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Some remarks on s-convex functions

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Summary. Two kinds of s-convexity ($0 < s \leq 1$) are discussed. It is proved among others that s-convexity in the second sense is essentially stronger than the s-convexity in the first, original, sense whenever $0 < s < 1$. Some properties of s-convex functions in both senses are considered and various examples and counterexamples are given.

1. Introduction

Two definitions of s-convexity ($0 < s \leq 1$) of real-valued functions are known in the literature.

A function f: $R_+ \to R$, where $R_+ = [0, \infty)$, is said to be *s-convex in the first sense* if

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v) \tag{1}$$

for all $u, v \in R_+$ and all $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. We denote this by $f \in K_s^1$. This definition of s-convexity, for so called φ-functions, was introduced by Orlicz in [4] and was used in the theory of Orlicz spaces (cf. [2], [3], [5]). A function f: $R_+ \to R_+$ is said to be a *φ-function* if $f(0) = 0$ and f is nondecreasing and continuous.

A function f: $R_+ \to R$ is said to be *s-convex in the second sense* if inequality (1) holds for all $u, v \in R_+$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. We denote this by $f \in K_s^2$. This definition of s-convexity may be found in [1], where the problem, when the rationally s-convex functions are s-convex, was considered.

Of course, both s-convexities mean just the convexity when $s = 1$.

This paper is divided into two parts. The first part is devoted to the s-convex functions in both senses. It consists of necessary conditions on the functions to be

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in the classes K_s^1 or K_s^2 and a theorem about superposition of functions from class K_s^1. For example, if $f, g \in K_s^1$ then the functions $f+g$ and $\max(f, g)$ are also in K_s^1. If, moreover, we know that $f(0)=0$ then the s-convexity in the second sense implies s-convexity in the first sense but not conversely and if s decreases then the classes K_s^1 and K_s^2 increase. In the second part, s-convex non-negative functions are considered. We will prove a theorem about the superposition and the product of two functions in classes $K_{s_1}^1$ and $K_{s_2}^1$, respectively. On an example we will see that functions in K_s^1 need not be continuous on $(0, \infty)$. On another example we will see that s-convexity (with $0<s<1$) of a φ-function f need not imply that f is of the form $f(u)=\Phi(u^s)$ or $f(u)=\Phi(u)^s$, where Φ is a convex φ-function. Some other properties of s-convex functions are also given in this second part.

2. On s-convex functions

We start with the following properties of s-convex functions.

THEOREM 1. *Let* $0<s<1$.
(*a*) *If* $f \in K_s^1$ *then* f *is non-decreasing on* $(0, \infty)$ *and*

$$f(0^+) := \lim_{u \to 0^+} f(u) \le f(0).$$

(*b*) *If* $f \in K_s^2$ *then* f *is non-negative on* $[0, \infty)$.

Proof. (a) We have, for $u>0$ and $\alpha \in [0, 1]$,

$$f[(\alpha^{1/s}+(1-\alpha)^{1/s})u] \le \alpha f(u)+(1-\alpha)f(u)=f(u).$$

The function

$$h(\alpha)=\alpha^{1/s}+(1-\alpha)^{1/s}$$

is continuous on $[0, 1]$, decreasing on $[0, 1/2]$, increasing on $[1/2, 1]$ and $h([0, 1])=[h(1/2), h(1)]=[2^{1-1/s}, 1]$. This yields that

$$f(tu) \le f(u) \qquad \forall u>0,\ t \in [2^{1-1/s}, 1]. \tag{2}$$

If now $t \in [2^{2(1-1/s)}, 1]$ then $t^{1/2} \in [2^{1-1/s}, 1]$. Therefore, by the fact that (2) holds for all $u>0$, we get

$$f(tu)=f(t^{1/2}(t^{1/2}u)) \le f(t^{1/2}u) \le f(u)$$

for all $u > 0$. By induction we then obtain that

$$f(tu) \leq f(u) \qquad \forall u > 0,\ t \in (0, 1]. \tag{3}$$

Therefore, taking $0 < u \leq v$ and applying (3), we get

$$f(u) = f((u/v)v) \leq f(v),$$

which means that f is non-decreasing on $(0, \infty)$.

