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Some Remarks on the Global Transforms of Noetherian Rings

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In his paper [4], Matijevic generalized the Krull-Akizuki theorem on the intermediate rings between a noetherian domain of Krull dimension one and its quotient field to the case of general noetherian rings, using the notion of the global transform of a noetherian ring. The aim of this paper is to give some interesting properties of the global transforms of noetherian rings. For a noetherian ring A, the global transform A^g is the set $\{x \in Q(A);\ x \in A \ \mathit{or}\ \dim(A/(A: x))=0\}$, where $Q(A)$ is the total quotient ring of A. Now A^g coincides with the $\mathscr{C}$-divisorial envelope of A in $Q(A)$, where $\mathscr{C}$ is the Serre subcategory of $\mathrm{Mod}(A)$ consisting of all A-modules M such that $\mathrm{Supp}(M) \subseteq \mathrm{Max}(A)$ (for $\mathscr{C}$-divisorial envelope, see [2]). On the other hand, a $\mathscr{C}$-divisorial module M is characterized by $\mathrm{Ext}_A^1(N, M)=0$ for every object N in $\mathscr{C}$. In other words, the relation between A^g and A depends deeply on the set $\{\mathrm{depth}(A_{\mathfrak{m}});\ \mathfrak{m} \in \mathrm{Max}(A)\}$. Among our results in this paper, we shall show that the canonical homomorphism $A \to A^g$ is a flat epimorphism for any noetherian normal domain A, and also that the global transform of A^g is A^g itself if A is reduced. Finally we give an example which shows that the Corollary to the Theorem in [4] is not true if we drop the assumption of reducedness of A.

Throughout this paper, all rings are commutative with unit. We use the following notations: for a ring A,

$\mathrm{Max}(A)$ = the set of all maximal ideals in A,

$z(A)$ = the set of all zero divisor of A,

$(A: x)=\{a \in A;\ ax \in A\}$, where x is any element of $Q(A)$.

PROPOSITION 1. *Let A be a noetherian ring. Then the following statements hold;*

a) *If $\dim(A) \leq 1$, then $A^g = Q(A)$.*

b) *$S^{-1}(A^g) \subseteq (S^{-1}A)^g$ holds for any multiplicatively closed subset S of A.*

c) *Suppose that $ht(\mathfrak{p}) \leq 1$ for any associated prime ideal $\mathfrak{p}$ in A. Let S be a multiplicatively closed subset of A such that $\mathrm{Max}(S^{-1}A) \subseteq ({}^a i)^{-1}(\mathrm{Max}(A))$, where ${}^a i$ is the morphism of $\mathrm{Spec}(S^{-1}A)$ to $\mathrm{Spec}(A)$ defined by the canonical homomorphism i of A to $S^{-1}A$. Then $S^{-1}(A^g)=(S^{-1}A)^g$. In particular, $(A^g)_{\mathfrak{m}}=(A_{\mathfrak{m}})^g$ for any maximal ideal $\mathfrak{m}$ in A.*

PROOF. a) Let a/s be any element of $Q(A)$, where $a \in A$, $s \in A - z(A)$. We may assume that s is a non-unit. Hence $\dim(A/sA)=0$; this implies that a/s is an element of A^g because $(A: a/s) \subseteq sA$.

b) This follows easily from the relation $S^{-1}Q(A) \subseteq Q(S^{-1}A)$.

c) We have $S^{-1}Q(A)=Q(S^{-1}A)$ by the remark (2) given after the proof. Let x/s be any element of $(S^{-1}A)^g$, where $x \in Q(A)$, $s \in S$. We may assume that $(A: x) \subsetneq A$. *Let* $(A: x)=\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ be a primary decomposition of $(A: x)$. Then $(S^{-1}A: x/s)=S^{-1}\mathfrak{q}_1 \cap \cdots \cap S^{-1}\mathfrak{q}_m$. Since x/s is an element of $(S^{-1}A)^g$, we may assume that $S^{-1}\sqrt{\mathfrak{q}_1},\dots, S^{-1}\sqrt{\mathfrak{q}_r}$ are maximal in $S^{-1}A$ and that $S^{-1}\mathfrak{q}_{r+1} = \cdots = S^{-1}\mathfrak{q}_m = S^{-1}A$. By our assumption, $\sqrt{\mathfrak{q}_i}$ is maximal in A for $i \leq r$. Let t be an element of $S \cap \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_m$. Then $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \subseteq (A: tx)$. Therefore $tx \in A^g$. Thus $x/s = tx/ts$ is an element of $S^{-1}(A^g)$.

