Some remarks on the eigenvalue spectrum of a large symmetric Wigner random-sign matrix

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Some remarks on the eigenvalue spectrum of a large symmetric Wigner random-sign matrix

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The replica method is used to calculate the averaged eigenvalue spectrum as $N \longrightarrow \infty$, of the ensemble of Wigner random-sign real symmetric $N \times N$ matrices. Results are presented for the cases where the individual matrix elements have a mean value of zero and also where the mean value of the individual matrix elements has a finite nonzero value. It is shown that the replica method provides a straightforward framework within which it is possible to verify the Wigner conjecture that any reasonably well-behaved distribution of matrix elements must lead to the well-known semicircular averaged eigenvalue spectrum of the Gaussian orthogonal ensemble of random matrices. Some numerical simulations of the averaged eigenvalue spectrum of these random-sign matrices are presented and they lend support to the prediction that if the individual matrix elements have sufficiently large a mean value, then a single eigenvalue will split off from the main semicircular band of eigenvalues.

On a utilisé la méthode de replication pour calculer le spectre des moyennes des valeurs propres, lorsque $N \longrightarrow \infty$, de l'ensemble des matrices $N \times N$ de Wigner symétriques, réelles et à signe aléatoire. Des résultats sont présentés pour les cas où les éléments de matrice individuels ont une valeur moyenne nulle et aussi où la valeur moyenne des éléments de matrice individuels est finie et non nulle. On montre que la méthode de replication fournit directement un cadre dans lequel il est possible de vérifier la conjecture de Wigner, selon laquelle toute distribution à comportement normal d'éléments de matrice doit conduire au spectre semicirculaire bien connu de moyenne de valeurs propres de l'ensemble orthogonal gaussien de matrices aléatoires. Quelques simulations numériques du spectre des moyennes de valeurs propres de ces matrices à signe aléatoire sont présentées; ces simulations confirment la prédiction que si les éléments de matrice individuels ont une valeur moyenne suffisamment grande, une valeur propre s'écartera de la bande semicirculaire principale de valeurs propres.

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1. Introduction

Recent years have seen a revival of interest in problems associated with random matrices in physics. The importance of the study of random-matrix ensembles in the investigation of the highly excited states of large nuclei was recognised by Dyson (1) who realised that in such complex systems it would be more profitable to study the statistical properties of such spectra rather than attempt detailed *ab initio* calculations. In such problems one has only global information on the symmetry of the Hamiltonian, which because of the large number of particles involved, could be represented as a large matrix. In the absence of other informations, one then constructs a statistical ensemble of such matrices (just as one constructs an ensemble in statistical mechanics), subject to the requirement that each member of the ensemble obeys some physically useful symmetry requirement. One such useful ensemble is the Gaussian orthogonal ensemble (GOE) in which the requirement that the ensemble of $N \times N$ real symmetric matrices be invariant under orthogonal transformations leads to a description of a typical member of the ensemble as a real symmetric $N \times N$ matrix in which each element is a normally distributed random variable. It was shown by Wigner (2) that the ensemble averaged eigenvalue density $\rho(\lambda)$ (for which $\rho(\lambda)\, d\lambda$ gives the average number of eigenvalues between λ and $\lambda + d\lambda$) is a semicircle whose radius as $N \longrightarrow \infty$ is proportional to the standard deviation associated with a matrix element of an individual member of the ensemble. A useful compendium of early results and papers in this area is to be found in the reprint collection edited by Porter (3) and in the book by Mehta (4), which latter has become a standard text on important calculations and ideas in random-matrix physics. A more recent and exhaustive review of work in this area is provided by Brody *et al.* (5).

Use of random-matrix ensembles has now found a firm place in areas outside nuclear physics. They have been used in condensed matter physics by Kosterlitz *et al.* (6) to describe problems associated with certain models of spin glasses and more recently in the description of atomic and molecular spectra (7, 8). Some recent and fascinating work by Bohigas and Giannoni (9) has shown that the eigenvalue spectra of the quantum counterparts of nonintegrable classically chaotic systems show many of the features associated with the GOE described earlier.

The earliest calculations of the averaged eigenvalue density (aed) for the GOE when $N \longrightarrow \infty$ are to be found in the texts by Porter (3) and Mehta (4). These generally rely either on elaborate moment expansions or on the properties of Hermite polynomials and oscillator wave functions that arise naturally from the many integrations against a Gaussian weight that occur in the GOE. However, a radically different method was presented by Edwards and Jones (EJ) (10) for calculating the aed, and this permitted an extensions of previous work to consideration of a GOE in which each matrix element had a nonzero mean. The methods used by these latter authors rely on the so-called replica method, first used by Edwards (11) in the study of polymer physics. The method of EJ (10) predicted that under certain circumstances, a single isolated eigenvalue would appear outside the Wigner semicircular band of eigenvalues; this result was, following some controversy, confirmed by Jones *et al.* (12) who used completely different methods based on techniques used to describe impurity modes in a disordered lattice. The work on ensembles with mean zero was extended to Hermitian matrices by Edwards and Warner (13).

