LE MATEMATICHE
Vol. LXII (2007) - Fasc. I, pp. 55-65

SOME REMARKS ON THE CANTOR PAIRING FUNCTION

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In this paper, some results and generalizations about the Cantor pairing function are given. In particular, it is investigated a very compact expression for the n-degree generalized Cantor pairing function (g.C.p.f., for short), that permits to obtain $n-$tupling functions which have the characteristics to be n-degree polynomials with rational coefficients. A recursive formula for the n-degree g.C.p.f. is also provided.

1. Introduction.

In mathematics a pairing function is a process to uniquely encode two natural numbers into a single natural number. Any pairing function can be used in set theory to prove that integers and rational numbers have the same cardinality as natural numbers. In theoretical computer science, pairing functions are used to encode a function defined on a vector of natural numbers $f : N^k \to N$ into a new function, [6], [9], [12], [13].

Definition 1.1. *The function* $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$, *such that*

$$\langle x_1, x_2 \rangle = \frac{(x_1 + x_2)^2 + 3x_1 + x_2}{2}, \qquad \forall\ (x_1, x_2) \in \mathbb{N}^2, \tag{1}$$

is called the Cantor "pairing" function. □

Entrato in redazione il 28 Novembre 2006.

The fundamental property of the Cantor pairing function is given by the following theorem.

Theorem 1.1. *The Cantor pairing function is a bijection from* $\mathbb{N}^2$ *onto* $\mathbb{N}$.

Proof. In order to prove the theorem, consider the straight lines $x_1 + x_2 = k$, with $k \in \mathbb{N}$. It is clear that the "point" $(\bar{x}_1, \bar{x}_2)$ belongs to $x_1+x_2 = \bar{x}_1+\bar{x}_2$, or, more precisely, to the intersection of $x_1+x_2 = \bar{x}_1+\bar{x}_2$ with the first quadrant of the euclidean plane.

Hence, for any $(\bar{x}_1 + \bar{x}_2)$ fixed, moving along the line $x_1 + x_2 = k$ towards increasing ordinates $(k = 0, \dots, \bar{x}_1 + \bar{x}_2)$, let us list all the couples of natural numbers we "meet" and associate to each one of these an increasing natural number, starting from zero.

Thanks to a simple (pascal) algorithm, we have a bijective correspondence between $\mathbb{N}^2$ and $\mathbb{N}$. The variable $output$ represents the natural number associated to (x_1, x_2):

```
program pairing
   var i, j, k, x1, x2, output: 0..maxint;
   begin
      output := 0
      for k = 0 to (x1 + x2) do
      begin
         for (i = 0 to k) and (j = k to 0) do
         begin
            if (x1 = i) and (x2 = j)
               then write (output);
                else
                  begin
                       i := i + 1;
                       j := j − 1;
                       output := output + 1;
                  end;
         end;
         k := k + 1;
       end.
    end.
```
□

By using the previous algorithm, we can build Table 1, where we list the couples of natural numbers (x_1, x_2). We stop at $(3, 0)$.

Remark 1.1. In what follows we shall use the following notation: $N_n(k)$ indicates the number of n-tuples satisfying condition $\sum_{i=1}^{n} x_i = k$. For the pairing function, we have:

$$N_2(k) = k + 1. \tag{2}$$ □

(x_1,x_2)	(0,0)	(0,1)	(1,0)	(0,2)	(1,1)	(2,0)	(0,3)	(1,2)	(2,1)	(3,0)
output	0	1	2	3	4	5	6	7	8	9
k	0	1		2			3			
$N_2(k)$	1	2		3			4			

Table 1

The Cantor pairing function is a second degree polynomial, with rational coefficients, [1], [2], [3], [7]. Rudolph Fueter proposed in 1923 four conjectures about the set of polynomial pairing functions, [11].

