SOME REMARKS ON S. WEINGRAM: ON THE TRIANGULATION OF A SEMISIMPLICIAL COMPLEX [8]

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In this note we use the terminology and notation of [8].

1. Intention

Weingram's paper is concerned with a proof of the following theorem [8, Theorem 1.1].

THEOREM 1. *Let X be a semisimplicial complex and $|X|$ its geometric realization (in the sense of* [6] *or* [7]). *Then there is a functor D from the category of semisimplicial complexes and semisimplicial maps to that of ordered simplicial complexes and weak order-preserving maps, a transformation of functors $\lambda : D \to 1$, and, for each X, a map $t_x : |DX| \to |X|$ such that*

(i) *t_x is a homeomorphism (and therefore a triangulation of $|X|$);*
(ii) *t_x defines a subdivision of the CW complex $|X|$; and*
(iii) *$|\lambda(X)|$ is homotopic to t_x by a homotopy F such that for each cell $|e|$ of $|DX|$, F maps $|e| \times I$ into the smallest cell $|x|$ of $|X|$ which contains $t_x(|e|)$.*

The statement of this theorem is correct, also the idea of the proof given at the end of §1 in Weingram's paper. But in the details there are several mistakes which shall be corrected in the following.

2. On the barycentric subdivision functor

Weingram needs the following

THEOREM 2. *For any semisimplicial complex X, there is a homeomorphism $t : |\mathrm{Sd}\, X| \to |X|$ identifying $|\mathrm{Sd}\, X|$ with a subdivision of the CW-complex $|X|$* [8, Proposition 2.5].

In order to prove this theorem, Weingram wants—analogous to M. G. Barratt [1]—to subdivide $|X|$ by a modified star-subdivision process. To do so, he has to choose in each face of a topological simplex which is the source of the realization of the characteristic map of a simplex of X an interior point—called pseudo-barycenter—and to subdivide these topological simplices by starring. Having done this Weingram states without proof that there is a "consistent subdivision of $|X|$". However in general this is impossible, as it is shown by the counterexample described at the end of [4]. Nevertheless the statement of theorem 2 is correct, proofs can be found in [2] and [4], which also contain proposition 2.6 of [8].

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But it is impossible to choose the homeomorphisms $t : |\mathrm{Sd}\, X| \to |X|$ in a natural way, that means such that for each semisimplicial map $f : X \to X'$ the following diagram commutes:

$$\begin{array}{ccc} |\mathrm{Sd}\, X| & \xrightarrow{|\mathrm{Sd}\, f|} & |\mathrm{Sd}\, X'| \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ |X| & \xrightarrow[|f|]{} & |X'| \end{array} .$$

Since many people assert something other[1], this fact shall be shown here by the following trivial counterexample: Assume that there are homeomorphisms

$$t : |\mathrm{Sd}\, \Sigma(2)|^{2} \to |\Sigma(2)| \quad \text{and} \quad t' : |\mathrm{Sd}\, \Sigma(1)| \to |\Sigma(1)|$$

such that the diagram

$$\begin{array}{ccc} |\mathrm{Sd}\, \Sigma(2)| & \xrightarrow{|\mathrm{Sd}\, s_i^*|} & |\mathrm{Sd}\, \Sigma(1)| \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ |\Sigma(2)| & \xrightarrow[|s_i^*|]{} & |\Sigma(1) \end{array}$$

commutes for $i = 0$ and $i = 1$. Let b be the inner vertex of $|\mathrm{Sd}\, \Sigma(2)|$ and b' the inner vertex of $|\mathrm{Sd}\, \Sigma(1)|$; then

$$|\mathrm{Sd}\, s_0^*|\ (b) = b' = |\mathrm{Sd}\, s_1^*|\ (b)$$

and

$$t(b) \in |s_0^*|^{-1}(t'(b')) \cap |s_1^*|^{-1}\ (t'(b')).$$

The intersection on the right of the last formula contains only boundary points of $|\Sigma(2)|$, but since t shall be a homeomorphism $t(b)$ must be a interior point of $|\Sigma(2)|$, thus we have a contradiction; cf. Figure 1.

3. On regulated semisimplicial complexes

The definitions of a "regulated simplex" and a "regulated semisimplicial complex" given in §3 of [8] seem to be convenient. We repeat (using other words):

DEFINITION. Let x be a simplex of the semisimplicial complex X and let $\varphi : \Sigma(n) \to X$ be its characteristic map. x is *regulated* if the restriction of φ on $\Sigma(n) - d_0^*(\Sigma(n-1))$ is injective. X is *regulated* if each nondegenerate simplex of X is regulated.