Not all the published work on random matrices has been

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directed at the GOE. Wigner (14) addressed the problem of calculating the aed of an ensemble of large symmetric random matrices that were either bordered (having certain integers down the diagonal and random numbers equal to plus or minus some constant v down the super- and subdiagonals) or that had zeros on the diagonal and entries that (subject to the symmetry requirement) were either $+v$ or $-v$ in the off-diagonal elements. In each case, moment calculations again showed that when the size of the matrix $N \longrightarrow \infty$, the aed again became a semicircle as in the GOE of matrices. In a short article, Wigner (2) then conjectured that a semicircular distribution of eigenvalues would be the limiting distribution obtained as $N \longrightarrow \infty$ for an ensemble of symmetric matrices in which the probability density function of any diagonal element is reasonably well behaved and in which the second moment of all such off-diagonal elements should have the same constant value.

In this paper we apply the replica method used by EJ (10) to an ensemble of large real symmetric matrices with zeros down the diagonal and with off-diagonal elements chosen to have the same magnitude but random signs. We show first how easily the replica formalism reproduces the semicircular aed and then extend the calculation to describe the case where each element has a finite nonzero mean. In this case also we are able to show that under suitable circumstances a single eigenvalue splits off from the main semicircular band and that the magnitude of this eigenvalue is the same as that predicted by EJ (10) for the corresponding problem in the GOE. We then use the replica method to give a simple demonstration, based on the central limit theorem, that any reasonable probability density function of individual matrix elements must lead to a Wigner semicircular eigenvalue density and that the isolated eigenvalue predicted for both the GOE and the random-sign matrix should again be a feature of any large symmetric matrix with the same finite mean value for individual elements. Finally, we present some numerical simulations of the aed for the random sign ensemble with both zero and finite means and for the GOE with finite means. Such simulation on ensembles with finite means are new and would appear to support the analytical predictions of an isolated eigenvalue and lay to rest some of the controversy which has surrounded these predictions (15).

2. The technique

Consider a real symmetric $N \times N$ matrix $\underline{\underline{J}}$ with eigenvalues $\{J_i\}$. The density $\nu(\lambda)$ of such eigenvalues is given by the expression

$$[2.1] \qquad \nu(\lambda) = N^{-1} \sum_i \delta(\lambda - J_i)$$

and $\nu(\lambda)$ has been chosen as normalized to unity.

If we use the result that

$$\det(\underline{\underline{I}}\,\lambda - \underline{\underline{J}}) = \prod_i (\lambda - J_i)$$

and understand λ to have the usual infinitesimal imaginary part $-i\varepsilon$, then [2.1] may be written in the form

$$[2.2] \qquad \nu(\lambda) = \frac{1}{N\pi} \operatorname{Im} \frac{\partial}{\partial\lambda} \ln \det(\underline{\underline{I}}\,\lambda - \underline{\underline{J}})$$

The method developed by EJ (10) is to use the result that

$$[2.3] \qquad \ln x = \lim_{n \to 0} \left[\frac{x^n - 1}{n}\right]$$

to write [2.2] as

$$[2.4] \qquad \nu(\lambda) = -\frac{2}{N\pi} \operatorname{Im} \frac{\partial}{\partial\lambda} \lim_{n \to 0} \frac{1}{n} [\det{}^{-1/2}(\underline{\underline{I}}\,\lambda - \underline{\underline{J}})^n - 1]$$

The determinant can be parametrized as a multiple Fresnel integral of the form

$$[2.5] \qquad \det{}^{-1/2}(\underline{\underline{I}}\,\lambda - \underline{\underline{J}}) = \left[\frac{e^{i\pi/4}}{\pi^{1/2}}\right]^N \int_{-\infty}^{+\infty} \prod_i \mathrm{d}x_i \exp\left[-i \sum_{i,j} x_i(\underline{\underline{I}}\,\lambda - \underline{\underline{J}})_{ij} x_j\right]$$

We now substitute [2.5] into [2.4] and assume that this latter results holds for integer values of n and may then be continued to $n = 0$. We thus obtain the basic result that

$$[2.6] \qquad \nu(\lambda) = -\frac{2}{N\pi} \operatorname{Im} \frac{\partial}{\partial\lambda} \lim_{n \to 0} \frac{1}{n} \left\{\left[\frac{e^{i\pi/4}}{\pi^{1/2}}\right]^{Nn} \int_{-\infty}^{+\infty} \prod_{i,\alpha} \mathrm{d}x_i^\alpha \left\{\exp\left[-i \sum_{ij;\alpha} x_i^\alpha(\lambda\delta_{ij} - J_{ij})x_j^\alpha\right] - 1\right\}\right\}$$

The integration is now over the Nn variables $\{x_i^\alpha\}$ where the indices i and α range from 1 to N and from 1 to n, respectively; the limit $n \longrightarrow 0$ is to be taken at the end of the calculations. The averaged density of eigenvalues $\rho(\lambda)$ of an ensemble of real symmetric matrices, from which a typical matrix element J_{ij} has a probability density function (pdf) $p(J_{ij})$ is then obtained by caculating

$$[2.7] \qquad \rho(\lambda) = \int \nu(\lambda;\ \{J_{ij}\}) \prod_{ij} p(J_{ij})\, \mathrm{d}J_{ij}$$