By composition, we can obtain “tripling” functions, “quadrupling” functions and so on. By using (1), a possible tripling function is given by $\langle \cdot, \cdot, \cdot \rangle \colon \mathbb{N}^3 \to \mathbb{N}$, such that:

$$\langle x_1, x_2, x_3 \rangle = \langle x_1, \langle x_2, x_3 \rangle \rangle = \tag{3}$$

$$\frac{\left\{ x_1 + \dfrac{[(x_2+x_3)^2 + 3x_2 + x_3]}{2} \right\}^2 + 3x_1 + \dfrac{[(x_2+x_3)^2 + 3x_2 + x_3]}{2}}{2}.$$

However, note that, listing the terns of natural numbers (x_1, x_2, x_3) in a way similar to that used for the pairing function (in this case, we consider the planes $x_1 + x_2 + x_3 = k$, with $k \in \mathbb{N}$), we are not able to identify a generalized rule of order. In fact, by means of an analogous algorithm, we can build Table 2 (we stop at $(2, 0, 0)$):

(x_1,x_2,x_3)	(0,0,0)	(0,0,1)	(1,0,0)	(0,1,0)	(1,0,1)	(2,0,0)	(0,0,2)	(1,1,0)	(2,0,1)	(3,0,0)
output	0	1	2	3	4	5	6	7	8	9

Table 2

Note also that the tripling function (3), obtained by means of a composition, is a fourth degree polynomial. Spontaneous questions arise:

is it possible to have tripling functions which are third degree polynomials, quadrupling functions which are fourth degree polynomials and so on? And moreover: does a generalized formula exist, that permits to obtain these polynomials in a simple way?

2. The *n*-degree generalized Cantor pairing function.

Theorem 2.1. *For any $n \in \mathbb{N}$, the so-called "n-tupling" function* $, \langle \cdot, ..., \cdot \rangle : \mathbb{N}^n \to \mathbb{N}$, *such that*

$$(4) \quad \langle x_1, ..., x_n \rangle = \sum_{h=1}^{n} \left\{ \frac{1}{h!} \prod_{j=0}^{h-1} \left[\left(\sum_{i=1}^{h} x_i \right) + j \right] \right\}, \quad \forall (x_1, ..., x_n) \in \mathbb{N}^n,$$

is an n-degree polynomial, with rational coefficients. Moreover it is a one-to-one correspondence from $\mathbb{N}^n$ onto $\mathbb{N}$.

Proof. We can prove the theorem by induction on the degree n. In order to do this, before studying the inductive case, we analyze some particular cases.

Case $n = 1$

From definition (4), we have $\langle x_1 \rangle = x_1$.

In this case, $\langle \cdot \rangle$ is trivially a bijection from $\mathbb{N}$ to $\mathbb{N}$.

Case $n = 2$

From definition (4), we have:

$$\langle x_1, x_2 \rangle = \sum_{h=1}^{2} \left\{ \frac{1}{h!} \prod_{j=0}^{h-1} \left[\left(\sum_{i=1}^{h} x_i \right) + j \right] \right\}$$
$$= x_1 + \frac{(x_1 + x_2)(x_1 + x_2 + 1)}{2} = \frac{(x_1 + x_2)^2 + 3x_1 + x_2}{2}.$$

This is the Cantor pairing function, we have just proved in Theorem 1.1 to be a bijection from $\mathbb{N}^2$ to $\mathbb{N}$.

Case $n = 3$

Let us build a bijection from $\mathbb{N}^3$ to $\mathbb{N}$, by defining a way of ordering terns similar to that followed for couples of the pairing function, at the previous step: first organize terns of natural numbers (x_1, x_2, x_3) according to their plane of belongings $\sum_{i=1}^{3} x_i = k$, with increasing $k \geq 0$ ($k \in \mathbb{N}$). Then, for any plane, put the “points” in order as follows:

$$(x_1, x_2, x_3) < (y_1, y_2, y_3) \quad \text{if} \quad x_3 > y_3 \quad \text{or if} \quad x_3 = y_3$$

$$\text{and} \quad < x_1, x_2 > \quad \text{is smaller than} \quad < y_1, y_2 >,$$

where the way of organizing $< x_1, x_2 >$ is that defined for the pairing function $< y_1, y_2 >$. This permits to build Table 3. We stop at $(2, 0, 0)$.

