Some remarks on stability of cones for the one-phase free boundary problem

SOME REMARKS ON STABILITY OF CONES FOR THE ONE-PHASE FREE BOUNDARY PROBLEM

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ABSTRACT. We show that stable cones for the one-phase free boundary problem are hyperplanes in dimension 4. As a corollary, both one and two-phase energy minimizing hypersurfaces are smooth in dimension 4.

1. INTRODUCTION

We investigate stable homogenous solutions

$$u : \overline{\Omega} \to \mathbb{R}, \qquad \Omega \subset \mathbb{R}^n,$$

to the one-phase free boundary problem

$$\triangle u = 0 \quad \text{in } \Omega, \qquad |\nabla u| = 1 \quad \text{on } \partial\Omega \setminus \{0\}. \tag{1.1}$$

Here u is a homogenous of degree one function which is positive in Ω, and Ω is a conical domain generated by a smooth domain Ω_S on the unit sphere, i.e.

$$\Omega_S := \Omega \cap \partial B_1.$$

We are interested in solutions u which are stable with respect to the Alt-Caffarelli (see [AC]) energy functional

$$E(u, B) = \int_B |\nabla u|^2 + \chi_{\{u>0\}}\, dx. \tag{1.2}$$

We consider those solutions which are stable with respect to compact domain deformations that do not contain the origin. Precisely we require that for any smooth vector field $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ with $0 \notin supp\ \Psi \subset B_R$ we have

$$\frac{d^2}{dt^2}\, E\left(u(x + t\Psi(x)), B_R\right) \geq 0 \qquad \text{at } t = 0. \tag{1.3}$$

There is a vast literature concerning the one-phase free boundary problem, see for example the book of Caffarelli and Salsa [CS]. Many results in the regularity theory of the free boundary $\partial\{u > 0\}$ parallel the corresponding statements in the regularity theory of minimal surfaces, see [C1, C2, DJ2, W].

Our main result is the following.

Theorem 1.1. *The only stable homogenous solutions in dimension $n \leq 4$ are the one-dimensional solutions $u = (x \cdot \nu)^+$.*

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For dimension $n = 3$ this result was obtained by Caffarelli, Jerison and Kenig in [CJK], and they conjectured that it remains true up to dimension $n \le 6$. On the other hand De Silva and Jerison provided in [DJ1] an example of a nontrivial minimal solution in dimension $n = 7$.

The main consequence of Theorem 1.1 is that it implies the smoothness of the free boundary for minimizers in both the one-phase and *two-phase problem* in dimension $n \le 4$. Moreover, by the dimension reduction arguments of Weiss [W], we obtain the following regularity result.

Corollary 1.2. *Let v be a minimizer of the energy functional*

$$J(v) := \int_{B_1} \left(|\nabla v|^2 + Q_+(x)\chi_{\{v>0\}} + Q_-(x)\chi_{\{v\le 0\}}\right) dx$$

with $Q_\pm$ smooth functions satisfying $Q_+ > Q_-$. Then the free boundary

$$F(v) := \partial\{v > 0\} \cap B_1$$

is a smooth hypersurface except possibly on a closed singular set $\Sigma \subset F(v)$ of Hausdorff dimension $n - 5$, and

$$(v_\nu^+)^2 - (v_\nu^-)^2 = Q_+ - Q_- \qquad \text{on } F(v) \setminus \Sigma.$$

The proof of Theorem 1.1 is similar to Simons proof of rigidity of stable minimal cones in low dimensions: we find a function involving the second derivatives of u which satisfies a differential inequality for the linearized equation.

The paper is organized as follows. In Section 2 we collect some basic facts about stability and the linearized equation of u. In Section 3 we obtain the differential inequality for a function involving $\|D^2u\|$ and deduce the rigidity result in dimension $n = 3$. Finally in Section 4 we prove Theorem 1.1 by modifying slightly the function considered in Section 3.

2. Preliminaries and stability

In this section we recall some facts about stability of solutions u of (1.1) that were obtained in [CJK]. We insist a bit more on the non-variational approach to stability.

2.1. **Normals for second derivatives at the boundary.** Fix a point

$$x_0 \in \partial\Omega \setminus \{0\}$$

and choose a system of coordinates at x_0 such that

$$e_n = \nu_{x_0} \quad \text{the interior normal at } x_0$$

and $\partial\Omega$ is given locally by the graph of a function g

$$\Omega = \{x_n > g(x')\}, \quad \text{with} \quad \nabla_{x'} g(x_0') = 0, \quad D^2_{x'} g(x_0') \quad \text{diagonal.}$$

By differentiating $u(x', g(x')) = const.$ in the i, j directions, i,$j < n$, we obtain

$$u_i = 0, \qquad u_{ij} = -u_n\, g_{ij} = -g_{ij} \qquad \text{at } x_0. \tag{2.1}$$

If we apply these equalities for the function $\frac{1}{2}|\nabla u|^2 = \frac{1}{2}u_k^2$ instead of u we obtain

$$u_k\, u_{ki} = 0, \qquad u_n\, u_{nij} + u_{ki}\, u_{kj} = -u_n\, u_{nn}\, g_{ij},$$

and we use throughout the convention of summation over repeated indices. Here k runs over the indices $1, 2, .., n$. In conclusion at x_0 we satisfy (here $i, j < n$)

$$u_{in} = 0, \quad \text{thus} \quad D^2u \quad \text{is diagonal}$$

and

$$u_{ijn} = 0 \qquad \text{if } i \neq j,$$
$$(2.2) \qquad u_{iin} = u_{nn}\, u_{ii} - u_{ii}^2 \qquad \text{for each } i < n,$$
$$u_{nnn} = u_{nn}\, u_{nn} + \sum_{i \neq n} u_{ii}^2,$$

where the last equation follows from the previous one and $\triangle u_n = 0$.