3. The random-sign symmetric matrix

We consider an $N \times N$ real symmetric matrix with zero down the diagonal, but whose off-diagonal elements J_{ij} have the values $+J/\sqrt{N}$ and $-J/\sqrt{N}$ with equal probability 1/2, and J is a constant of order unity. We see that each element has a mean value $\langle J_{ij}\rangle = 0$ and that the variance of each element is J^2/N. This model (previously considered by Wigner (14)) is formally described by the pdf

$$[3.1]\quad p(J_{ij}) = \frac{1}{2}\left\{\delta\left(J_{ij} - \frac{J}{\sqrt{N}}\right) + \delta\left(J_{ij} + \frac{J}{\sqrt{N}}\right)\right\} \text{ with } J_{ij} = J_{ji}$$

We substitute [3.1] into [2.6] and [2.7] and perform the integrals over the $\{J_{ij}\}$; the result is

$$[3.2]\quad \rho(\lambda) = -\frac{2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to 0}\left\{\left[\frac{e^{1\pi/4}}{\pi^{1/2}}\right]^{Nn}\int_{-\infty}^{+\infty}\prod_{i,\alpha}\mathrm{d}x_i^\alpha \exp\left[-i\lambda\sum_{i,\alpha}(x_i^\alpha)^2\right]\prod_{i<j}\cos\left(\frac{2J}{\sqrt{N}}\sum_\alpha x_i^\alpha x_j^\alpha\right) - 1\right\}$$

The product of the cosines is rewritten as

$$[3.3]\quad \prod_{i<j}\cos\left(\frac{2J}{\sqrt{N}}\sum_\alpha x_i^\alpha x_j^\alpha\right) = \exp\left\{\frac{1}{2}\sum_{i,j}\ln\left[\cos\left(\frac{2J}{\sqrt{N}}\sum_\alpha x_i^\alpha x_j^\alpha\right)\right]\right\}$$

and the argument of the exponential is expanded in powers of J to give

$$[3.4]\quad \prod_{i<j}\cos\left(\frac{2J}{\sqrt{N}}\sum_\alpha x_i^\alpha x_j^\alpha\right) = \exp\left\{-\frac{J^2}{N}\sum_{i,j}\left(\sum_\alpha x_i^\alpha x_j^\alpha\right)^2 + O(J^4) + \cdots\right\}$$

If we now write

$$\sum_{ij}\left(\sum_\alpha x_i^\alpha x_j^\alpha\right)^2 = \sum_\alpha\left[\sum_i (x_i^\alpha)^2\right]^2 + \sum_{\substack{\alpha\neq\beta\\ i,j}} x_i^\alpha x_j^\alpha x_i^\beta x_j^\beta,$$

then [3.2] can be written as

$$[3.5]\quad \rho(\lambda) = -\frac{2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{N}\left\{\left[\frac{e^{i\pi/4}}{\pi^{1/2}}\right]^{Nn}\int_{-\infty}^{+\infty}\prod_{i,\alpha}\mathrm{d}x_i^\alpha \exp\left[-i\lambda\sum_{i\alpha}(x_i^\alpha)^2 - \frac{J^2}{N}\sum_\alpha\left[\sum_i (x_i^\alpha)^2\right]^2\right]\exp\left[-\frac{J^2}{N}\sum_{ij;\alpha\neq\beta} x_i^\alpha x_j^\alpha x_i^\beta x_j^\beta\right] - 1\right\}$$

Some discussion is needed to justify the use of the approximation in [3.4] and to decide which terms in [3.5] it is appropriate that we retain. Since we shall eventually be taking the limits $N \to \infty$ and $n \to 0$, it is important that we retain terms in the exponent that are of dominant order in both N and n: it is easily seen that we must retain the terms with the highest power of N and of the lowest power in n. Thus, in [3.5] we must keep terms that are of order Nn in the exponent. It is easily seen that whereas the term of order J^2 that has been retained in [3.4] contains a contribution whose size is of order Nn, the terms of order J^4 and higher powers are of smaller order in N and so may be neglected in the limit $N \to \infty$. Analysis of the terms that remain in [3.5] is rather subtle. This latter equation is very similar to equation [3.3] of ref. 10 and may be treated in the same way; it was argued in ref. 10 that the term in the exponent of [3.5] with $\alpha \neq \beta$ has zero mean but a square that is of order n. A careful diagramatic analysis by Edwards and Warner (13) has confirmed that indeed in the limit $N \to \infty$ and $n \to 0$ the term with $\alpha \neq \beta$ gives a contribution of order $n1$ (rather than $O(nN)$) and so may be neglected. Thus retaining terms in the exponent of [3.5] that are of order nN, we may write [3.5] as

$$[3.6]\quad \rho(\lambda) = -\frac{2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to 0}\left\{\left[\frac{e^{i\pi/4}}{\pi^{1/2}}\right]^{Nn}\int_{-\infty}^{+\infty}\prod_{i,\alpha}\mathrm{d}x_i^\alpha \exp\left[-i\lambda\sum_{i,\alpha}(x_i^\alpha)^2 - \frac{J^2}{N}\sum_\alpha\left[\sum_i (x_i^\alpha)^2\right]^2\right] - 1\right\}$$