(x_1,x_2,x_3)	(0,0,0)	(0,0,1)	(0,1,0)	(1,0,0)	(0,0,2)	(0,1,1)	(1,0,1)	(0,2,0)	(1,1,0)	(2,0,0)
$\langle x_1,x_2,x_3\rangle$	0	1	2	3	4	5	6	7	8	9
k	0	1			2					
$N_3(k)$	1	3			6					

Table 3

Note that

$$N_3(k) = \sum_{h=0}^{k} (h+1) = \frac{1}{2!} \prod_{j=1}^{2} (k+j) \tag{5}$$

indicates the number of terns satisfying $\sum_{i=1}^{3} x_i = k$. In particular, from (2), we have:

$$N_3(k) = \sum_{h=0}^{k} N_2(h). \tag{6}$$

Hence, to identify a given $z_3 = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, with $\sum_{i=1}^{3} \bar{x}_i = m$ $(\bar{x}_i, m \in \mathbb{N})$, we have to take into account all those points “coming before” it. First, we have to consider all those “points” such that $\sum_{i=1}^{3} x_i = g$, for any $x_i, g \in \mathbb{N}$, with $g < m$. They are:

$$\sum_{l=0}^{m-1} N_3(l) = \sum_{l=0}^{m-1} \sum_{h=0}^{l} N_2(h) = \sum_{l=0}^{m-1} \sum_{h=0}^{l} (h+1)$$
$$= \sum_{l=0}^{m-1} \frac{1}{2!} \prod_{j=1}^{2} (l+j) = \frac{1}{3!} \prod_{j=0}^{2} (m+j),$$

where we used (5), (6) and relation (10) proved in Appendix.

Then, we have to consider all those terns satisfying $\sum_{i=1}^{3} \bar{x}_i = m$ but "smaller" than z_3. Since the rule of ordering the first two terms of (x_1, x_2, x_3) is that used for the Cantor pairing, we have that, starting from 0, the natural number associated to $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is:

$$\frac{1}{3!} \prod_{j=0}^{2} \left[\left(\sum_{i=1}^{3} \bar{x}_i \right) + j \right] + \frac{1}{2!} \prod_{j=0}^{1} \left[\left(\sum_{i=1}^{2} \bar{x}_i \right) + j \right] + \bar{x}_1$$

$$= \frac{(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + 1)(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + 2)}{3!} +$$
$$+ \frac{(\bar{x}_1 + \bar{x}_2)(\bar{x}_1 + \bar{x}_2 + 1)}{2!} + \bar{x}_1$$

$$= \frac{(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)^3 + 3(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)^2 + 2(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)}{6} +$$
$$+ \frac{(\bar{x}_1 + \bar{x}_2)^2 + (\bar{x}_1 + \bar{x}_2)}{2} + \bar{x}_1.$$

Since

$$\langle x_1, x_2, x_3 \rangle = \sum_{h=1}^{3} \left\{ \frac{1}{h!} \prod_{j=0}^{h-1} \left[\left(\sum_{i=1}^{h} x_i \right) + j \right] \right\}, \tag{7}$$

we proved that the tripling function given by relation (7) represents a third degree polynomial with rational coefficients and it turns out to be a bijection from $\mathbb{N}^3$ to $\mathbb{N}$.