2.2. **The linearized equation.** A smooth function $v : \overline{\Omega} \to \mathbb{R}$ solves the linearized equation for a solution u if

$$(2.3) \qquad \begin{cases} \triangle v = 0 & \text{in } \Omega, \\ \\ v_\nu = u_{\nu\nu}\, v & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

Notice that from $\triangle u = 0$ and (2.1) it follows that

$$-u_{\nu\nu} = H$$

where H denotes the mean curvature of $\partial\Omega$ oriented towards the complement of Ω. Thus the second equation in (2.3) can be rewritten as

$$v_\nu + H\, v = 0 \qquad \text{on } \partial\Omega.$$

In the case when Ω is a cone different from a half-space, it easily follows that

$$H > 0.$$

Indeed, $|\nabla u|^2/2$ is a subharmonic function homogenous of degree 0, and its maximum occurs on the boundary. Then either $|\nabla u|^2/2$ is constant or by Hopf lemma its normal derivative on $\partial\Omega$, which equals $-H$, is negative.

The linearized equation (2.3) is obtained by requiring that $(u + \epsilon v)^+$ solves the original equation up to an error of order $O(\epsilon^2)$ (here we think that u and v are extended smoothly in a neighborhood of $\partial\Omega$). Thus the function v above represents the infinitesimal vertical distance between the graph of a perturbed solution and the graph of the original solution u of (1.1).

We deduce briefly (2.3). The interior condition for v is obvious. For the boundary condition we see that the free boundary of $(u + \epsilon v)^+$ lies in $O(\epsilon^2)$ of the surface Γ_ϵ obtained as

$$x \in \Gamma_0 := \partial\Omega \quad \longmapsto \quad x_\epsilon \in \Gamma_\epsilon, \qquad x_\epsilon := x - \epsilon\, v(x)\, \nu_x.$$

Thus

$$\begin{aligned} \nabla(u + \epsilon v)(x_\epsilon) =& \nabla u(x) - D^2 u(x)(x_\epsilon - x) + \epsilon\, \nabla v(x_\epsilon) + O(\epsilon^2) \\ =& \nu - \epsilon\, w\, D^2 u\, \nu + \epsilon\, \nabla v(x) + O(\epsilon^2) \end{aligned}$$

and

$$|\nabla(u + \epsilon v)(x_\epsilon)|^2 = 1 + 2\epsilon(v_\nu - v\, u_{\nu\nu}) + O(\epsilon^2),$$

which gives the second condition in (2.3).

Clearly the derivatives u_e, $|e| = 1$, solve the linearized equation and this can be seen also from (2.1).

2.3. **Criteria for stability and instability.** Let u be a homogeneous one-phase free boundary solution u as in (1.1) supported on the cone Ω. Consider the annulus

$$\mathcal{U} = \{x \in \mathbb{R}^n : 0 < c_1 < |x| < c_2\}$$

The main lemma of [CJK] says that the stability (1.3) under perturbations in $\mathcal{U}$ implies that for all smooth functions f supported in $\mathcal{U}$,

$$\int_{\partial\Omega} H f^2 d\sigma \le \int_{\Omega} |\nabla f|^2 dx \,. \tag{2.4}$$

We will deduce from (2.4) a criterion for instability in the form we will need, that is, expressed in terms of subsolutions.

We say that v is a subsolution to the linearized equation 2.3 in $\Omega \cap \mathcal{U}$ if

$$\begin{cases} \triangle v \ge 0 & \text{in } \Omega \cap \mathcal{U}, \\ \\ v_\nu + H\, v \ge 0 & \text{on } \mathcal{U} \cap \partial\Omega, \end{cases} \tag{2.5}$$

with

$$v \ge 0 \quad \text{on } \Omega \cap \mathcal{U}, \qquad v = 0 \qquad \text{on } \Omega \cap \partial\mathcal{U}.$$

It follows from integration by parts that if there is a *strict* subsolution v as in (2.5), then u is unstable.[1] Indeed,

$$\int_\Omega |\nabla v|^2 dx = -\int_\Omega v \triangle v\, dx - \int_{\partial\Omega} v v_\nu \, d\sigma \le -\int_{\partial\Omega} v v_\nu \, d\sigma \le \int_{\partial\Omega} H v^2 \, d\sigma \tag{2.6}$$

If at any step, the inequality is strict, then we have violated the condition for stability.

One can construct such subsolutions in the form $v := f(r)\bar{v}$ for a homogeneous degree 0 function $\bar{v}$ as follows.