For proving lemma 3.3, Corollary 3.4 and Proposition 3.6 of [8] it does not

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[1] Though Weingram says nothing about naturality one finds in the Mathematical Reviews, vol. 38 (1969), #6579, the assertion that Weingram proved the existence of natural homeomorphisms!

[2] In the definition of Sd $\Sigma(n)$ in [8] there is a misprint: the word "nonincreasing" [8, Definition 2.1, 3rd line] is to be replaced by "nondecreasing".

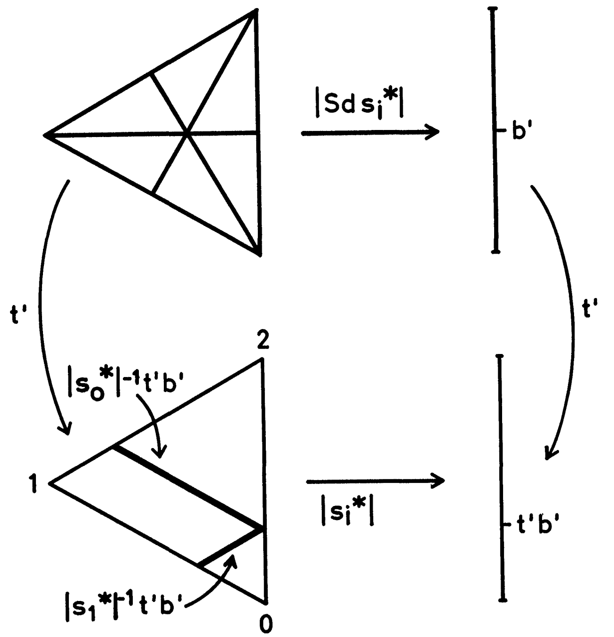

FIGURE 1

suffice to assume that the simplex under consideration is regulated as Weingram does; one needs that the semisimplicial subcomplex which is generated by this simplex also is regulated. Moreover the conclusions in [8] for proving Lemma 3.3 do not lead really to a proof. Thus we give here a formulation and a proof of this lemma

LEMMA 1. *Let X be a semisimplicial complex and x a nondegenerate n-simplex of X such that the subcomplex $\{x\}^*$ of X which is generated by x is regulated; further let $\varphi : \Sigma(n) \to X$ denote the semisimplicial characteristic map of x. Then there is an integer p and a face operator F such that $|\varphi|$ is injective on all open cells outside of the p-dimensional face $F^*\Delta^p$ of Δ^n. On this face, there is a face $F'^*\Delta^q$ such that the restriction of $|\varphi|$ to $F^*\Delta^p$ is $|\varphi'|\cdot D^*$, where D is the identity operator or a suitable degeneracy operator, $F'x$ a nondegenerate simplex of X and φ' its characteristic map.*

Proof. Since x is regulated, φ is injective on $\Sigma(n) - d_0^*(\Sigma(n-1))$. If $d_0 x$ is nondegenerated, then it also is regulated by the assumption on $\{x\}^*$ and therefore φ must be injective on $\Sigma(n) - (d_0)^*(\Sigma(n-2))$. This conclusion can be continued such that we obtain either

(i) φ is injective on all simplices of $\Sigma(n)$—then is nothing to prove—or
(ii) there is an integer p' such that $(d_0^{p'})^*x$ is degenerated.

In this case let $\bar{p}$ denote the smallest integer with this property

$$p := n - \bar{p} \quad \text{and} \quad F := d_0^{\bar{p}}$$

It follows that φ is injective on $\Sigma(n) - F^*(\Sigma(p))$; therefore $|\varphi|$ also is injective on $\Delta^n - F^*\Delta^p$. Further there is a unique degeneracy operator D and a unique nondegenerate simplex $y \in X$ such that

$$Fx = Dy$$

Now choose any face operator F'' such that $F''D$ is the identity operator and define

$$F' = F''F$$

Then one has

$$|\varphi|\cdot F^* = |\varphi\cdot F^*| = |\varphi'\cdot D^*| = |\varphi'|\cdot D^*$$

where φ' is the characteristic map of $F'x = y$.

4. The effect of one degeneration map

One essential step in Weingram's proof of Theorem 1 is the following [8, Lemma 3.5].