Consider the inductive case: assume $< x_1, \ldots, x_n >$ to be a bijection from $\mathbb{N}^n$ to $\mathbb{N}$, for any $n \geq 1$ and that the n-tuples of natural numbers satisfying $\sum_{i=1}^{n} x_i = k$ are:

$$(8) \qquad \begin{aligned} N_n(k) &= \sum_{h=0}^{k} N_{n-1}(h) = \sum_{h=0}^{k} \frac{1}{(n-2)!} \prod_{j=1}^{n-2} (h+j) \\ &= \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (k+j). \end{aligned}$$

Now, we want to prove that

$$\langle x_1, \ldots, x_{n+1} \rangle = \sum_{h=1}^{n+1} \left\{ \frac{1}{h!} \prod_{j=0}^{h-1} \left[\left(\sum_{i=1}^{h} x_i \right) + j \right] \right\}$$

is a polynomial of degree $(n+1)$, with rational coefficients and a bijection from $\mathbb{N}^{n+1}$ to $\mathbb{N}$.

By using a rule of ordering the $(n+1)-$tuples similar to that followed for the $n-$tupling function, we build a bijection from $\mathbb{N}^{n+1}$ to $\mathbb{N}$. In particular, ordinate the $(n+1)$-tuples of natural numbers according to relation $\sum_{i=1}^{n+1} x_i = k$, with increasing $k \geq 0, k \in \mathbb{N}$. Organize the "points" satisfying $\sum_{i=1}^{n+1} x_i = k$ as follows: $(x_1, ..., x_{n+1}) < (y_1, ..., y_{n+1})$ if $x_{n+1} > y_{n+1}$ or if $x_{n+1} = y_{n+1}$ and $< x_1, ..., x_n >$ is smaller than $< y_1, \ldots, y_n >$. Hence:

$$(9) \qquad \begin{aligned} N_{n+1}(k) &= \sum_{h=0}^{k} N_n(h) = \sum_{h=0}^{k} \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (h+j) \\ &= \frac{1}{n!} \prod_{j=0}^{n-1} (k+j+1) = \frac{1}{n!} \prod_{j=1}^{n} (k+j), \end{aligned}$$

where we used the inductive assumption (8) and relation (10) of Appendix.

To "identify" $z_{n+1} = (\bar{x}_1, \ldots, \bar{x}_{n+1})$, with $\sum_{i=1}^{n+1} \bar{x}_i = m$, we have to count all those $(n+1)-$tuples of natural numbers, satisfying the condition $\sum_{i=1}^{n+1} x_i = k$, with $k = 0, \ldots, m-1$:

$$\sum_{l=0}^{m-1} N_{n+1}(l) = \sum_{l=0}^{m-1}\sum_{h=0}^{l} N_n(h) = \sum_{l=0}^{m-1} \frac{1}{n!}\prod_{j=1}^{n}(l+j) = \frac{1}{(n+1)!}\prod_{j=0}^{n}(m+j),$$

where we used (9) and relation (10) proved in Appendix.

Then, we have to add all those $(n+1)-$tuples satisfying $\sum_{i=1}^{n+1} x_i = k$, but "smaller" than z_{n+1}. However, since the rule of ordering the first n terms of $(x_1, \ldots x_{n+1})$ is that followed for the n-tupling function, we have that, starting to count from 0, the natural number associated to $(\bar{x}_1, \ldots, \bar{x}_{n+1}), \bar{x}_i \in \mathbb{N}$, is given by:

$$\frac{1}{(n+1)!}\prod_{j=0}^{n}\left[\left(\sum_{i=1}^{n+1}\bar{x}_i\right)+j\right]+\sum_{h=1}^{n}\left\{\frac{1}{h!}\prod_{j=0}^{h-1}\left[\left(\sum_{i=1}^{h}\bar{x}_i\right)+j\right]\right\}.$$

Since

$$\langle x_1, ..., x_{n+1}\rangle = \sum_{h=1}^{n+1}\left\{\frac{1}{h!}\prod_{j=0}^{h-1}\left[\left(\sum_{i=1}^{h} x_i\right)+j\right]\right\},$$

we have that it represents a polynomial of degree $(n+1)$, with rational coefficients and which is a one-to-one correspondence from $\mathbb{N}^{n+1}$ onto $\mathbb{N}$. The theorem is so proved. □