Denote by Ω_S the intersection of Ω with the unit sphere and write $\triangle_S$ for the Laplacian on the sphere.

Restricting the Let $\Lambda > 0$ be the first eigenvalue of

The criterion we use to show that no cones other than the hyperplane are stable in dimension 4 is expressed in terms of homogeneous strict subsolutions. (This result is implicit in [CJK], but not stated or used directly there.)

Proposition 2.1. *If there exists $\bar{v} \ge 0$, homogeneous of degree $-\mu$ on Ω, that is a strict subsolution for the following problem*

$$\begin{cases} \triangle \bar{v} \ge \gamma\, \bar{v}/|x|^2 & \text{in } \Omega, \\ \\ \bar{v}_\nu + H\, v \ge 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases} \tag{2.7}$$

and the constant γ satisfies

$$\gamma \ge \left(\frac{n}{2} - 1 - \mu\right)^2, \tag{2.8}$$

[1] I DID NOT THINK THROUGH WHERE TO PLACE THIS NOR EDIT IT: Infinitesimally this corresponds to the case when u and a perturbed solution "cross each other" in the region $\mathcal{U}$. It is well known that the stability of a solution u in a region $\mathcal{U}$ is equivalent to the existence of positive solutions to the linearized equation in the region $\mathcal{U}$. In fact in non variational elliptic problems this characterization can be taken as the definition of stability. When such a positive solution exists then, in a neighborhood of the graph of u in $\mathcal{U} \times \mathbb{R}$, the space can be foliated by perturbed solutions, and then u is stable in this region $\mathcal{U}$.

then u is unstable in the sense that (1.3) *fails for some perturbation Ψ in an annulus. (By strict subsolution, we mean that equality cannot hold everywhere in* (2.7)*.)*

Proof. Then

$$\begin{cases} \triangle_S \bar{v} \geq (\gamma + \mu(n-2-\mu))\bar{v} & \text{in } \Omega_S, \\ \bar{v}_\nu + H\,\bar{v} \geq 0 & \text{on } \partial\Omega_S. \end{cases}$$

□

As suggested implicitly in [CJK], this can expressed in terms of homogeneous subsolutions.

The criterion for instability is the existence of a non-negative function v that is a strict subsolution to the linearized equation **??** in $\mathcal{U}$.

Conversely, u is unstable if there is a strict subsolution $v \geq 0$ to (2.3) in a region $\mathcal{U}$, $(0 \notin \mathcal{U})$,

In order to obtain a pair $(v, \mathcal{U})$ as above it suffices to find a homogenous function $\bar{v} \geq 0$ which does not vanish identically such that:

$$\bar{v} \text{ is homogenous of degree } -\mu \text{ for some } \mu > 0,$$

$$\begin{cases} \triangle \bar{v} \geq \gamma \, \frac{\bar{v}}{|x|^2} & \text{in } \Omega, \\ \bar{v}_\nu + H\,\bar{v} \geq 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases} \tag{2.9}$$

and

$$\gamma > \left(\frac{n}{2} - 1 - \mu\right)^2. \tag{2.10}$$

Indeed, then

$$v := f(r)\bar{v}, \qquad \text{with } f \geq 0 \text{ a radial function, } r := |x|,$$

satisfies the boundary condition in (2.5) and

$$\begin{aligned} \triangle v &= \bar{v}\triangle f + 2\nabla\bar{v}\cdot\nabla f + f\triangle\bar{v} \\ &\geq \bar{v}\left(f'' + (n-1)\frac{f'}{r}\right)\bar{v} - 2\mu\frac{\bar{v}}{r}f' + \gamma f\frac{\bar{v}}{r^2} \\ &\geq \bar{v}\left(f'' + (n-1-2\mu)\frac{f'}{r} + \gamma\frac{f}{r^2}\right). \end{aligned}$$

On the other hand it is straightforward to check that if f satisfies the constant coefficients ODE

$$f'' + \alpha\frac{f'}{r} + \beta\frac{f}{r^2} = 0,$$

then f oscillates around 0 if and only if

$$4\beta > (\alpha - 1)^2.$$

In view of (2.10) we can choose α, β accordingly such that v satisfies the interior condition in (2.5), with $\mathcal{U}$ being the annular region between two consecutive zeros of f (where f is positive).

We prove Theorem 1.1 by constructing an explicit subsolution $\bar{v}$ to (2.9)-(2.10) which depends on the second derivatives of u.

In order to maximize γ, we take $\bar{v}$ to be the first eigenfunction for $\triangle_S$ on Ω_S with respect to the Neumann boundary condition above, i.e.

$$\triangle_S\,\bar{v} = \Lambda\,\bar{v}, \qquad \bar{v}_\nu + H\,\bar{v} = 0 \quad \text{on } \partial\Omega_S,$$

where $\Lambda > 0$ denotes the first eigenvalue. Notice that variationally, Λ is given by

$$-\Lambda := \min_{\bar{v}} \frac{\int_{\Omega_S} |\nabla \bar{v}|^2 - \int_{\partial\Omega_s} H\bar{v}^2}{\int_{\Omega_S} \bar{v}^2}.$$

Then, from (2.10), we obtain

$$\Lambda > \left(\frac{n}{2} - 1\right)^2 \quad \Longrightarrow \quad u \text{ is unstable.} \tag{2.11}$$

Conversely, by the discussion above, it follows that if

$$\Lambda \le \left(\frac{n}{2} - 1\right)^2$$

then u is stable in any bounded domain that does not contain the origin.