The second part we can prove in the following way. For $u > 0$ we have

$$f(\alpha u) = f(\alpha u + \beta 0) \leq \alpha^s f(u) + \beta^s f(0)$$

and taking $u \to 0^+$ we get

$$\lim_{u \to 0+} f(u) \leq \lim_{u \to 0+} f(\alpha u) \leq \alpha^s \lim_{u \to 0+} f(u) + \beta^s f(0),$$

and so

$$\lim_{u \to 0+} f(u) \leq f(0).$$

(b) We have for $u \in R_+$,

$$f(u) = f(u/2 + u/2) \leq f(u)/2^s + f(u)/2^s = 2^{1-s} f(u).$$

Therefore, $(2^{1-s} - 1) f(u) \geq 0$ and so $f(u) \geq 0$.

REMARK 1. The above results do not hold, in general, in the case of convex functions, i.e. when $s = 1$, because a convex function f: $R_+ \to R$, need not be either non-decreasing or non-negative.

REMARK 2. If $0 < s < 1$, then the function $f \in K_s^1$ is non-decreasing on $(0, \infty)$ but not necessarily on $[0, \infty)$.

EXAMPLE 1. Let $0 < s < 1$ and $a, b, c \in R$. Defining, for $u \in R_+$,

$$f(u) = \begin{cases} a & \text{if } u = 0, \\ bu^s + c & \text{if } u > 0, \end{cases}$$

we have the following.

(i) If $b \geq 0$ and $c \leq a$ then $f \in K_s^1$.

(ii) If $b \geq 0$ and $c < a$ then f is non-decreasing on $(0, \infty)$ but not on $[0, \infty)$.

(iii) If $b \geq 0$ and $0 \leq c \leq a$ then $f \in K_s^2$.

(iv) If $b > 0$ and $c < 0$ then $f \notin K_s^2$.

In the proof of (i) there are two non-trivial cases:

1^0. $u, v > 0$. Then $\alpha u + \beta v > 0$ and

$$\begin{aligned} f(\alpha u + \beta v) &= b(\alpha u + \beta v)^s + c \leq b(\alpha^s u^s + \beta^s v^s) + c \\ &= b(\alpha^s u^s + \beta^s v^s) + c(\alpha^s + \beta^s) \\ &= \alpha^s(bu^s + c) + \beta^s(bv^s + c) = \alpha^s f(u) + \beta^s f(v). \end{aligned}$$

2^0. $v > u = 0$ and $\beta > 0$. Then

$$\begin{aligned} f(\alpha 0 + \beta v) &= f(\beta v) = b\beta^s v^s + c = b\beta^s v^s + c(\alpha^s + \beta^s) \\ &= \alpha^s c + \beta^s(bv^s + c) = \alpha^s c + \beta^s f(v) \leq \alpha^s a + \beta^s f(v) \\ &= \alpha^s f(0) + \beta^s f(v). \end{aligned}$$

Similarly we can prove (iii). Property (ii) is obvious and property (iv) immediately follows from Theorem 1(b) since for sufficiently small u the function f is negative.

From known examples of s-convex functions we can build up other s-convex functions using the following composition property.

THEOREM 2. *Let* $0 < s \leq 1$. *If* $f, g \in K_s^1$ *and if* $F: R^2 \to R$ *is a convex and non-decreasing function in each variable then the function* $h: R_+ \to R$ *defined by*

$$h(u) = F(f(u), g(u))$$

is in K_s^1. *In particular, if* $f, g \in K_s^1$ *then* $f + g$, $\max(f, g) \in K_s^1$.

Proof. If $u, v \in R_+$ then for all $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$ we have

$$\begin{aligned} h(\alpha u + \beta v) &= F(f(\alpha u + \beta v), g(\alpha u + \beta v)) \\ &\leq F(\alpha^s f(u) + \beta^s f(v), \alpha^s g(u) + \beta^s g(v)) \\ &\leq \alpha^s F(f(u), g(u)) + \beta^s F(f(v), g(v)) = \alpha^s h(u) + \beta^s h(v). \end{aligned}$$

Since $F(u, v) = u + v$ and $F(u, v) = \max(u, v)$ are non-decreasing convex functions on R^2, therefore they yield particular cases of our theorem.

It is important to know when the condition $\alpha^s + \beta^s = 1$ $(\alpha + \beta = 1)$ in the definition of K_s^1 (K_s^2) can be equivalently replaced by the condition $\alpha^s + \beta^s \le 1$ $(\alpha + \beta \le 1$, respectively).

