is a real number A such that $\sigma(J^{i\gamma})$ lies in the annulus $A \leqslant |\lambda| \leqslant \exp(\pi|\gamma|/2)$. Since $\sum_{n=0}^{\infty} \lambda^{-n-1} J^{in\gamma}$ converges to $(\lambda - J^{i\gamma})^{-1}$ on $|\lambda| > \exp(\pi|\gamma|/2)$ and since $-\sum_{n=0}^{\infty} \lambda^n J^{-i(n+1)\gamma}$ converges to $(\lambda - J^{i\gamma})^{-1}$ in $|\lambda| < \exp(-\pi|\gamma|/2)$, we have the desired results.

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Some remarks on the Gurarij space

by

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Abstract. Complementably universal properties of the Gurarij space of universal disposition are proved. Some linearly isomorphic equivalences between Banach spaces whose duals are L_1 spaces are stated.

A predual of L_1 is a Banach space X such that X^* is linearly isometric to $L_1(\mu)$ for some measure μ.

DEFINITION. A separable space X is a *space of universal disposition* iff for every finite dimensional Banach spaces $F \supset E$ and every isomorphism $T\colon E \to X$ and every $\varepsilon > 0$ there is an isomorphism $\tilde{T}\colon F \to X$ such that $\tilde{T}|E = T$ and $\|\tilde{T}\| \cdot \|\tilde{T}^{-1}\| \leqslant (1+\varepsilon)\|T\|\|T^{-1}\|$.

Such a space was first constructed by Gurarij [1] and next by Lazar and Lindenstrauss [3].

In this note we prove the following

THEOREM. *Let X be a separable predual of L_1. Then there exists a Banach space of universal disposition Γ_X, $\Gamma_X \supset X$ and there is a projection of norm one from Γ_X onto X.*

The proof of this Theorem is a slight modification of Gurarij's proof [1].

By [5], Theorem 4.2 there exists a Banach space Y such that:

(*) Y is a separable predual of L_1 and for any separable predual of L_1, say X, and any $\varepsilon > 0$ there exist an embedding $T\colon X \to Y$, $\|T\|\|T^{-1}\| \leqslant 1+\varepsilon$ and a projection of norm one from Y onto $T(X)$.

By [4] Remark c after Theorem 4 there exists a separable predual of L_1, say W, such that any separable predual of L_1 is a quotient space of W.

If we apply the above Theorem for $X = Y$ or $X = W$ we obtain

COROLLARY 1. *The spaces Y and W can be choosen to be of universal disposition.*

COROLLARY 2. *Every space which satisfies (*) is isomorphic to every space of universal disposition.*

The result follows from the isomorphic uniqueness of spaces satisfying (*) (cf. [5]) and the following fact due to Gurarij [1].

For any two spaces of universal disposition Γ_1, Γ_2 and any $\varepsilon > 0$ there exists an isomorphism $U(\varepsilon)\colon \Gamma_1 \xrightarrow[\text{onto}]{} \Gamma_2$ such that $\|U(\varepsilon)\|\|U(\varepsilon)^{-1}\| \leqslant 1+\varepsilon$.

In connection with this we have

COROLLARY 3. *The following alternative holds:*

1) *either there are two non-isometric spaces of universal disposition;*

2) *or there exists a space Y_0 which satisfies (*) for $\varepsilon = 0$.*

Proof. Suppose that 2) does not hold. Take Y satisfying (*) and consider Γ_Y. Then there exists a separable predual of L_1, X, such that X is not isometric to any subspace X_0 of Γ_Y so that there is a projection of norm one from Γ_Y onto X_0. Then the spaces Γ_Y and Γ_X are two non-isometric spaces of universal disposition.

Now we pass to the proof of the Theorem. The following lemmas are well-known.

LEMMA 1 [1]. *The following metric spaces are compact:*

a) *$\mathscr{G}(B)$ the set of all subspaces of a finite dimensional Banach space B, equipped with the metric*

$$\mathscr{L}(P_1, P_2) = \max(\sup\{\operatorname{dist}(x, S_{P_1})\colon x \in S_{P_2}\}, \sup\{\operatorname{dist}(x, S_{P_2})\colon x \in S_{P_1}\}),$$

where $S_X = \{x \in X\colon \|x\| = 1\}$.