REMARK. (1) If A is a Hilbert ring, then the multiplicatively closed subset $\{1, s, s^2,\dots\}$ of A satisfies the assumption of *c*).

(2) Let A be a noetherian ring. Then it is easy to see that $S^{-1}Q(A) = Q(S^{-1}A)$ holds for any multiplicatively closed subset S of A if and only if $ht(\mathfrak{p}) \leq 1$ for any associated prime ideal $\mathfrak{p}$ in A.

Assume that A is a local ring and $Q(A)$ is A-injective. Then A^g is the $\mathscr{C}$-divisorial envelope of A, where $\mathscr{C}$ is the Serre subcategory of Mod (A) consisting of all A-modules M such that $\mathrm{Supp}(M) \subseteq \mathrm{Max}(A)$. Therefore $A^g = A$ if and only if $\mathrm{Ext}^1_A(N, A)=0$ for every object N in $\mathscr{C}$, i.e., $\mathrm{depth}(A)=1$. In general, weh ave the following:

PROPOSITION 2. *Let A be a noetherian ring. Then the following statements are equivalent*:

a) $A^g = A$.

b) $\mathrm{depth}(A_\mathfrak{m}) \neq 1$ *for any maximal ideal* $\mathfrak{m}$ *in* A.

PROOF. Suppose that $A^g \supsetneq A$. Let a/b be an element of $A^g - A$, where $a \in A$, $b \in A - z(A)$. Since $\dim(A/(bA: a))=0$, $\mathfrak{m}=((bA: a): c)=(bA: ac)$ is maximal in A for some $c \in A$. Then we have $\mathrm{depth}(A_\mathfrak{m})=1$. Conversely suppose that $\mathrm{depth}(A_\mathfrak{m})=1$ for some maximal ideal $\mathfrak{m}$ in A. Then $\mathfrak{m}=(bA: a)$ for some $b \in A - z(A)$ and $a \in A$; hence $a/b \in A^g - A$. Therefore $A^g \supsetneq A$.

COROLLARY. *Let A be a noetherian normal domain. Then the following statements hold*:

a) *$A^g = A$ if and only if there exists no height one maximal ideal in A.*

b) *The canonical homomorphism $A \to A^g$ is a flat epimorphism.*

c) *If the class group of A is a torsion group, then A^g is a localization of A.*

PROOF. a) This follows immediately from Prop. 2.

b) Let $\mathfrak{m}$ be any maximal ideal in A. If $ht(\mathfrak{m})=1$, then $(A^g)_{\mathfrak{m}}=(A_{\mathfrak{m}})^g=Q(A)$. If $ht(\mathfrak{m})\geq 2$, then $(A^g)_{\mathfrak{m}}=(A_{\mathfrak{m}})^g=A_{\mathfrak{m}}$ by Prop. 2. Therefore $A_{\mathfrak{m}}\to(A^g)_{\mathfrak{m}}$ is a flat epimorphism; hence so is $A\to A^g$.

c) This follows from Cor. 4.4 in [3].

Remark. Let $A=k[X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3]$ be a subring of the polynomial ring $k[X, Y]$, where k is a field. We see easily that $A\to A^g=k[X, Y]$ is neither an epimorphism nor a flat homomorphism from Prop. 1.7 in [3] and Prop. 2 in [5].