In this form we easily see that the replica label α can be removed and the integral replaced by

$$\left[\int_{-\infty}^{+\infty}\prod_i \mathrm{d}x_i \ldots\right]^n,$$

which then gives

$$[3.7]\quad \rho(\lambda) = -\frac{2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\left\{\left[\frac{e^{i\pi/4}}{\pi^{1/2}}\right]^{Nn} J^n - 1\right\}$$

where

[3.8] $$J = \int_{-\infty}^{+\infty} \prod_i \mathrm{d}x_i \exp\left[-i\lambda\sum_i x_i^2 - \frac{J^2}{N}\left(\sum_i x_i^2\right)^2\right]$$

This can be evaluated by using a polar coordinate R in the space of the $\{x_i\}$, with R defined by $R^2 = \sum_i x_i^2$. We define Ω_N as the usual solid angle in N dimensions and so have

[3.9] $$J = \Omega_N \int_0^{\infty} \mathrm{d}R \exp\left[(N-1)\ln R - i\lambda R^2 - \frac{J^2R^4}{N}\right]$$

For large N, the integral is straightforwardly performed by steepest descents: The integrand has saddle points at $R = \bar{R}_\pm$ where

[3.10] $$\bar{R}_\pm^2 \equiv -i\frac{\lambda N}{4J^2} \pm \frac{N}{4J^2}\left[4J^2\left(1-\frac{1}{N}\right) - \lambda^2\right]^{1/2}$$

The contour of integration may be deformed to pass through one of these saddle points but not both: the details for a similar integral are given in ref. 10 and will not be reproduced again here. We now see that our basic "replica" identity [2.3] can be used in [3.7] to yield a term in $\ln J$ when the limit $n \longrightarrow 0$ is taken. J is calculated by using its asymptotic value when $N \longrightarrow \infty$ by substituting [3.10] into [3.9]. The final result is very simple, and after carrying out the derivative with respect to λ in [3.7] we find that

[3.11] $$\rho(\lambda) = \frac{2}{N\pi}\,\mathrm{Im}\,(i\bar{R}_+^2)$$

where we have chosen the saddle point yielding a positive density, $\rho(\lambda)$. Thus, using [3.10] and taking the limit $N \longrightarrow \infty$ we find that

[3.12] $$\rho(\lambda) = \frac{1}{2J^2\pi}(4J^2 - \lambda^2)^{1/2} \qquad \text{for } |\lambda| < 2J$$
$$\rho(\lambda) = 0 \qquad \text{for } |\lambda| > 2J$$

This is the Wigner semicircle law for the aed of the random-sign ensemble of matrices. It was first derived by Wigner (14) for this problem, using moment calculations. The result is identical to that which obtains for the GOE of symmetric matrices in which each off-diagonal element is normally distributed with mean zero and variance J^2/N. We shall see in Sect. 5 that the replication method used here easily allows us to infer that [3.12] must be the aed of any reasonable matrix ensemble in which individual matrix elements have mean zero.

4. The random-sign symmetric matrix when each element has a finite nonzero mean

We consider now an $N \times N$ real symmetric matrix with zeros down the diagonal but whose off-diagonal elements J_{ij} take the values p and $q(< p)$ with equal probability. Anticipating the eventual thermodynamic limit $N \longrightarrow \infty$, we write

$$p = \frac{M_0}{N} + \frac{J}{\sqrt{N}} \text{ and } q = \frac{M_0}{N} - \frac{J}{\sqrt{N}}$$

with both J and M_0 constants of order unity.

This system has matrix elements described by a pdf

[4.1] $$\rho(J_{ij}) = \frac{1}{2}\{\delta(J_{ij} - p) + \delta(J_{ij} - q)\} \text{ with } J_{ij} = J_{ji}$$

From this we see that each element of the matrix has a mean $\langle J_{ij}\rangle$ equal to $(p+q)/2$ whose value is M_0/N. The variance $\langle J_{ij}^2\rangle - \langle J_{ij}\rangle^2$ is $(p-q)^2/4$ and has a value J^2/N. Substituting [4.1] into [2.7] yields the result that

[4.2] $$\rho(\lambda) = -\frac{2}{N\pi}\,\mathrm{Im}\,\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\left\{\left[\frac{e^{i\pi/4}}{\pi^{1/2}}\right]^{Nn}\int_{-\infty}^{+\infty}\prod_{i,\alpha}\mathrm{d}x_i^\alpha \exp\left[-i\lambda\sum_{i,\alpha}(x_i^\alpha)^2\right]\prod_{i,j}F_{ij} - 1\right\}$$

where

[4.3] $$F_{ij} = e^{2iM_0/N\sum_\alpha x_i^\alpha x_j^\alpha}\cos\frac{2J}{\sqrt{N}}\left(\sum_\alpha x_i^\alpha x_j^\alpha\right)$$