THEOREM 3. *(a) Let $f \in K_s^1$. Then inequality* (1) *holds for all $u, v \in R_+$ and all $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s \le 1$ if and only if $f(0) \le 0$.*

(b) Let $f \in K_s^2$. Then inequality (1) *holds for all $u, v \in R_+$ and all $\alpha, \beta \ge 0$ with $\alpha + \beta \le 1$ if and only if $f(0) = 0$.*

Proof. (a) Necessity is obvious by taking $u = v = 0$ and $\alpha = \beta = 0$. Therefore assume that $u, v \in R_+$, $\alpha, \beta \ge 0$ and $0 < \gamma = \alpha^s + \beta^s < 1$. Put $a = \alpha\gamma^{-1/s}$ and $b = \beta\gamma^{-1/s}$. Then $a^s + b^s = \alpha^s/\gamma + \beta^s/\gamma = 1$ and so we have sufficiency:

$$\begin{aligned}
f(\alpha u + \beta v) &= f(a\gamma^{1/s}u + b\gamma^{1/s}v) \\
&\le a^s f(\gamma^{1/s}u) + b^s f(\gamma^{1/s}v) \\
&= a^s f[\gamma^{1/s}u + (1-\gamma)^{1/s}0] + b^s f[\gamma^{1/s}v + (1-\gamma)^{1/s}0] \\
&\le a^s[\gamma f(u) + (1-\gamma)f(0)] + b^s[\gamma f(v) + (1-\gamma)f(0)] \\
&= a^s\gamma f(u) + b^s\gamma f(v) + (1-\gamma)f(0) \le \alpha^s f(u) + \beta^s f(v).
\end{aligned}$$

(b) Necessity. Taking $u = v = \alpha = \beta = 0$ we obtain $f(0) \le 0$ and using Theorem 1(a) we get $f(0) \ge 0$, therefore $f(0) = 0$.

Sufficiency. Let $u, v \in R_+$ and $\alpha, \beta \ge 0$ with $0 < \gamma = \alpha + \beta < 1$. Put $a = \alpha/\gamma$ and $b = \beta/\gamma$. Then $a + b = \alpha/\gamma + \beta/\gamma = 1$ and so

$$\begin{aligned}
f(\alpha u + \beta v) &= f(a\gamma u + b\gamma v) \\
&\le a^s f(\gamma u) + b^s f(\gamma v) = a^s f[\gamma u + (1-\gamma)0] + b^s f[\gamma v + (1-\gamma)0] \\
&\le a^s[\gamma^s f(u) + (1-\gamma)^s f(0)] + b^s[\gamma^s f(v) + (1-\gamma)^s f(0)] \\
&= a^s\gamma^s f(u) + b^s\gamma^s f(v) + (1-\gamma)^s f(0) = \alpha^s f(u) + \beta^s f(v).
\end{aligned}$$

Using the above theorem we can compare both definitions of the s-convexity.

THEOREM 4. *(a) Let $0 < s \le 1$. If $f \in K_s^2$ and $f(0) = 0$ then $f \in K_s^1$.*
(b) Let $0 < s_1 \le s_2 \le 1$. If $f \in K_{s_2}^2$ and $f(0) = 0$ then $f \in K_{s_1}^2$.
(c) Let $0 < s_1 \le s_2 \le 1$. If $f \in K_{s_2}^1$ and $f(0) \le 0$ then $f \in K_{s_1}^1$.

Proof. (a) Assume that $f \in K_s^2$ and $f(0) = 0$. For $u, v \in R_+$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, we have $\alpha + \beta \leq \alpha^s + \beta^s = 1$ and, by Theorem 3(b), we obtain

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v),$$

which means that $f \in K_s^1$.

(b) Assume that $f \in K_{s_2}^2$ and that $u, v \geq 0, \alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then we have

$$f(\alpha u + \beta v) \leq \alpha^{s_2} f(u) + \beta^{s_2} f(v) \leq \alpha^{s_1} f(u) + \beta^{s_1} f(v),$$

which means that $f \in K_{s_1}^2$.

(c) Assume that $f \in K_{s_2}^1$ and that $u, v \geq 0, \alpha, \beta \geq 0$ with $\alpha^{s_1} + \beta^{s_1} = 1$. Then $\alpha^{s_2} + \beta^{s_2} \leq \alpha^{s_1} + \beta^{s_1} = 1$ and according to Theorem 3(a) we have

$$f(\alpha u + \beta v) \leq \alpha^{s_2} f(u) + \beta^{s_2} f(v) \leq \alpha^{s_1} f(u) + \beta^{s_1} f(v),$$

which means that $f \in K_{s_1}^1$.