LEMMA 2. *Let $\tau \subset \sigma \subset \Delta^n$ be proper faces, let $D^* : \sigma \to \tau$ be a degeneration map, let L be the quotient of Δ^n by the identifications of D^* and let $\varphi : \Delta^n \to L$ be the quotient map. Then there is a homeomorphism $h : \Delta^n \to L$ such that $h \mid \tau = \varphi \mid \tau$.*

In order to prove this Weingram first defines a continuous map ρ by the following procedure. Let σ' be the face of Δ^n opposite σ. Each point P of Δ^n has a representation

$$P = (1 - t)Q + tQ',$$

where $Q \in \sigma$, $Q' \in \sigma'$, $t \in I$. In this representation of a point P, the real number $t \in I$ is always uniquely determined, the point Q iff $t \neq 1$ and Q' iff $t \neq 0$. Then ρ is defined by

$$\begin{aligned} \rho(P) &= p && \text{if } \tfrac{1}{2} \leq t \leq 1 \\ &= t(Q + Q') + (1 - 2t)D^*Q && \text{if } 0 \leq t \leq \tfrac{1}{2}. \end{aligned}$$

Then clearly

(i) $\rho(P) = \rho(P')$ iff $\varphi(P) = \varphi(P')$,
(ii) image ρ is a compact subset of Δ^n,
(iii) $\rho(P) = P$ for all $P \in \tau$.

These conditions imply that image ρ is homeomorphic to L by means of a homeomorphism h' : image $\rho \to L$ such that

$$h' \mid \tau = \varphi \mid \tau.$$

In order to proceed Weingram asserts without proof the image ρ also is a convex subset of Δ^n. But in general this does not happen. We consider the following counter-example: $n = 3$, $\sigma = d_0^*\Delta^2$, $\tau = (d_0^2)^*\Delta^1$ and $D = s_0$.

In the boundary of Δ^3 we look at the trapezium

$$T = \{P = (1 - t)Q + tQ' \mid 0 \leq t \leq \tfrac{1}{2}, Q \in d_0^* d_2^* \Delta^1, Q' \in \sigma'\}$$

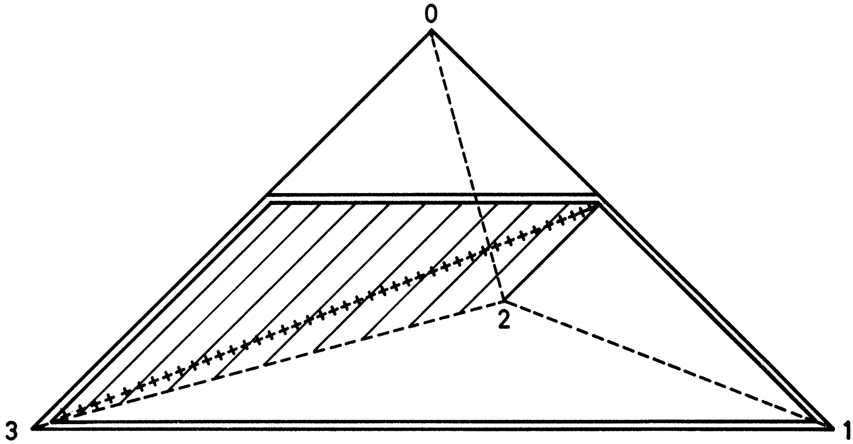

FIGURE 2

Clearly $\rho(\tau)$ is a subset of the boundary of image ρ, but it also is a part of a ruled surface. Figure 2 shows that image ρ cannot be convex.

(The double lines in Figure 2 indicate the boundary of T, $\rho(T)$ is hatched, the crossed line segment joins points of image ρ, but its interior points do not belong to image ρ).

Nevertheless the statement of Lemma 2 is true. In order to show this it suffices to prove the following

ASSERTION. *There is a homeomorphism* $h'' : \Delta_n \to$ image ρ *such that*

$$h'' \mid \tau = \rho \mid \tau$$

Proof. For each triple (Q, Q', t) such that $Q \in \sigma$, $Q' \in \sigma'$, $t \in I$, let $w(Q, Q', t)$ denote the subset of Δ^n which is given by

$$w(Q, Q', t) = \{P = (1 - \bar{t})Q + \bar{t}Q' \mid t \leq \bar{t} \leq 1\}.$$

Then

(iv) $w(Q, Q', \frac{1}{2}) \subset$ image ρ

(v) image $\rho \cap w(Q, Q', 0)$ is connected for all $Q \in \sigma$, $Q' \in \sigma'$.

Now, let H denote the convex hull of image ρ. H is also a compact subset of Δ'' and

(vi) $H \cap \sigma = \tau$.