These results were obtained in [CJK] and we summarize them in the following proposition.

Proposition 2.2. *If there exists $\bar{v} \ge 0$, homogenous of degree $-\mu$, which is a strict subsolution for the following problem*

$$\begin{cases} \triangle \bar{v} \ge \left(\frac{n}{2} - 1 - \mu\right)^2 \frac{\bar{v}}{|x|^2} & \text{in } \Omega, \\ \\ \bar{v}_\nu + H\,\bar{v} \ge 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases} \tag{2.12}$$

then u is unstable.

Indeed, since $\bar{v}$ is a strict subsolution it follows that Λ satisfies the strict inequality (2.11).

Finally we remark that (2.7) is equivalent to

$$\begin{cases} \triangle(\log \bar{v}) + |\nabla(\log \bar{v})|^2 \ge \left(\frac{n}{2} - 1 - \mu\right)^2 \frac{1}{|x|^2} & \text{in } \Omega \cap \{\bar{v} > 0\}, \\ \\ \frac{1}{H}(\log \bar{v})_\nu \ge -1 & \text{on } \partial\Omega \cap \{\bar{v} > 0\}. \end{cases} \tag{2.13}$$

3. The case $w = \|D^2 u\|$

In this section we show that

$$\bar{v} = w^\alpha$$

satisfies an inequality of the type (2.9) where $w := \|D^2 u\|$, that is

$$w^2 := \|D^2 u\|^2 = u_{ij}^2,$$

where in the last term the summation is over all indices i, j from 1 to n.

3.1. The interior inequalty.

First we obtain an inequality for harmonic functions which is similar to Simons inequality for minimal surfaces.

Proposition 3.1. *Assume u is harmonic and homogenous of degree 1. Then*

$$w\,\triangle w \ge \frac{2}{n-1}\,|\nabla w|^2 + 2\,\frac{n-2}{n-1}\,\frac{w^2}{|x|^2},$$

in the set $\{w > 0\}$.

Proof. We have

$$w\, w_k = u_{ij}\, u_{ijk} \quad \text{for each } k = 1, .., n,$$

and

$$w \triangle w + |\nabla w|^2 = u_{ijk}^2 + u_{ij}\, u_{ijkk} = u_{ijk}^2. \tag{3.1}$$

Since u is homogenous of degree one, the radial direction $x/|x|$ is an eigenvector for D^2u. We choose a system of coordinates such that e_1 points in the radial direction at x. Then

$$u_{1i} = 0 \quad \text{for each } i = 1, .., n,$$

and since u_{ij} are homogenous of degree -1 we obtain

$$u_{1ij} = -\frac{u_{ij}}{|x|}, \quad u_{11i} = 0.$$

We have

$$w^2 = u_{ii}^2, \quad w_k = \frac{u_{ii}}{w}\, u_{iik},$$

thus by Holder inequality, for each k,

$$w_k^2 \le \sum_i \left(\frac{u_{ii}}{w}\right)^2 \sum_i u_{iik}^2 = \sum_i u_{iik}^2.$$

Then

$$u_{ijk}^2 = u_{iik}^2 + \sum_{i \ne j} u_{ijk}^2 \ge |\nabla w|^2 + 2 \sum_{i \ne k} u_{iik}^2. \tag{3.2}$$

Next we estimate for each k the sum in the last term above.

If $k = 1$ then

$$\sum_{i \ne k} u_{iik}^2 = \sum_{i \ne 1} u_{ii1}^2 = \sum_{i \ne 1} \left(\frac{u_{ii}}{|x|}\right)^2 = \frac{w^2}{|x|^2}. \tag{3.3}$$

If $k \ne 1$ then we use that $\triangle u_k = 0$ and $u_{11k} = 0$ and obtain

$$\sum_{i \ne k} u_{iik} = -u_{kkk} \quad \Longrightarrow \quad (n-2) \sum_{i \ne k} u_{iik}^2 \ge u_{kkk}^2.$$

We use

$$u_{iik}^2 = \frac{1}{n-1} u_{iik}^2 + \frac{n-2}{n-1} u_{iik}^2$$

thus

$$\sum_{i \ne k} u_{iik}^2 \ge \frac{1}{n-1} \sum_{i \ne k} u_{iik}^2 + \frac{1}{n-1} u_{kkk}^2 \ge \frac{1}{n-1} w_k^2. \tag{3.4}$$

Now the conclusion follows by using (3.2)-(3.4) into (3.1), and also remarking that

$$w_1 = -\frac{w}{|x|},$$

since w is homogenous of degree -1.