Proposition 3. *Let $A\subseteq B$ be noetherian rings. Then the following statements hold:*

a) *Suppose $Q(A)\subseteq Q(B)$. If B is integral over A, then $A^g\subseteq B^g$.*

b) *Suppose $B\subseteq Q(A)$. If the going down theorem holds for $A\subseteq B$, then $A^g\subseteq B^g$.*

c) *Suppose $B\subseteq A^g$. If every maximal ideal in B contracts to a maximal ideal in A, then $B^g\subseteq A^g$.*

Proof. a) Let x be any element of A^g. Since we have $(A: x)\supseteq\mathfrak{a}$ for some ideal $\mathfrak{a}$ in A such that $\dim(A/\mathfrak{a})=0$, $(B: x)\supseteq(A: x)B\supseteq\mathfrak{a}B$. Hence x is an element of B^g because $\dim(B/\mathfrak{a}B)=0$ by our assumption.

b) Let $\mathfrak{m}$ be a maximal ideal in A and let $\mathfrak{n}$ be a prime ideal in B such that $\mathfrak{m}=\mathfrak{n}\cap B$. We see easily that $ht(\mathfrak{n})\leq ht(\mathfrak{m})$ holds by our assumption $B\subseteq Q(A)$. Since the going down theorem holds for $A\subseteq B$, $ht(\mathfrak{n})\geq ht(\mathfrak{m})$. Therefore $ht(\mathfrak{n})=ht(\mathfrak{m})$; this implies that $\mathfrak{n}$ is maximal in B. Hence b) can be proved similarly as a).

c) Let x be any element of B^g; then $(B: x)\supseteq\mathfrak{n}_1\cdots\mathfrak{n}_r$ for some maximal ideals $\mathfrak{n}_1,\dots, \mathfrak{n}_r$ in B. Set $\mathfrak{m}_i=\mathfrak{n}_i\cap A$. By our assumption, $\mathfrak{m}_i$ is maximal in A. Let $x_1,\dots, x_n$ generate $\mathfrak{m}_1,\dots, \mathfrak{m}_r$. Since x_ix is an element of $B\subseteq A^g$, $(A: x_ix)\supseteq\prod_{j=1}^{r(i)}\mathfrak{m}_{ij}$ for some maximal ideals $\mathfrak{m}_{i1},\dots, {}_{ir(i)}$ in A. Then we see that $(A: x)\supseteq\mathfrak{m}_1\cdots\mathfrak{m}_r\cdot\prod_{ij}\mathfrak{m}_{ij}$. Therefore $x\in A^g$. Thus $B^g\subseteq A^g$.

Remark. (1) Even if $A\subseteq B\subseteq Q(A)$, the relation $A^g\subseteq B^g$ does not always hold. In fact, let $A=k[X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3]$ and let $B=A[X/Y]$, where k is a field and X, Y are indeterminates. Let $\mathfrak{m}=(X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3)A$ and let $\mathfrak{n}=(X/Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3)B$. For any positive integer n, $(B_{\mathfrak{n}}: Y)\not\ni(X/Y)^n$. Hence Y is not contained in $(B_{\mathfrak{n}})^g$. Therefore $(A_{\mathfrak{m}})^g\nsubseteq(B_{\mathfrak{n}})^g$.

(2) Let A be a noetherian domain. If A is a universally catenarian and if A has no height one maximal ideal, then A^g is integral over A. In fact, $ht(\mathfrak{P}\cap A)=1$ for any height one prime ideal $\mathfrak{P}$ in $\bar{A}$, where $\bar{A}$ is the derived normal ring of A; hence $A^g\subseteq\bigcap_{\mathfrak{P}\in Ht_1(\bar{A})}A_{\mathfrak{P}\cap A}\subseteq\bigcap_{\mathfrak{P}\in Ht_1(\bar{A})}\bar{A}_{\mathfrak{P}}=\bar{A}$. Therefore A^g is integral over A.

PROPOSITION 4. *Let A be a noetherian ring. Then the following statements hold*:

a) *If $\mathfrak{M}$ is a maximal ideal in A^g such that $\mathfrak{M} \not\subseteq z(A^g)$, then $\mathfrak{M} \cap A$ is maximal in A.*

b) *If A is reduced, we have $(A^g)^g = A^g$.*

PROOF. Since $\mathfrak{m} = \mathfrak{M} \cap A \not\subseteq z(A)$ by our assumption, $\mathfrak{m}$ contains a non zero divisor x. From the Theorem in [4] it follows that A^g/xA^g is a finite A/xA-module. Therefore $\mathfrak{m}$ is maximal in A.