Since we once more wish to retain terms of order nN in an exponent in [4.2], we re-exponentiate the cosine term of [4.3] and carry out the expansion as in [3.3] and [3.4]; after retaining terms in the exponent that are of order nN exactly as before we find that

$$[4.4] \quad \rho(\lambda) = -\frac{2}{N\pi} \,\mathrm{Im}\, \frac{\partial}{\partial\lambda} \lim_{n\to 0} \frac{1}{n} \left\{ \left[\frac{e^{i\pi/4}}{\pi^{1/2}}\right]^{Nn} \int_{-\infty}^{+\infty} \prod_{i,\alpha} dx_i^\alpha \exp\left[-i\lambda \sum_{i,\alpha} (x_i^\alpha)^2\right] \exp\left[\frac{iM_0}{N} \sum_\alpha \left[\sum_i x_i^\alpha\right]^2\right] \times \exp\left[-\frac{J^2}{N} \sum_\alpha \left[\sum_i (x_i^\alpha)^2\right]^2\right] - 1 \right\}$$

This integral is identical to that of [4.2] in ref. 10 and cannot be performed by the straightforward methods of Sect. 3; instead we must use the auxiliary field (or Hubbard–Stratanovich) identity twice over as in ref. 10. Since [4.4] is identical to [4.2] of ref. 10 the lengthy calculation from this point proceeds exactly as in this latter paper and we shall not reproduce the details here that are given in full in ref. 10. The result of a saddle-point integration is that when $|M_0| > J$, the spectrum of eigenvalues consists of the Wigner semicircle for $-2J < \lambda < +2J$ together with a single isolated eigenvalue situated at $\lambda = M_0 + (J^2/M_0)$ outside the main semicircular band of eigenvalues. For $|M_0| < J$ we simply have the semicircular band of eigenvalues. The presence of an isolated eigenvalue outside the semicircle band was confirmed for the GOE by Jones *et al.* (12) who demonstrated once again the presence of such an eigenvalue and pointed out the strong analogy between this and the eigenvalue associated with the localized vibrational state that a single impurity atom may have in an otherwise perfect crystal lattice (see also Kosterlitz *et al.* (6)).

To summarize, we have found for our random-sign matrix ensemble with finite means that the aed is given by

$$[4.5] \quad \rho(\lambda) = \frac{(4J^2 - \lambda^2)^{1/2}}{2\pi J^2} + \frac{1}{N}\delta\left[\lambda - \left(M_0 + \frac{J^2}{M_0}\right)\right], \quad |M_0| > J$$

$$= \frac{(4J^2 - \lambda^2)^{1/2}}{2\pi J^2}, \quad |M_0| < J$$

where $J^2 = N(p-q)^2/4$ and $M_0 = N(p+q)/2$

5. Some comments on arbitrary matrix ensembles: the Wigner conjectures within the replica framework

Our formulation of the calculation of the aed of a random-matrix ensemble using the "replica" method based on [2.3] yields a ready demonstration of the Wigner conjecture (2) that if reasonable conditions are placed on the pdf (J_{ij}) of matrix elements, then provided that each element has mean zero and all matrix elements have the same variance J^2/N, the resulting aed must be the Wigner semicircle [3.12] when $N \to \infty$. An extension of this argument will show that if the pdf is such that each element has mean M_0/N the aed must be the semicircle together with the isolated eigenvalue predicted by EJ in ref. 10 and demonstrated in [4.5] for the random-sign ensemble with finite mean. Let us, like Wigner (2), assume that the matrix element J_{ij} is a random variable with mean zero and variance $\sigma^2(\equiv J^2/N)$; further we assume that for all $m > 2$ the mth moment of $p(J_{ij})$ is bounded by some number b_m that may depend on m, but that is independent of i and j. The average implied in [2.6] and [2.7] has the form

$$[5.1] \quad \left\langle \exp i \sum_{i,j} J_{ij} \left(\sum_\alpha x_i^\alpha x_j^\alpha\right) \right\rangle$$

where the angular brackets denote an average over each independent element of the symmetric matrix $\underline{\underline{J}}$. This quantity [5.1] is nothing other than the characteristic function for a weighted sum of $O(N^2)$ random variables each with mean zero. We may then use a form of the central limit theorem. (see, for example, Whittle (16) equation 8.5.10) to deduce that for large N

$$[5.2] \quad \left\langle \exp\left[i \sum_{i,j} J_{ij} \left(\sum_\alpha x_i^\alpha x_j^\alpha\right)\right] \right\rangle \longrightarrow \exp\left[-\frac{J^2}{N} \sum_{i,j} \left(\sum_\alpha x_i^\alpha x_j^\alpha\right)^2\right]$$

It should be noted that we have ignored the diagonal elements J_{ii} of which there are N compared with the off-diagonal elements of which there are $N(N-1)/2$ independent terms (see the discussion following [3.3] in ref. 10). Equation [5.2] represents the results of carrying out the averaging in [2.7] and we see immediately that the integral that gives the averaged eigenvalue spectrum is identical to [3.5] and hence the rest of the calculation follows exactly as in Sect. 3 and yields the Wigner semicircle, [3.12], for any reasonably behaved pdf $p(J_{ij})$, satisfying the conditions given earlier and for which each J_{ij} has mean zero and variance J^2/N.