∎

The conclusion can be written as

$$\triangle(\log w) \geq \left(\frac{2}{n-1} - 1\right) |\nabla(\log w)|^2 + 2\frac{n-2}{n-1}\frac{1}{|x|^2},$$

or

$$\triangle(\alpha \log w) + |\nabla(\alpha \log w)|^2 \geq \alpha \left(\frac{2}{n-1} - 1 + \alpha\right) |\nabla(\log w)|^2 + 2\alpha\frac{n-2}{n-1}\frac{1}{|x|^2}.$$

Since

$$|\nabla(\log w)| \geq \frac{w_1^2}{w^2} = \frac{1}{|x|^2},$$

we obtain further that

$$\begin{aligned}\triangle(\alpha \log w) + |\nabla(\alpha \log w)|^2 &\geq \alpha \left(\frac{2}{n-1} - 1 + \alpha + 2\frac{n-2}{n-1}\right) \frac{1}{|x|^2} \\ &\geq \alpha(\alpha+1)\frac{1}{|x|^2},\end{aligned}$$

provided that

$$\alpha \geq 1 - \frac{2}{n-1}.$$

In conclusion the function $\bar{v} = w^\alpha$, which is homogenous of degree $-\alpha$, satisfies

$$\triangle \bar{v} \geq \alpha(\alpha+1)\frac{\bar{v}}{|x|^2} \qquad \text{for all} \quad \alpha \geq 1 - \frac{2}{n-1}, \tag{3.5}$$

and this clearly holds also on the set $\{\bar{v} = 0\}$.

3.2. **The boundary inequality.** We have

$$w^2 = u_{ij}^2 \quad \Longrightarrow \quad w\, w_n = u_{ij}\, u_{ijn}.$$

Fix a point x_0 on $\partial\Omega \setminus \{0\}$ and we choose a system of coordinates as in Section 2, i.e. such that $e_n = \nu_{x_0}$, $D^2u(x_0)$ is diagonal and, say, e_1 coincides with the radial direction $x_0/|x_0|$. We recall (2.2) together with $u_{nn} = -H$

$$\begin{aligned} u_{iin} &= -H u_{ii} - u_{ii}^2 \qquad\qquad \text{for all } i < n, \\ u_{nnn} &= -H u_{nn} + w^2 - u_{nn}^2, \end{aligned}$$

thus

$$w\, w_n = -2Hw^2 - \sum u_{ii}^3,$$

or

$$\frac{1}{H}(\log w)_n = -\left(2 + \frac{\sum u_{ii}^3}{Hw^2}\right).$$

Define

$$L := \max_{\partial\Omega_S} \left(2 + \frac{\sum u_{ii}^3}{Hw^2}\right), \tag{3.6}$$

and we see that the function $\bar{v} = w^\alpha$ satisfies

$$\frac{1}{H}(\log \bar{v})_\nu \geq -1 \qquad \text{if} \quad \alpha \leq \frac{1}{L}. \tag{3.7}$$

From (3.5), (3.7) and Proposition 2.1 we see that u is unstable if there exists α such that

$$\alpha \geq 1 - \frac{2}{n-1}, \qquad \alpha \leq \frac{1}{L},$$

and

$$\alpha(\alpha+1) > \left(\frac{n}{2} - 1 - \alpha\right)^2 \qquad \Longleftrightarrow \qquad \alpha > \frac{(n-2)^2}{4(n-1)}.$$

Notice that this second lower bound on α guarantees the first lower bound since

$$\frac{(n-2)^2}{4(n-1)} \geq 1 - \frac{2}{n-1}.$$

We summarize these results in the next proposition.

Proposition 3.2. *Let u be a solution to* (1.1) *which is not one-dimensional. Then u is unstable if*

$$\frac{(n-2)^2}{4(n-1)} < \frac{1}{L}, \tag{3.8}$$

with L given by (3.6).

Moreover, u is unstable also in case of equality in (3.8) *provided that equality does not hold at all points in* (3.5), (3.7) .

The quantity L at a point on $x_0 \in \partial\Omega$ depends only on the proportion of the $n-2$ nonvanishing curvatures of $\partial\Omega$ at that point. Let κ_l, $l = 2, .., n-1$, denote the curvatures of $\partial\Omega$ with respect to the outer normal, ($\kappa_1 = 0$ since e_1 is the radial direction). Then

$$H = \sum_l k_l > 0, \quad \text{and let} \quad \mu_l := \frac{k_l}{H},$$

and

$$L_{x_0} = 2 + \frac{\sum_l \mu_l^3 - 1}{1 + \sum \mu_l^2}, \qquad \sum \mu_l = 1. \tag{3.9}$$

When $n = 3$ then $L = 2$ and (3.8) holds. We obtain

Corollary 3.3. *If u is a stable solutions to* (1.1) *in dimension $n = 3$ then u is one-dimensional.*

Unfortunately (3.8) does not always apply in dimension $n = 4$. It is not difficult to check that in this case the optimal upper bound for L is $\frac{7}{2}$, and the left hand side in (3.8) is $\frac{1}{3}$. Moreover, if $n \geq 5$ then the algebraic quantity in (3.9) is no longer bounded above.