b) *$\mathscr{R}(k, n)$, $n \geqslant k$, the set of all pairs of Banach spaces (P, R), $P \subset R$, $\dim P = k$, $\dim R = n$, equipped with the metric*

$$\mathscr{L}((P_1, R_1), (P_1, R_2)) = \ln\inf\|T\|\|T^{-1}\|,$$

where the inf *is taken over all isomorphisms $T\colon R_1 \to R_2$ such that $T(P_1) = P_2$.*

c) *$\mathscr{F}(B_1, B_2, c)$ the set of all isomorphisms from the finite dimensional Banach space B_1 onto the Banach space B_2 such that $\|T\|\|T^{-1}\| \leqslant c$ equipped with the metric*

$$\mathscr{N}(T_1, T_2) = \max(\|T_1 - T_2\|, \|T_1^{-1} - T_2^{-1}\|).$$

LEMMA 2 [1]. *Let (P, R), $(\tilde{P}, \tilde{R}) \in \mathscr{R}(k, n)$ and let T be an isomorphism from R onto $\tilde{R}$ such that $U = T|P\colon P \xrightarrow[\text{onto}]{} \tilde{P}$. Then for any $\tilde{U}\colon P \xrightarrow[\text{onto}]{} \tilde{P}$ there exists an isomorphism $\tilde{T}\colon R \xrightarrow[\text{onto}]{} \tilde{R}$ such that $\tilde{T}|P = \tilde{U}$ and $\mathscr{N}(T, \tilde{T}) \leqslant k\mathscr{N}(U, \tilde{U})$.*

DEFINITION. A finite dimensional subspace E of a Banach space X is called a subspace of *a-universal disposition* iff for any pair of finite dimensional Banach spaces $P \subset R$ and isomorphism $T\colon P \xrightarrow[\text{onto}]{} E$ there exists an isomorphism $\tilde{T}\colon R \to X$ such that $\tilde{T}|P = T$ and $\|\tilde{T}\|\|\tilde{T}^{-1}\| \leqslant (1+a)\|T\|\|T^{-1}\|$.

LEMMA 3 [1]. *Let E be a subspace of a-universal disposition in X. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \dim P) > 0$ such that any $\tilde{P} \subset X$, $\mathscr{L}(P, \tilde{P}) < \delta$ is a-subspace of $(a+\varepsilon)$-universal disposition.*

LEMMA 4 [1]. *Let Banach spaces $P \subset E$, $\tilde{P} \subset \tilde{E}$ and an isomorphism $T\colon P \to \tilde{P}$ be given. Then there exists a Banach space $B \supset \tilde{E}$, $\dim B \leqslant \dim \tilde{E} + \dim E/P$ and an isomorphic embedding $\tilde{T}\colon E \to B$ such that $\tilde{T}|P = T$, $\|\tilde{T}\| = \|T\|$ and $\|\tilde{T}^{-1}\| = \|T^{-1}\|$.*

The following lemma is an improvement of Lemma 4. It is an obvious reformulation of Lemma 3.3 of [5].

LEMMA 5. *Let Banach spaces $P \subset E$, $\tilde{P} \subset \tilde{E}$, projections $Q\colon E \to P$, $\tilde{Q}\colon \tilde{E} \to \tilde{P}$ and an isomorphism $T\colon P \to \tilde{P}$ be given. Then there exists a Banach space B, $B \supset \tilde{E}$, $\dim B \leqslant \dim \tilde{E} + \dim E/P$ an isomorphic embedding $U\colon E \to B$, $U|P = T$, $\|U\| = \|T\|$, $\|T^{-1}\| = \|U^{-1}\|$ and projections $S\colon B \to T(E)$, $\tilde{S}\colon B \to \tilde{E}$,* $\ker S = \ker \tilde{Q}$, $\ker \tilde{S} = U(\ker Q)$, $\|S\| = \|Q\|$, $\|\tilde{S}\| = \|\tilde{Q}\|$.

Proof of the Theorem. By [2] we can choose a sequence of finite dimensional subspaces $X_1 \subset X_2 \subset X_3 \subset \ldots \subset X$ such that $\overline{\bigcup_n X_n} = X$ and X_n is isometric to l_n^∞. Let us choose two sequences of positive numbers $\varepsilon_n \to 0$ and $a_n \to \infty$. Consider sets $\mathscr{R}_i = \bigcup_{n=1}^{i+1} \bigcup_{k=1}^{n-1} \mathscr{R}(k, n)$. They are compact metric spaces.

We construct a sequence of finite dimensional spaces (B_n), $n = 1, 2, \ldots$, satisfying the following conditions:

(i) $B_n \subset B_{n+1}$ with $X_n \subset B_n$ for $n = 1, 2, \ldots$,

(ii) there are projections of norm one $\pi_n\colon B_n \to X_n$, $n = 1, 2, \ldots$,

(iii) $\pi_{n+1}|B_n = \pi_n$ for $n = 1, 2, \ldots$,

(iv) for any $(P, R) \in \mathscr{R}_n$ and any isomorphic embedding $T\colon P \to B_n$ with $\|T\|\|T^{-1}\| \leqslant a_n$ there exists $\tilde{T}\colon R \to B_{n+1}$, $\|\tilde{T}\|\|\tilde{T}^{-1}\| \leqslant (1+\varepsilon_n)\|T\|\|T^{-1}\|$ and $\tilde{T}|P = T$.