The extension of this argument to the case where each matrix element has mean M_0/N and variance J^2/N and the pdf satisfies the same conditions as before, is straightforward within the replica method. The average that is implied in [2.6] and [2.7] is now

$$\left\langle \exp\left[i \sum_{ij} J_{ij} \left(\sum_{\alpha} x_i^{\alpha} x_j^{\alpha} \right) \right] \right\rangle = \exp\left[\frac{M_0}{N} \sum_{ij} \left(\sum_{\alpha} x_i^{\alpha} x_j^{\alpha} \right) \right] \left\langle \exp\left[i \sum_{i,j} (J_{ij} - \langle J_{ij} \rangle) \left(\sum_{\alpha} x_i^{\alpha} x_j^{\alpha} \right) \right] \right\rangle \tag{5.3}$$

Once again, the term in angular brackets on the right-hand side of [5.3] is the characteristic function for a weighted sum of $O(N^2)$ random variables, $J_{ij} - \langle J_{ij} \rangle$, each with mean zero and variance J^2/N. Application of the central limit theorem once more as $N \longrightarrow \infty$ shows that

$$\left\langle \exp\left[i \sum_{i,j} J_{ij} \left(\sum_{\alpha} x_i^{\alpha} x_j^{\alpha} \right) \right] \right\rangle \longrightarrow \exp\left\{ i \frac{M_0}{N} \sum_{i,j} \left(\sum_{\alpha} x_i^{\alpha} x_j^{\alpha} \right) - \frac{J^2}{N} \sum_{i,j} \left(\sum_{\alpha} x_i^{\alpha} x_j^{\alpha} \right)^2 \right\} \tag{5.4}$$

Equation [5.4] once more represents the result of carrying out the averaging in [2.7] and we can immediately see that the integral that gives the averaged eigenvalue spectrum is identical to [4.4] and hence the rest of the calculation follows exactly that outlined in Sect. 4 and which is given in full in ref. 10. The resulting averaged eigenvalue spectrum will be that given in [4.5] with an isolated eigenvalue lying outside the band at $\lambda = M_0 + J^2/M_0$ (when $|M_0| > J$) for any reasonable pdf of the J_{ij} that has mean M_0/N and variance J^2/N.

Thus, within the framework of the replica method we have given a simple demonstration of the results of the Wigner conjecture and their extension to the case of random-matrix ensembles with finite means.

6. Numerical simulations of certain averaged eigenvalue spectra

In this section we compare the predictions made earlier in this paper with the results of numerical simulations of the aed of ensembles of 50×50 and 100×100 matrices.

We have chosen to perform simulations first on the random-sign symmetric matrix ensemble described in Sect. 3. For convenience we have chosen the scaled variance J^2 of each off-diagonal element to have the value $J^2 = 1/2$ and we have drawn the value of each such element from the random-sign pdf given by [3.1] in which the mean of each element is zero. We have diagonalized 250 such real symmetric square matrices of size 100×100 and 500 such matrices of size 50×50. From the eigenvalues thus generated it is easy to construct an averaged spectrum for both the 50×50 matrices and the 100×100 matrices. Figure 1 shows the aed produced by averaging the spectra of 500 of the 50×50 matrices and Fig. 2 shows the corresponding result for 250 matrices of size 100×100. On each such histogram we have shown the theoretical Wigner semicircular aed that should obtain as $N \longrightarrow \infty$. With our choice of $J^2 = 0.5$, the edges of this Wigner semicircular band of eigenvalues should occur at $\lambda = \pm\sqrt{2}$. It is clear from these figures that the Wigner semicircle provides a very good description of the aed for both $N = 50$ and $N = 100$ although a small tail in the numerical aed is apparent just outside the Wigner band edges. Since we have included only those eigenvalues between $\lambda = \pm 2.5$, we have checked by calculating the area under the histogram that indeed we have included in excess of the 99% available eigenvalues between these two limits. For comparison we show in Fig. 3 the corresponding aed obtained from 250 simulations of 100×100 matrices drawn from a GOE with $J^2 = 0.5$ and indeed we see that for all practical purposes there is no discernible difference between the aed produced by the GOE and the random-sign ensemble, in accord with the predictions of [3.12], Sect. 5, and of the Wigner conjecture.

