J commutes with the Ricci operator Q for a Kaehler manifold, we find

$$J\nabla_X V - \nabla_{JX} V = (FJ - JF)X. \tag{22}$$

At this point, we note that

$$\begin{aligned}(\mathcal{L}_V J)X &= \mathcal{L}_V JX - J\mathcal{L}_V X \\ &= \nabla_V JX - \nabla_{JX} V - J\nabla_V X + J\nabla_X V \\ &= J\nabla_X V - \nabla_{JX} V\end{aligned}$$

where we have used the fact that J is parallel for a Kaehler structure. The use of the foregoing equation in (22) gives

$$(\mathcal{L}_V J)X = (FJ - JF)X. \tag{23}$$

Now using the equation (11), the Kaehlerian properties: $JQ = QJ$, $g(JX, JY) = g(X, Y)$, $g(JX, Y) = -g(X, JY)$, skew-symmetry of F, and a straightforward computation we obtain

$$2[(dv)(JX, JY) - (dv)(X, Y)] = g(J(FJ - JF)X, Y).$$

The use of (23) in the above equation provides

$$(dv)(JX, JY) - (dv)(X, Y) = \frac{1}{2} g(J(\mathcal{L}_V J)X, Y).$$

This shows that $\mathcal{L}_V J = 0$ if and only if $(dv)(JX, JY) = (dv)(X, Y)$, i.e. dv is J-invariant, completing the proof.

Remark 4. For a gradient Ricci soliton, $v = df$ and hence $dv = 0$, and Theorem 2 implies $\mathcal{L}_V J = 0$ and so recovers the known result (mentioned earlier) that the gradient Ricci soliton on a Kaehler manifold is indeed Kaehler-Ricci soliton. Non-gradient examples satisfying the Kaehler-Ricci soliton condition $(dv)(JX, JY) = (dv)(X, Y)$ are the cases when (i) $dv = \Omega$ and (ii) $dv = \rho$ where Ω is the Kaehler 2-form defined by $\Omega(X, Y) = g(X, JY)$, and ρ is the Ricci 2-form defined by $\rho(X, Y) = g(QX, JY)$. We note that, both Ω and ρ are closed and J-invariant.

3. An Example Supporting Theorem 1.1

Let us consider R^3 with Euclidean metric δ_{ij} for which the Ricci soliton equation is

$$\partial_i v_j + \partial_j v_i = 2\lambda\delta_{ij}.$$

It can be verified easily that a solution of this equation is

$$\begin{aligned} v &= (\lambda x_1 + x_2 - x_3)dx_1 + (\lambda x_2 + x_3 - x_1)dx_2 \\ &+ (\lambda x_3 + x_1 - x_2)dx_3. \end{aligned} \tag{24}$$

Computing its exterior derivative we get

$$dv = -2(dx_1 \wedge dx_2 + dx_2 \wedge dx_3 + dx_3 \wedge dx_1). \tag{25}$$

We also compute

$$\delta dv = *d * (dv) = -2 * d(dx_3 + dx_1 + dx_2) = 0.$$

$$\begin{aligned} i_V dv &= (dv)(V) = -2[(\lambda(x_3 - x_2) + 2x_1 - x_2 - x_3)dx_1 \\ &+ (\lambda(x_1 - x_3) + 2x_2 - x_3 - x_1)dx_2 \\ &+ (\lambda(x_2 - x_1) + 2x_3 - x_1 - x_2)dx_3]. \end{aligned}$$

Re-arranging the terms we obtain

$$\begin{aligned} \delta dv + i_V dv &= 2\lambda[(x_2 - x_3)dx_1 + (x_3 - x_1)dx_2 + (x_1 - x_2)dx_3] \\ &- d[(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2]. \end{aligned}$$

Let us denote the 1-form $\delta dv + i_V dv$ by θ. It turns out that

$$d\theta = -4\lambda(dx_1 \wedge dx_2 + dx_2 \wedge dx_3 + dx_3 \wedge dx_1).$$

Use of equation (25) in the above shows $d\theta = 2\lambda dv$. Thus, for $\lambda \neq 0$, we see that $\theta = (\delta + i_V)dv$ is not exact because v is not exact [evident from equation (25)]. This is in agreement with the conclusion of Theorem 1.1. We also note that the Ricci soliton of this example is not gradient.

4. Acknowledgment

The authors are immensely grateful to the referee for many valuable suggestions.

References

1. Chow, B., Chu, S., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F, and Ni, L., *The Ricci flow: Techniques and Applications, Part I: Geometric Aspects, Mathematical Surveys and Monographs* 135, American Math. Soc., 2004.
2. Feldman, M., Ilmanen, T. and Knopf, D., Rotationally symmetric shrinking and expanding gradient Kaehler-Ricci solitons, *J. Diff. Geom.* **65** (2003), 169-209.
3. Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, Preprint, http:arXiv.org abs math.DG/02111159.
4. Petersen, P. and Wylie, W., Rigidity of gradient Ricci solitons, *Pacific J. Math* **241** (2009), 329-345.
5. Yano, K., *Differential geometry of complex and almost complex spaces*, Pergamon Press, New York, 1965.
6. Yano, K.,*Integral formulas in Riemannian geometry*,Marcel Dekker, New York, 1970.
7. Yano, K. and Nagano, T. On geodesic vector fields in a compact orientable Riemannian space, *Comm. Math. Helv.* **35**, 55-64.