We remark however that condition (3.8) gives the sharp result in the case when all curvatures are equal, i.e. the axis symmetric case. Then $L = (n-1)/(n-2)$ and (3.8) holds for $n \leq 5$. When $n = 6$ the inequality becomes equality, but in this case equality cannot hold in (3.5), (3.8) at all points. Indeed, otherwise we choose

$$\alpha = \frac{1}{L} > 1 - \frac{2}{n-1},$$

and from the computation at the end of Section 3.1 we obtain

$$\triangle(\alpha \log w) + |\nabla(\alpha \log w)|^2 \geq \alpha(\alpha+1)\frac{1}{|x|^2} + \alpha\left(\frac{2}{n-1} - 1 + \alpha\right)\frac{w_n^2}{w^2},$$

and the second term is positive on $\partial\Omega$ since $w_n/w = -HL < 0$.

Finally we point out the main difference with the minimal surface theory. The lower bound for the exponent α in (3.8) is essentially maximized when D^2u has only one negative eigenvalue and the remaining ones are positive and equal (as in the axis symmetric case). On the other hand the upper bound is minimized when on the boundary $\partial\Omega$ one tangential eigenvalue is positive and the remaining ones are negative. In other words the bounds for α that come from the interior inequality respectively boundary inequalities are nearly optimal but they are achieved for different configurations. This is the reason why (3.8) does not provide the conjectured optimal dimension $n \leq 6$.

Our computation is somewhat consistent with the findings of G. Hong in [H] where he studied the stability of Lawson-type cones for (1.1) in low dimensions. It turns out that in dimension $n = 7$ there are in fact two different stable cones corresponding precisely to the two situations described above.

4. The case $w^2 = \sum_{\lambda_k>0} \lambda_k^2 + a \sum_{\lambda_k<0} \lambda_k^2$.

In this section we proceed as in Section 3 for a slightly different choice of w i.e.

$$w^2 := \sum_{\lambda_k>0} \lambda_k^2 + a \sum_{\lambda_k<0} \lambda_k^2, \tag{4.1}$$

for some constant $a > 0$. Here λ_i represent the eigenvalues of D^2u.

When $a = 1$ then w coincides with the function considered in Section 3. We show that when $a = 4$ and $n = 4$, the interior inequality remains the same as in Section 3, however the boundary inequality improves to $L \leq 3$ and allows us to prove Theorem 1.1.

4.1. **Functions of the eigenvalues.**

Assume

$$F(D^2u) = f(\lambda_1, .., \lambda_n),$$

with $f \in C^1$ a symmetric function of its arguments. We choose a system of coordinates at a point $x \in \Omega$ such that

$$D^2u = diag(\lambda_1, .., \lambda_n),$$

and we use the following orthonormal basis in the space of symmetric matrices

$$e_{ii} := e_i \otimes e_i, \qquad e_{ij} := \frac{1}{\sqrt{2}}(e_i \otimes e_j + e_j \otimes e_i) \quad \text{for } i < j.$$

Then one can check that

$$F_{e_{ii}}(D^2u) = f_{\lambda_i} \quad \text{and} \quad F_{e_{ij}}(D^2u) = 0. \tag{4.2}$$

Moreover, if $f \in C^2$ then

$$F_{e_{ii},e_{kk}}(D^2u) = f_{\lambda_i\lambda_k},$$

$$F_{e_{ij},e_{ij}}(D^2u) = \begin{cases} \frac{f_{\lambda_i} - f_{\lambda_j}}{\lambda_i - \lambda_j} & \text{if } \lambda_i \neq \lambda_j, \\ f_{\lambda_i\lambda_i} & \text{if } \lambda_i = \lambda_j. \end{cases}$$

$$F_{e_{ij},e_{kl}}(D^2u) = 0 \quad \text{if} \quad e_{ij} \neq e_{kl},\ i < j.$$

These can be checked from the fact that the eigenvalues of the matrix

$$\begin{pmatrix} \lambda_1 & \epsilon \\ \epsilon & \lambda_2 \end{pmatrix}$$

are

$$\lambda_1 + \frac{\epsilon^2}{\lambda_1 - \lambda_2} + O(\epsilon^3) \quad \text{and} \quad \lambda_2 + \frac{\epsilon^2}{\lambda_2 - \lambda_1} + O(\epsilon^3) \quad \text{if } \lambda_1 \neq \lambda_2$$

or

$$\lambda_1 + \epsilon, \quad \lambda_2 - \epsilon \quad \text{if } \lambda_1 = \lambda_2.$$

4.2. **The interior inequality.** We show that the function w defined in (4.1) satisfies the same differential inequality as in Proposition 3.1. Rather surprinsingly we can prove a more general statement: any convex, symmetric, homogenous of degree one function of the eigenvalues satisfies the same conclusion as Proposition 3.1.

Theorem 4.1. *Assume $\triangle u = 0$ and let*

$$w = F(D^2u) := f(\lambda_1, .., \lambda_n),$$

with f a convex, symmetric, homogenous of degree one function. Then

$$w \triangle w \geq \frac{2}{n} |\nabla w|^2.$$

Moreover, if u is homogenous of degree 1, the inequality can be improved as

$$w \triangle w \geq \frac{2}{n-1}|\nabla w|^2 + 2\,\frac{n-2}{n-1}\frac{w^2}{|x|^2}.$$

The inequalities above are understood in the viscosity sense.