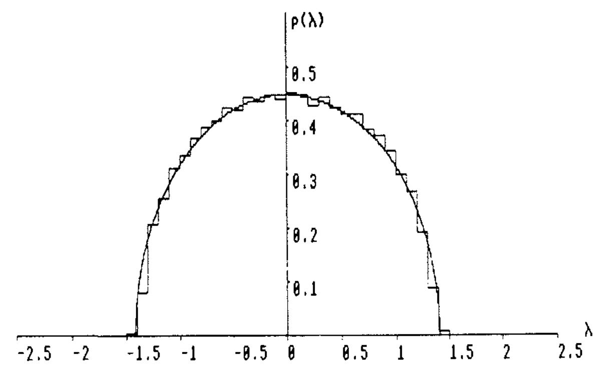

FIG. 1. The histogram shows the aed for $J^2 = 0.5, M_0 = 0$ calculated from 500 samples when $N = 50$ for the random sign ensemble. The continuous curve is the corresponding Wigner semicircle.

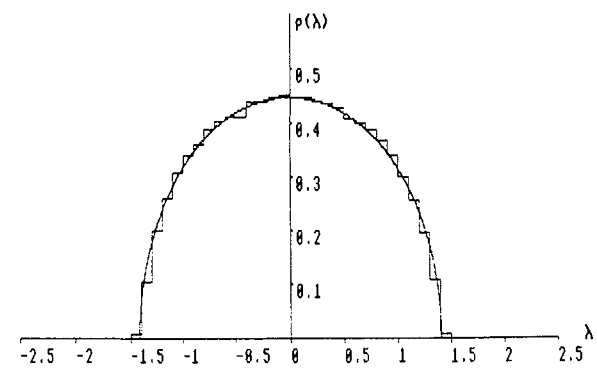

FIG. 2. The histogram shows the aed for $J^2 = 0.5, M_0 = 0$, calculated from 250 samples when $N = 100$ for the random sign ensemble. The continuous curve is the corresponding Wigner semicircle.

We have also performed simulations on the random-sign ensemble described in Sect. 4 by the pfd [4.1] in which each element of the random-sign matrix has a finite (scaled) mean $M_0 = 1.7$ (where we use the notation of Sect. 4). It is shown in Sect. 4 that when $|M_0| > J$, the spectrum for large N should consist of the Wigner semicircle together with an isolated eigenvalue at $\lambda_0 = M_0 + J^2/M_0$ of weight $1/N$ compared with that of the Wigner semicircle. For $J^2 = 0.5$ and $M_0 = 1.7$ we have $\lambda_0 \approx 1.99$. In Fig. 4 we show the aed produced for such a random-sign ensemble with 500 samples of a 50×50 matrix with M_0 chosen for convenience to be 1.7 and $J^2 = 0.5$. Figure 5 shows the corresponding results for 250

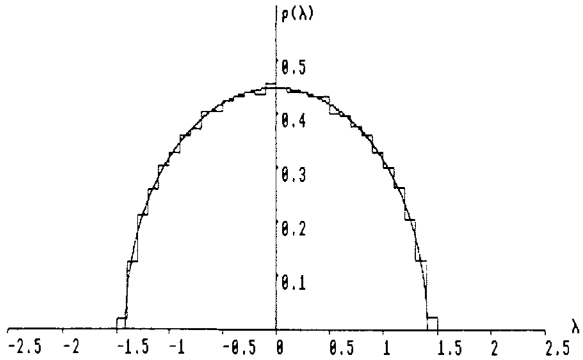

FIG. 3. The histogram shows the aed for $J^2 = 0.5, M_0 = 0$, calculated from 250 samples when $N = 100$ for the GOE. The continuous curve is the corresponding Wigner semicircle.

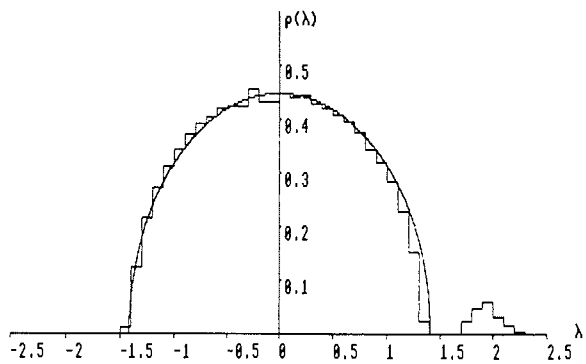

FIG. 4. The histogram shows the aed for $J^2 = 0.5, M_0 = 1.7$, calculated from 500 samples when $N = 50$ for the random sign ensemble. The continuous curve is the corresponding Wigner semicircle.