3. On non-negative s-convex functions

Let us note first that, if f is a non-negative function from K_s^1 and $f(0) = 0$, then f is right continuous at 0, i.e., $f(0^+) = f(0) = 0$.

We prove now the following other important observation.

THEOREM 5. *Let $0 < s < 1$ and let $p: R_+ \to R_+$ be a non-decreasing function. Then the function f defined for $u \in R_+$ by*

$$f(u) = u^{s/(1-s)} p(u) \tag{4}$$

belongs to K_s^1.

Proof. Let $v \geq u \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. We will consider two cases.

1^0. Let $\alpha u + \beta v \leq u$. Then

$$f(\alpha u + \beta v) \leq f(u) = (\alpha^s + \beta^s) f(u) \leq \alpha^s f(u) + \beta^s f(v).$$

2^0. Let $\alpha u + \beta v > u$. This yields $\beta v > (1-\alpha)u$ and so $\beta > 0$.

Since $\alpha \le \alpha^s$ for $\alpha \in [0, 1]$, we get $\alpha - \alpha^{s+1} \le \alpha^s - \alpha^{s+1}$ and then

$$\alpha/(1-\alpha) \le \alpha^s/(1-\alpha^s) = (1-\beta^2)/\beta^s,$$

i.e.,

$$\alpha\beta/(1-\alpha) \le \beta^{1-s} - \beta. \tag{5}$$

We have also

$$\alpha u + \beta v \le (\alpha + \beta)v \le (\alpha^s + \beta^s)v = v,$$

and, in view of (5),

$$\alpha u + \beta v \le \alpha\beta v/(1-\alpha) + \beta v \le (\beta^{1-s} - \beta)v + \beta v = \beta^{1-s}v,$$

whence

$$(\alpha u + \beta v)^{s/(1-s)} \le \beta^s v^{s/(1-s)}. \tag{6}$$

Applying (6) and the monotonicity of p, we get

$$\begin{aligned} f(\alpha u + \beta v) &= (\alpha u + \beta v)^{s/(1-s)} p(\alpha u + \beta v) \\ &\le \beta^s v^{s/(1-s)} p(\alpha u + \beta v) \le \beta^s v^{s/(1-s)} p(v) \\ &= \beta^s f(v) \le \alpha^s f(u) + \beta^s f(v), \end{aligned}$$

which finishes the proof.

REMARK 3. For $0 < s < 1$ functions in K_s^1 need not be continuous on $(0, \infty)$.

EXAMPLE 2. Let $0 < s < 1$ and $k > 1$. Define, for $u \in R_+$,

$$f(u) = \begin{cases} u^{s/(1-s)} & \text{if } 0 \le u \le 1, \\ ku^{s/(1-s)} & \text{if } u > 1. \end{cases}$$

The function f is non-negative, discontinuous at $u = 1$ and belongs to K_s^1 but not to K_s^2.

Proof. It was proved in Theorem 5 that $f \in K_s^1$. Now, we will prove that $f \notin K_s^2$. Take an arbitrary $a > 1$ and put $u = 1$. Consider all $v > 1$ such that $\alpha u + \beta v = \alpha + \beta v = a$, where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. In the case when $f \in K_s^2$ it must be

$$ka^{s/(1-s)} \leq \alpha^s + k(1-\alpha)^s[(a-\alpha)/(1-\alpha)]^{s/(1-s)} \tag{7}$$

for all $a > 1$ and all $0 \leq \alpha \leq 1$.

Define the functions

$$f_\alpha(a) = \alpha^s + k(1-\alpha)^s[(a-\alpha)/(1-\alpha)]^{s/(1-s)} - ka^{s/(1-s)}.$$

These functions are continuous on the interval (α, ∞) and

$$g(\alpha) = f_\alpha(1) = \alpha^s + k(1-\alpha)^s - k.$$

The function g is continuous on $[0, 1]$ and $g(1) = 1 - k < 0$. Therefore, there is a number α_0, $0 < \alpha_0 < 1$, such that $g(\alpha_0) = f_{\alpha_0}(1) < 0$. The continuity of f_{α_0} yields that $f_{\alpha_0}(a) < 0$ for a certain $a > 1$, i.e. inequality (7) does not hold, which means that $f \notin K_s^2$.