For each $(Q, Q') \in \sigma \times \sigma'$ let

$$h_{Q,Q'} : w(Q, Q', 0) \cap H \to w(Q, Q', 0) \cap \text{image } \rho$$

be the homeomorphism which maps its domain linearly onto its range. Especially we have

(vii) $h_{Q,Q'} =$ identity for all $Q \in \tau$.

Then it is possible to define a homeomorphism $h_H : H \to \text{image } \rho$ by

$$h_H P = h_{Q,Q'} P \quad \text{if } P \in \text{domain } h_{Q,Q'} ;$$

from (vi) and (vii) it follows that h_H is single-valued. Further we have

$$h_H Q = Q \quad \text{for all } Q \in \tau.$$

Now, let $h'_H : \Delta^n \to H$ be any homeomorphism which is constructed by a radial contraction to the barycenter of σ'. Each such homeomorphism maps any point of τ onto itself.

Then we obtain the desired h'' by

$$h'' = h_H \circ h'_H .$$

After these corrections one can use Weingram's method to prove Theorem 1.

5. Simplicial approximation

As a corollary of Theorem 1 Weingram formulates a "simplicial approximation theorem" which shall lead to a semisimplicial approximation map. To prove this he refers to the ordinary simplicial approximation theorem. But the latter only leads to simplicial maps between unordered simplicial complexes, and Weingram says nothing how to get a semisimplicial map, that means an order-preserving simplicial map between ordered simplicial complexes. Thus one also has to refer to the following

LEMMA 3. *Let K, L be (unordered) simplicial complexes and $f : K \to L$ a simplicial map. Then K and L can be ordered such that f is a weak order-preserving map.*

Proof. We take any ordering $\leq_L$ for L. Now we construct an ordering for K, that means a partial ordering for K^0, the set of vertices of K, such that each subset of K^0 which spans a simplex of K is totally ordered.

To this end, for each $u \in L^0$, we take any total ordering $\leq_u$ of the set vertices of K which are mapped on u under f. Then, for $v, w \in K^0$, we define

$$v \leq_K w \quad \text{if } f(v) \leq_L f(w)$$

and either

$$f(v) \neq f(w) \qquad \text{or} \qquad f(v) = f(w) = u \quad \text{and} \quad v \leq_u w.$$

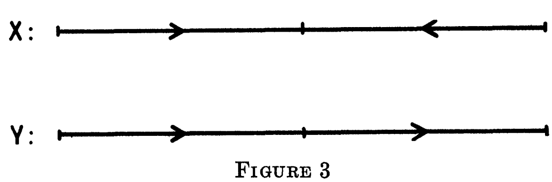

FIGURE 3

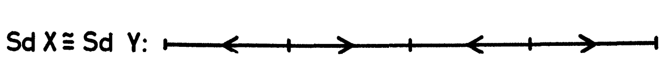

FIGURE 4

Clearly this gives an ordering for K and f becomes a weak-order-preserving map.

Lemma 3 is almost trivial; one could think its mentioning is superfluous. But there is one point which should be insisted. Let X, Y be semissimplicial complexes and $f : |X| \to |Y|$ a continuous map. Then the approximation theorem leads to subdivisions X' and Y' of X, resp. Y, and a semisimplicial map

$$f' : X' \to Y'.$$

The proof shows that one can assume X' as ordered simplicial complex, however the ordering of X' depends not on X but only on f, resp. f'. In the context of Weingram's paper this doesn't matter, since here the subdivision processes start with a kind of subdivision which does not preserve something about the original ordering. We make this clear by means of the following example.

We consider the one-dimensional semisimplicial complexes X, Y which are described by Figure 3. X and Y are simplicially, but not semisimplicially isomorphic, they differ in their orderings. But Sd X and Sd Y also are semisimplicially isomorphic (see Figure 4). Thus already from Sd X one gets no information about the ordering of X and therefore it is not of importance which ordering one has on a certain further subdivision X' of X. But there are other subdivision processes (cf. [3]) which preserve the original ordering in a certain sense and then it might be of interest to construct semisimplicial approximations with respect to given orderings. In this case Weingram's method does not work immediately. Perhaps one has to construct approximations similar to the ideas of Kan in [5].

Added in Proof. Meanwhile the content of [8] has been taken over nearly unchanged in the book of Albert T. Lundell and Stephen Weingram: *The Topology of CW Complexes* (Van Nostrand Reinhold Company, New York–Cincinnati–Toronto–London–Melbourne, 1969).

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