b) If every maximal ideal in A^g is not an element of Ass (A^g), then $(A^g)^g = A^g$ holds by the assertion c) of Prop. 3. (Note that A^g is noetherian by the Corollary in [4]). Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the height zero maximal ideals in A^g and let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the minimal prime ideals which are not maximal. By the Chinese Remainder Theorem, we have $A^g \simeq A^g/\mathfrak{m}_1 \times \cdots \times A^g/\mathfrak{m}_r \times A^g/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$. Set $\mathfrak{q}_i = \mathfrak{p}_i \cap A$ for $i = 1, \dots, t$. We see easily that $A^g/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = (A/\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t)^g$. Since $A^g/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$ has not a height zero maximal ideal, $(A^g/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t)^g = A^g/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$. Therefore $(A^g)^g \simeq (A^g/\mathfrak{m}_1)^g \times \cdots \times (A^g/\mathfrak{m}_r)^g \times (A^g/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t)^g \simeq A^g/\mathfrak{m}_1 \times \cdots \times A^g/\mathfrak{m}_r \times A^g/\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t \simeq A^g$. Thus $(A^g)^g = A^g$.

PROPOSITION 5. *Let A be a noetherian ring. Suppose that* $\operatorname{depth}(A_\mathfrak{m}) \geq 1$ *for any maximal ideal $\mathfrak{m}$ in A. Then the following statements hold*:

a) $A^g \simeq \varinjlim \operatorname{Hom}_A(\mathfrak{a}, A)$, *where $\mathfrak{a}$ runs over all ideals such that* $\dim(A/\mathfrak{a}) = 0$.

b) $(A(X))^g \simeq A^g \otimes_A A(X)$.

PROOF. a) Let $\mathfrak{a}$ be an ideal in A such that $\dim(A/\mathfrak{a}) = 0$. Since $\mathfrak{a}$ contains a non zero divisor, $\mathfrak{a}^{-1} \simeq \operatorname{Hom}_A(\mathfrak{a}, A)$ holds. Therefore $A^g = \cup\, \mathfrak{a}^{-1} \simeq \varinjlim \operatorname{Hom}_A(\mathfrak{a}, A)$, where $\mathfrak{a}$ runs over all ideals in A such that $\dim(A/\mathfrak{a}) = 0$.

b) Every maximal ideal in $A(X)$ is of the form $\mathfrak{m}A(X)$ for some maximal ideal $\mathfrak{m}$ in A. Therefore $(A(X))^g \simeq \varinjlim \operatorname{Hom}_{A(X)}(\mathfrak{a}A(X), A(X))$, where $\mathfrak{a}$ runs over all ideals in A such that $\dim(A/\mathfrak{a}) = 0$. Since $A(X)$ is flat over A, $\operatorname{Hom}_{A(X)}(\mathfrak{a}A(X), A(X)) \simeq \operatorname{Hom}_A(\mathfrak{a}, A) \otimes_A A(X)$ by Prop. 11 in [1], Chap. I, § 2. Thus $(A(X))^g \simeq A^g \otimes_A A(X)$.

The following proposition is a generalization of the Corollary in [4].

PROPOSITION 6. *Let A be a reduced ring. If $Q(A)$ and A/xA for any non zero divisor x are noetherian, then A is noetherian.*

PROOF. Since $Q(A)$ is noetherian, the number of minimal prime ideals in A is finite. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal prime ideals in A. Set $B = A/\mathfrak{p}_1 \times \cdots \times A/\mathfrak{p}_n$. Since A is reduced, A is contained in B. Let $\mathfrak{a}/\mathfrak{p}_1$ be any non zero ideal in $A/\mathfrak{p}_i$ and let x be an element of $\mathfrak{a} - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. By our assumption, A/xA

is noetherian. Therefore $\mathfrak{a}$ is finitely generated. Hence $A/\mathfrak{p}_i$ is noetherian and so is B. Thus by Eakin-Nagata's theorem, A is noetherian.