Remark 2.1. The Cantor pairing function is based on the idea of counting anti-diagonals $x + y = k$ and then of counting within a given diagonal by increasing ordinates. This geometrical device has been generalized to "Cantor n-tupling function" which is a bijection from N^n onto N. At first, we used the level k of the hyperplane H_k of the equation $x_1 + x_2 + \ldots + x_n = k$ and then the level h in the hyperplane H_k, having in turn for equation $x_1 + \ldots + x_{n-1} = h$, and so and up to obtaining the line $x_1 + x_2 =$ constant. Hence, Cantor n-tupling function can also be expressed via binomial coefficients as follows, [4]:

$$\langle x_1, ..., x_n\rangle = \binom{x_1 + ... + x_n + n - 1}{n} + \binom{x_1 + ... + x_{n-1} + n - 2}{n-1}$$

$$+\ldots+\binom{x_1+x_2+1}{2}+\binom{x_1}{1}.$$

Finally, the Cantor pairing function can be considered a "primitive recursive" function, [5], [8], [10]. For the n-degree generalized Cantor pairing function (4) (indicated as $\langle\cdot,\cdot,\ldots,\cdot\rangle_n$) a possible recursive formula could be given by the following:

$$\langle x_1,x_2,\ldots,x_n\rangle_n=\begin{cases}0, & \text{if } \sum_{i=1}^{n} x_i=0\\ \langle\sum_{i=1}^{n} x_i-1,0,...,0\rangle_n+\langle x_1,x_2,...,x_{n-1}\rangle_{n-1}+1, & \\ & \text{otherwise}\end{cases}.$$

□

3. Appendix.

We can prove by induction on the index m, that, for any $n\geq 2$, $m\geq 1$:

$$\sum_{k=0}^{m-1}\frac{1}{(n-1)!}\prod_{j=1}^{n-1}(k+j)=\frac{1}{n!}\prod_{j=0}^{n-1}(m+j). \tag{10}$$

In fact, if $m=1$, relation (10) is verified because, if $n\geq 2$:

$$\frac{1}{(n-1)!}\prod_{j=1}^{n-1}j=\frac{1\cdot 2\cdot\ldots\cdot(n-1)}{(n-1)!}=\frac{1\cdot 2\cdot\ldots\cdot n}{n!}=\frac{1}{n!}\prod_{j=0}^{n-1}(j+1).$$

Assume condition (10) to be satisfied for $m\geq 1$, for any $n\geq 2$, i.e.:

$$\sum_{k=0}^{m-1}\frac{1}{(n-1)!}\prod_{j=1}^{n-1}(k+j)=\frac{1}{n!}\prod_{j=0}^{n-1}(m+j). \tag{11}$$

Then, it is verified also for the index $m+1$. In fact, if $n\geq 2$:

$$\sum_{k=0}^{m}\frac{1}{(n-1)!}\prod_{j=1}^{n-1}(k+j)=\sum_{k=0}^{m-1}\frac{1}{(n-1)!}\prod_{j=1}^{n-1}(k+j)+\frac{1}{(n-1)!}\prod_{j=1}^{n-1}(m+j).$$

By using the inductive assumption (11), we get:

$$\frac{1}{n!}\prod_{j=0}^{n-1}(m+j)+\frac{1}{(n-1)!}\prod_{j=1}^{n-1}(m+j)=\frac{(m+n)}{n!}\prod_{j=1}^{n-1}(m+j),$$

and so, for any $n \geq 2$:

$$\sum_{k=0}^{m} \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (k+j) = \frac{1}{n!} \prod_{j=0}^{n-1} (m+j+1).$$

Acknowledgments.

The author would like to express her gratitude to Prof. A. Sorbi for his suggestions and to the referee who contributed to make the paper more readable. This work was partially supported by Par 2004 - Research Project of the University of Siena funds as well as by G.M.F.M. and M.U.R.S.T. research funds.

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