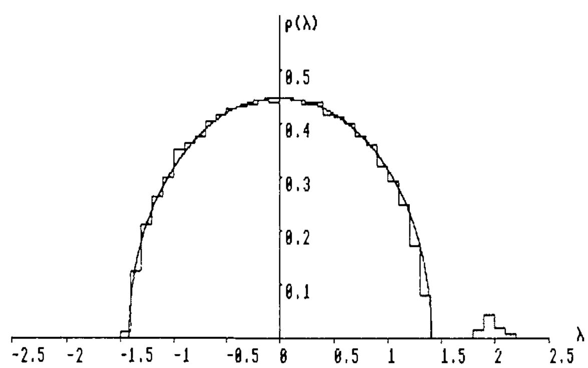

FIG. 5. The histogram shows the aed for $J^2 = 0.5, M_0 = 1.7$, calculated from 250 samples when $N = 100$ for the random sign ensemble. The continuous curve is the corresponding Wigner semicircle.

samples of a 100×100 matrix with $M_0 = 1.7$ and $J^2 = 0.5$. In each case we have also displayed the Wigner semicircle. We see the presence in each case not only of a band of states that closely fits the Wigner semicircle but also of a very narrow band of states quite detached from the main semicircle. In each case this narrow band of isolated states has a maximum of the histogram in the bin lying just below the value $\lambda = 2$. We also notice that this isolated band of states has a noticeably lower maximum in the case $N = 100$ than in the case $N = 50$. Further, by calculating the area under the histogram representing the main (semicirclelike) band of states, we find that when

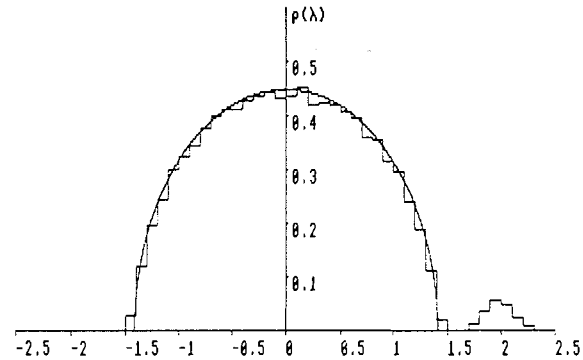

FIG. 6. The histogram shows the aed for $J^2 = 0.5, M_0 = 1.7$, calculated from 500 samples when $N = 50$ for the GOE. The continuous curve is the corresponding Wigner semicircle.

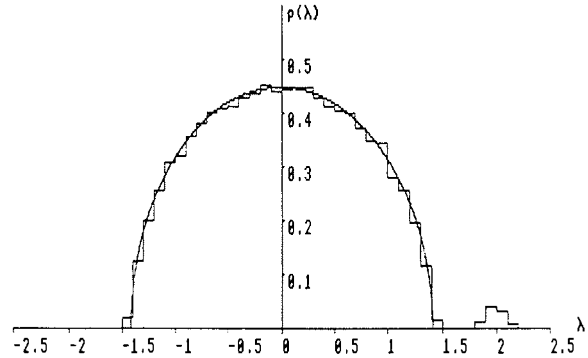

FIG. 7. The histogram shows the aed for $J^2 = 0.5, M_0 = 1.7$, calculated from 250 samples when $N = 100$ for the GOE. The continuous curve is the corresponding Wigner semicircle.

$N = 50$, exactly 2% of the eigenvalues lie in the isolated band and when $N = 100$ we find 1% of the eigenvalues lie in the isolated band exactly as is predicted by [4.5]. For comparison we show in Fig. 6 and 7 the numerical results obtained for the GOE when $N = 50$ and $N = 100$, respectively, with $M_0 = 1.7$ and $J^2 = 0.5$. Again these results are indistinguishable from those produced by the random-sign ensemble. The percentage of eigenvalues associated with the isolated bands are again 2 and 1% for $N = 50$ and $N = 100$, respectively. Although numerical simulations of the GOE with mean zero have been previously published by Porter and Rosenzweig (17) numerical results for ensembles with finite means have note previously been presented. It seems clear that the small isolated band of eigenvalues really is associated with what would be a single eigenvalue (giving a delta-function contribution to the aed) in the case where $N \longrightarrow \infty$ and it lends weight to the work of EJ (10) and the arguments of Sect. 5 of this paper for ensembles with finite means. A previous calculation (15), now known to be in error, had suggested that adding a finite mean M_0/N to each element of a matrix belonging to the GOE would simply produce a semicircular band of states shifted by an amount M_0 form the main Wigner semicircle.

7. Conclusions

In this paper we have shown how the replica method developed by EJ (10) in the context of a GOE of random matrices can be used to describe the aed of the Wigner random-sign ensemble. We have shown explicitly that as $N \longrightarrow \infty$, the aed of this ensemble is the Wigner semicircle, and in the case where each matrix element has a finite mean we have demonstrated the presence of an isolated eigenvalue that for suitable values of the mean will be outside the Wigner semicircle exactly as for the GOE with finite mean.

We have shown how within this replica framework, it is easy to reproduce the Wigner conjectures that would, for any reasonable ensemble with mean zero, lead to an aed that is again a band whose width is proportional to the standard deviation of an individual matrix element. Further we have demonstrated that if the ensemble has a finite mean, then an isolated eigenvalue may split off from the main band as was shown for the GOE in ref. 10.

We have presented numerical simulations of the random-sign ensemble with mean zero and with a finite mean and also of the GOE with finite mean; these simulations are in accord with our predictions.

Clearly investigations of the shape of the aed for large finite N is a difficult and subtle problem germane to the properties of any large symmetric, but finite, random matrix likely to occur in describing a real physical problem. We hope to address this in a future publication.

Acknowledgements

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