REMARK. The following example A shows that the reducedness of A in the above Prop. 6 is essential. Let k be a field and let X, Y be indeterminates. We put $B_0=k[X, Y]$, $B=k[X, Y, Y/X, Y/X^2,\dots]$, $B_1=k[X, Y, 1/X]$, $A_0=B_0/Y^2B_0$, $A=B/(Y^2B_1\cap B)$ and $A_1=B_1/Y^2B_1$. Let us denote X and Y mod Y^2B_0 by x and y respectively. Since $Q(A_0)=k[x, y]_{yk[x,y]}=Q(A_1)$, we have $Q(A)=k[x, y]_{yk[x,y]}$; hence $Q(A)$ is noetherian. We have $(A_0)^g=Q(A_0)$ by the assertion a) of Prop. 1. Therefore A/xA is noetherian for any non zero divisor x in A by the Theorem in [4]. Set $\mathfrak{n}=XB/(Y^2B_1\cap B)$ and set $j\colon A\to A_{\mathfrak{n}}$ the canonical homomorphism. As is easily seen, $j(y)$ is not zero in $(A_1)_{\mathfrak{n}}$. Since $\mathfrak{n}^nA_{\mathfrak{n}}\in j(y)$ for any n, $\mathfrak{n}^nA_{\mathfrak{n}}\neq\{0\}$. This implies that A is not noetherian.

Appendix

Matijevic has proved that A^g is noetherian if A is a reduced noetherian ring, and he has given a noetherian ring A such that A^g is not noetherian. We here give another example which is simpler than Matijevic's. To show this, we introduce the notion of the global transform of an arbitrary ring as follows: Let A be a ring; then the global transform of A is the set $A^g=\{x\in Q(A);\ \text{length}\ (A/(A\colon x))<\infty\}$. It is easy to see that A^g is a subring of $Q(A)$. The following proposition is corresponding one to Prop. 2 in the non-noetherian case. This can be proved by the same arguments as in the proof of Prop. 2.

PROPOSITION. *Let A be a ring. Then the following statements are equivalent*:

a) $A^g=A$.

b) *A has no maximal ideal of the form $(uA\colon a)$, where $u\in A-z(A)$ and $a\in A$.*

Now we give our desired example. Let k be a field, and let X, Y and Z be indeterminates. We put $C=k[X, Y, Z, 1/X]/(YZ, Z^2)$. Let x, y and z be the images of X, Y and Z in C respectively. Moreover we put $A=k[x, y, z/x]$, $B=k[x, y, z/x, z/x^2,\dots]$, $\mathfrak{m}=(x, y, z/x)A$ and $\mathfrak{n}=(x, y, z/x, z/x^2,\dots)B$. Since $B/xB\simeq k[Y]$, x, y is a $B_{\mathfrak{m}}=B_{\mathfrak{n}}$-regular sequence. On the other hand y, x is not a $B_{\mathfrak{m}}$-regular sequence. Therefore $B_{\mathfrak{m}}$ is not noetherian. $(A_{\mathfrak{m}})^g\supseteq B_{\mathfrak{m}}$ holds by the fact that $\sqrt{(A\colon z/x^t)}=\mathfrak{m}$ for any positive integer t. By the same proof as those of a) and c) of Prop. 3 we see that $(A_{\mathfrak{m}})^g=(B_{\mathfrak{m}})^g$ because $B_{\mathfrak{m}}$ is integral over $A_{\mathfrak{m}}$. Every regular element of $\mathfrak{n}$ is of the form $f=x^nv(x)+yc+(z/x^r)d$, where $v(x)\in k[x]$ such that $v(0)\neq 0$ and $c, d\in B$. Since $z/x^m=(z/x^{n+m})(x^nv(x)+yc+(z/x^r)d)(1/v(x))$ holds for every positive integer m in $B_{\mathfrak{m}}$, $B_{\mathfrak{m}}/(fB_{\mathfrak{m}})\simeq(k[X,$

$Y]/(F))_{(X,Y)}$ for some element F of $k[X, Y]$. Therefore $\mathfrak{n}B_{\mathfrak{m}}$ is not of the form $(uB_{\mathfrak{m}}: a)$, where $u \in B_{\mathfrak{m}} - z(B_{\mathfrak{m}})$ and $a \in B_{\mathfrak{m}}$; this implies that $(B_{\mathfrak{m}})^g = B_{\mathfrak{m}}$ by the above Prop.. Hence $(A_{\mathfrak{m}})^g = B_{\mathfrak{m}}$. Thus $(A_{\mathfrak{m}})^g$ is not noetherian.

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