THEOREM 6. *Let $f \in K_{s_1}^1$ and $g \in K_{s_2}^1$, where $0 < s_1, s_2 \leq 1$.*

(*a*) *If f is a non-decreasing function and g is non-negative function such that $f(0) \leq 0 = g(0)$ then the composition $f \circ g$ of f with g belongs to K_s^1, where $s = s_1 s_2$.*

(*b*) *Assume that $0 < s_1, s_2 < 1$. If f and g are non-negative functions such that either $f(0) = 0$ and $g(0^+) = g(0)$ or $g(0) = 0$ and $f(0^+) = f(0)$ then the product fg of f and g belongs to K_s^1, where $s = \min(s_1, s_2)$.*

Proof. (a) Let $u, v \in R_+$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, where $s = s_1 s_2$. Since $\alpha^{s_i} + \beta^{s_i} \leq \alpha^{s_1 s_2} + \beta^{s_1 s_2} = 1$ for $i = 1, 2$, therefore, according to Theorem 3(a) and the assumptions, we have

$$\begin{aligned} f \circ g(\alpha u + \beta v) &= f(g(\alpha u + \beta v)) \leq f(\alpha^{s_2} g(u) + \beta^{s_2} g(v)) \\ &\leq \alpha^{s_1 s_2} f(g(u)) + \beta^{s_1 s_2} f(g(v)) = a^s f \circ g(u) + \beta^s f \circ g(v), \end{aligned}$$

which means that $f \circ g \in K_s^1$.

(b) According to Theorem 1(a), both functions f and g are non-decreasing on $(0, \infty)$. Therefore

$$(f(u) - f(v))(g(v) - g(u)) \leq 0$$

or, equivalently,

$$f(u)g(v) + f(v)g(u) \le f(u)g(u) + f(v)g(v) \tag{8}$$

for all $v \ge u > 0$. If $v > u = 0$ then inequality (8) is still true because f, g are non-negative and either $f(0) = 0$ and $g(0^+) = g(0)$ or $g(0) = 0$ and $f(0^+) = f(0)$. Now let $u, v \in R_+$ and $\alpha, \beta \ge 0$ with $\alpha^s + \beta^s = 1$, where $s = \min(s_1, s_2)$. Then $\alpha^{s_i} + \beta^{s_i} \le \alpha^s + \beta^s = 1$ for $i = 1, 2$ and by Theorem 3(a) and inequality (8) we have

$$\begin{aligned}
&f(\alpha u + \beta v)g(\alpha u + \beta v)\\
&\quad \le (\alpha^{s_1} f(u) + \beta^{s_1} f(v))(\alpha^{s_2} g(u) + \beta^{s_2} g(v))\\
&\quad = \alpha^{s_1 + s_2} f(u)g(u) + \alpha^{s_1}\beta^{s_2} f(u)g(v) + \alpha^{s_2}\beta^{s_1} f(v)g(u) + \beta^{s_1 + s_2} f(v)g(v)\\
&\quad \le \alpha^{2s} f(u)g(u) + \alpha^s\beta^s(f(u)g(v) + f(v)g(u)) + \beta^{2s} f(v)g(v)\\
&\quad \le \alpha^{2s} f(u)g(u) + \alpha^s\beta^s(f(u)g(u) + f(v)g(v)) + \beta^{2s} f(v)g(v)\\
&\quad = \alpha^s f(u)g(u) + \beta^s f(v)g(v),
\end{aligned}$$

which means that $fg \in K_s^1$.

REMARK 4. From the above proof it is also easy to see that, if f is a non-decreasing function in K_s^2 and g is a non-negative convex function on $[0, \infty)$, then the composition $f \circ g$ of f with g belongs to K_s^2.

REMARK 5. Convex functions on $[0, \infty)$ need not be monotonic. However, if f and g are non-negative, convex and either both are non-decreasing or both are non-increasing on $[0, \infty)$ then the product fg is also a convex function.

COROLLARY 1. *If Φ is a convex φ-function and g is a φ-function from K_s^1 then the composition $\Phi \circ g$ belongs to K_s^1. In particular, the φ-function $h(u) = \Phi(u^s)$ belongs to K_s^1.*

COROLLARY 2. *If Φ is a convex φ-function and f is a φ-function from K_s^2 then the composition $f \circ \Phi$ belongs to K_s^2. In particular, the φ-function $h(u) = \Phi(u^s)$ belongs to K_s^2.*

EXAMPLE 3. Let $0 < s < 1$. Then there exists a φ-function f in the class K_s^2 which is neither of the form $\Phi(u^s)$ nor $\Phi(u)^s$ with a convex function Φ.