We remark that the hypotheses on f easily imply $f \geq 0$. Notice that the first inequality is equivalent to $w^{1-\frac{2}{n}}$ is subharmonic, or in the case $n = 2$ that $\log w$ is subharmonic.

Proof. We assume that f is smooth in $\mathbb{R}^n \setminus \{0\}$. Then the general case easily follows by approximation. Also, it suffices to show the inequality in the set $\{w > 0\}$ since it is obvious in $\{w = 0\}$.

Fix a point x with $D^2u(x) \neq 0$, and we choose a system of coordinates at x such that

$$D^2u = diag(\lambda_1, .., \lambda_n).$$

First we show that

$$(f_{\lambda_i} - f_{\lambda_j})(\lambda_i - \lambda_j) \geq 0, \tag{4.3}$$

and the inequality is strict if f is strictly convex and $\lambda_i \neq \lambda_j$.

Indeed, let $Z_0 := (\lambda_1, .., \lambda_n)$ and let Z_1 denote the vector obtained from Z_0 after interchanging λ_i with λ_j. Using the symmetry and convexity of f we obtain

$$0 = f(Z_1) - f(Z_0) \geq \nabla f(Z_0) \cdot (Z_1 - Z_0),$$

and this gives our claim (4.3).

In view of the section 4.1, for each k we have

$$w_k = f_{\lambda_i} u_{iik}, \tag{4.4}$$

and

$$w_{kk} = f_{\lambda_i} u_{iikk} + f_{\lambda_i \lambda_j} u_{iik} u_{jjk} + 2\, F_{e_{ij}, e_{ij}}\, u_{ijk}^2,$$

where we used the repeated indices summation convention for i, j (with $i < j$ in the last term). Summing over k and using that f is convex and $\triangle u_{ii} = 0$ we find

$$\triangle w \geq 2F_{e_{ij},e_{ij}} u_{ijk}^2,$$

with the indices i, j, k running over the set $\{1,..,n\}$, with $i < j$. Notice that all such terms are nonnegative since, by (4.3), $F_{e_{ij},e_{ij}} \geq 0$. We keep only the terms for which two of the indices i, j, k coincide and obtain

$$\triangle w \geq 2\sum_{i \neq k} F_{e_{ik},e_{ik}} u_{iik}^2,$$

where i, k run over $\{1,..,n\}$ with $i \neq k$.

In order to obtain our inequality it suffices to show that for each fixed k we have

$$\sum_{i \neq k} F_{e_{ik},e_{ik}} u_{iik}^2 \geq \frac{1}{n}\frac{w_k^2}{w}. \tag{4.5}$$

From (4.4) and $\triangle u_k = 0$ we find

$$w_k = \sum_{i \neq k} (f_{\lambda_i} - f_{\lambda_k}) u_{iik}.$$

Notice that from section 4.1 and the symmetry of f we have

$$f_{\lambda_i} - f_{\lambda_k} = (\lambda_i - \lambda_k) F_{e_{ik},e_{ik}}.$$

Hence by Chauchy-Schwartz we obtain

$$w_k^2 \leq \left(\sum_{i \neq k} F_{e_{ik},e_{ik}} u_{iik}^2\right)\left(\sum_{i \neq k} (\lambda_i - \lambda_k)(f_{\lambda_i} - f_{\lambda_k})\right),$$

thus, in order to prove (4.5) it suffices to show that

$$\sum_{i \neq k} (\lambda_i - \lambda_k)(f_{\lambda_i} - f_{\lambda_k}) \leq nf. \tag{4.6}$$

Indeed, using that $\sum \lambda_i = 0$, $\sum \lambda_i f_{\lambda_i} = f$, we obtain the identity

$$\sum_{i<j} (\lambda_i - \lambda_j)(f_{\lambda_i} - f_{\lambda_j}) = \sum_i f_{\lambda_i} \sum_{j \neq i} (\lambda_i - \lambda_j) = \sum_i f_{\lambda_i} n \lambda_i = nf. \tag{4.7}$$

Our claim (4.6) is proved since, by (4.3), the left hand side in (4.6) is bounded above by the left hand side of (4.7).

Remark: From the equality above and (4.3) we see that, if f is strictly convex in a neighborhood of $Z_0 = (\lambda_1, .., \lambda_n)$, we have equality in (4.6) only when all λ_i with $i \neq k$ are equal. In other words, the coefficient of $\frac{w_k^2}{w}$ in (4.5) can be replaced by $\frac{1}{n} + \epsilon$ in a neighborhood of x, if $\lambda_i \neq \lambda_j$ for some $i, j \neq k$ and f is strictly convex near Z_0 in the 2-dimensional plane generated by the λ_i, λ_j directions. Here $\epsilon > 0$ depends on $(\lambda_i - \lambda_j)(f_{\lambda_i} - f_{\lambda_j})$.