Define

$$f(u) = \begin{cases} u & \text{for } 0 \le u \le 1 \\ u^s & \text{for } u > 1. \end{cases}$$

We will prove that $f \in K_s^2$. Assume that $u, v \in R_+$, $\alpha, \beta \ge 0$ and $\alpha + \beta = 1$. We have obviously that for $u, v \ge 1$ as well as for $u, v \in [0, 1]$ inequality (1) holds. Let us consider the remaining cases.

1^0. $0 \le u \le 1$, $v \ge 1$ and $\alpha u + \beta v \le 1$. Then, since $\alpha \le 1$ and $\beta v \le 1$,

$$f(\alpha u + \beta v) = \alpha u + \beta v \le \alpha^s u + \beta^s v^s = \alpha^s f(u) + \beta^s f(v).$$

2^0. $0 \le u \le 1$, $v \ge 1$ and $\alpha u + \beta v > 1$. We need to prove the inequality

$$(\alpha u + \beta v)^s \le \alpha^s u + \beta^s v^s. \tag{9}$$

Fix an arbitrary $a > 1$ and assume that $\alpha u + \beta v = a$. Inequality (9) is then equivalent to

$$a^s \le \alpha^s u + (a - \alpha u)^s \qquad \forall u \in [0, 1]. \tag{10}$$

Define on the interval $[0, 1]$ the function

$$h(u) = \alpha^s u + (a - \alpha u)^s - a^s.$$

We want to prove that $h(u) \ge 0$ for all $u \in [0, 1]$.

Since $h''(u) = \alpha^2 s(s-1)(a - \alpha u)^{s-2} \le 0$ therefore h has no local minimum on the interval $[0, 1]$. Thus

$$\begin{aligned} \inf\{h(u) : u \in [0, 1]\} &= \min\{h(0), h(1)\} \\ &= \min\{0, \alpha^s + (a - \alpha)^s - a^s\} = 0, \end{aligned}$$

and so $h(u) \ge 0$ for all $u \in [0, 1]$, which finishes the proof of the fact that $f \in K_s^2$.

Now, for

$$\Phi(u) = \begin{cases} u^{1/s} & \text{for } 0 \le u \le 1, \\ u & \text{for } u > 1, \end{cases}$$

we have that $f(u) = \Phi(u^s) = \Phi(u)^s$ and Φ is non-convex φ-function.

THEOREM 7. *If* $0 < s < 1$ *and* $f \in K_s^1$ *is a* φ*-function then there exists a convex* φ*-function* Φ *such that*

$$f(2^{-1/s}u) \le \Phi(u^s) \le f(u)$$

for all $u \ge 0$.

Proof. By the s-convexity of the function f and by $f(0) = 0$, we obtain $f(\alpha u) \le \alpha^s f(u)$ for all $u \ge 0$ and all $\alpha \in [0, 1]$.

Assume now that $v > u \ge 0$. Then $f(u^{1/s}) \le f((u/v)^{1/s} v^{1/s}) \le (u/v) f(v^{1/s})$, i.e.,

$$f(u^{1/s})/u \le f(v^{1/s})/v. \tag{11}$$

Inequality (11) means that the function $f(u^{1/s})/u$ is a non-decreasing function on $(0, \infty)$. Define

$$\Phi(u) = \begin{cases} 0 & \text{for } u = 0, \\ \int_0^u f(t^{1/s})/t\, dt & \text{for } u > 0. \end{cases}$$

Then Φ is a convex φ-function and

$$\Phi(u^s) = \int_0^{u^s} f(t^{1/s})/t\, dt \le (f((u^s)^{1/s})/u^s) u^s = f(u),$$

$$\Phi(u^s) \ge \int_{u^s/2}^{u^s} f(t^{1/s})/t\, dt \ge (f((u^s/2)^{1/s}) 2u^{-s}) u^s/2 = f(2^{-1/s}u).$$

Therefore,

$$f(2^{-1/s}u) \le \Phi(u^s) \le f(u)$$

for all $u \ge 0$, which means that the function Ψ defined by $\Psi(u) = \Phi(u^s)$ is equivalent to f (this kind of equivalence is taken from the theory of Orlicz spaces—cf. [3], [5]), and the proof is complete.

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