We conclude with the case when u is homogenous of degree 1 and show that the inequalities above can be improved. We assume that at the point x, the e_1 direction represents the radial direction $x/|x|$, thus

$$\lambda_1 = 0, \quad u_{ij1} = -\frac{u_{ij}}{|x|}.$$

Let $k \neq 1$. Then, the coefficient of $\frac{w_k^2}{w}$ in (4.5) can be replaced by $\frac{1}{n-1}$. Indeed, $u_{11k} = 0$, $\lambda_1 = 0$, thus the index $i = 1$ can be ignored in the computations above, and we reduce the problem to $n-1$ variables.

When $k = 1$ the left hand side of (4.6) equals f since

$$\sum_{i\neq 1}(\lambda_i - \lambda_1)(f_{\lambda_i} - f_{\lambda_1}) = \sum_{i\neq 1}\lambda_i f_{\lambda_i} = f,$$

where we used $\lambda_1 = 0$, $\sum \lambda_i = 0$. This shows that the coefficient of $\frac{w_1^2}{w}$ in (4.5) can be replaced by 1. Since w is homogenous of degree -1 we also have $w_1 = -\frac{w}{|x|}$ and the second part of our theorem is complete.

□

4.3. **The boundary inequality.** We show that the function w defined in (4.1), when $a = 4$, $n = 4$ satisfies

$$\frac{1}{H}(\log w)_\nu \geq -3. \tag{4.8}$$

Notice that $w \in C^{1,1}$ in the set $\{w \neq 0\}$.

Let $x_0 \in \partial\Omega \setminus \{0\}$ and we choose a system of coordinates as before i.e. with $D^2u(x_0)$ diagonal, $e_n = \nu_{x_0}$ and $e_1 = x_0/|x_0|$. We also denote by i, s the indices for which $\lambda_i > 0$ respectively $\lambda_s < 0$.

From (4.4), (2.2) we have

$$\begin{aligned} w_n &= \frac{\lambda_i}{w}u_{iin} + a\frac{\lambda_s}{w}u_{ssn} \\ &= \frac{\lambda_i}{w}(-H\lambda_i - \lambda_i^2) + a\sum_{s\neq n}\frac{\lambda_s}{w}(-H\lambda_s - \lambda_s^2) \\ &\qquad + a\frac{\lambda_n}{w}(-H\lambda_n - \lambda_n^2 + \lambda_i^2 + \lambda_s^2). \end{aligned}$$

Using $\lambda_n = -H$ we obtain

$$-\frac{w_n}{Hw} = 1 + \frac{\lambda_i^3 + a\lambda_s^3 + aH(\lambda_i^2 + \lambda_s^2)}{Hw^2}.$$

Since $\lambda_1 = 0$ and $\lambda_4 < 0$, we distinguish two cases depending whether λ_2 and λ_3 are both positive or have opposite signs.

Case 1: $\lambda_2 > 0$, $\lambda_3 \leq 0$.

Let

$$\mu := -\frac{\lambda_3}{H} \quad \text{thus} \quad \frac{\lambda_2}{H} = \mu + 1, \quad \text{and } \mu \geq 0.$$

We need to show that

$$\frac{(1+\mu)^3 - a\mu^3 + a((1+\mu)^2 + \mu^2)}{(1+\mu)^2 + a\mu^2 + a} \leq 2.$$

This is equivalent to

$$(\mu-1)\left(a(\mu^2+\mu-1)-(\mu+1)^2\right)\geq 0,$$

or, since $a=4$,

$$(\mu-1)^2(3\mu+5)\geq 0,$$

which is obvious since $\mu\geq 0$.

Case 2: $\lambda_2>0$, $\lambda_3>0$.

Let

$$\mu:=\frac{\lambda_2}{H}\quad\text{thus}\quad\frac{\lambda_3}{H}=1-\mu,\quad\text{and}\quad\mu\in(0,1).$$

We need to show that

$$\frac{\mu^3+(1-\mu)^3+4(\mu^2+(1-\mu)^2)}{\mu^2+(1-\mu)^2+4}\leq 2.$$

This is obvious since the numerator is bounded above by 5 thus the fraction is bounded by $5/4<2$.

In conclusion (4.8) is proved and equality at a point holds only when

$$\lambda_2>0,\qquad\lambda_3=\lambda_4<0. \tag{4.9}$$

Proof of Theorem 1.1. Let $n=4$ and set

$$\bar v:=w^{\frac{1}{3}}$$

with w as in (4.1) and $a=4$. Assume that w is not identically 0, i.e. u is not a one-dimensional solution.

By Theorem 4.1 and (4.8) it follows as in Section 3 that $\bar v$ satisfies (2.7). In order to prove that u is not stable it remains to show that $\bar v$ is a strict subsolution.

We fix a point $x_0\in\partial\Omega$. If equality holds in (4.8) then, by (4.9), $\lambda_2\neq\lambda_3$ at x_0. Then, from the remark in the proof of Theorem 4.1, it follows that the differential inequality can be improved by adding a term ϵw_n^2 to the right hand side. Since $w_n=-3Hw<0$, we find that at x_0 we have strict inequality in Theorem 4.1 which in turn gives that $\bar v$ is a strict subsolution for the interior problem in a neighborhood of x_0.

□

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