The space $B = \overline{\bigcup_n B_n}$ has the desired properties. Obviously there is a projection of norm one from B onto X. To check that B is of universal disposition consider a pair of finite dimensional spaces $E \subset F$ and any positive number ε, and an embedding $T\colon E \to B$. We can choose n in such the way that $a_n > \|T\|\|T^{-1}\|$, $\varepsilon_n < \varepsilon/2$ and there is a subspace $\tilde{B} \subset B_n$ such that $\mathscr{L}(\tilde{B}, T(E)) \leqslant \delta(\varepsilon/2, \dim E)$ (cf. Lemma 3). By Lemma 3 $T(E)$ is a subspace of ε-universal disposition and our statement is proved.

Construction of spaces (B_n).

We set B_1 equal to the one dimensional space. Suppose we have constructed $B_1, \ldots, B_n$. Consider an $\frac{1}{n}\varepsilon_n$-net $(P_i, R_i)_{i=1}^{n_1}$ in $\mathscr{R}_n$ such that $(P_{n_1}, R_{n_1}) = (X_n, X_{n+1})$. Let $(E_i)_{i=1}^{n_2}$ be an $\frac{1}{n}\varepsilon_2$-net in $\mathscr{G}(B_n)$ and $E_{n_2} = X_n$. Let $(\varphi_k^{(i,j)})_{k=1}^{n_3(i,j)}$ be an $\frac{1}{n}\varepsilon_n$-net in $\mathscr{F}(P_i, E_j, a_n)$ and let $\varphi_{n_3(n_1,n_2)}^{(n_1,n_2)}$ be the identity map on X_n. We apply Lemma 4 for spaces P_i, E_j and isomorphisms $\varphi_k^{(i,j)}$ except for P_{n_1}, E_{n_2} and $\varphi_{n_3(n_1,n_2)}^{(n_1,n_2)}$. Thus we obtain the space $\tilde{B} \supset B_n$. Since $B_n \supset X_n$ and X_n is isometric to l_n^∞ the projection $\pi_n: B_n \to X_n$, $\|\pi_n\| = 1$ can be extended to a projection $\tilde{\pi}: \tilde{B} \to X_n$ of norm one. Thus we apply Lemma 5 to obtain the space B_{n+1} which contains X_{n+1} and there is a projection π_{n+1} of norm one from B_{n+1} onto X_{n+1} and $\pi_{n+1}|B_n = \pi_n$. The space B_{n+1} satisfies (iv) in view of Lemma 2. This completes the proof.

Remark. By the same method one may establish the following statement:

For any finite set of separable preduals of L_1, $X_1, \ldots, X_k$ there exists a space of universal disposition $\Gamma_{X_1,\ldots,X_k}$ such that $X_i \subset \Gamma_{X_1,\ldots,X_k}$, $i = 1, 2, \ldots, k$, and there are projections of norm one from $\Gamma_{X_1,\ldots,X_k}$ onto X_i for $i = 1, 2, \ldots, k$.

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Construction of an orthonormal basis in $C^m(I^d)$ and $W_p^m(I^d)$

by

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Abstract. The space $C^m(I^d)$ is equipped in the natural scalar product induced from $L_2(I^d)$. A special orthonormal set of functions in $C^m(I^d)$ is constructed. This set of functions turns out to be a basis for the Banach spaces $C^m(I^d)$ and $W_p^m(I^d)$.

1. Introduction. The sequence $(x_n, n = 1, 2, \ldots)$ of elements of a given Banach space X is called a *basis* whenever each $x \in X$ has unique expansion

$$x = \sum_{n=1}^{\infty} a_n x_n$$

convergent in the norm. It is known that the coefficients $a_n = a_n(x)$ are linear functionals over X.

In this paper we shall deal mainly with the following two real Banach spaces:

The space $C^m(I^d)$, $m \geqslant 0$, $d \geqslant 1$, of m times continuously differentiable functions on I^d, $I = \langle 0, 1 \rangle$, with the norm

$$\|f\|^{(m)} = \max_{|\boldsymbol{k}| \leqslant m} \max_{t \in I} |D^{\boldsymbol{k}} f(t)|,$$

where $\boldsymbol{k} = (k_1, \ldots, k_d)$, k_j, and $1 \leqslant f \leqslant d$, being non-negative integers, $|\boldsymbol{k}| = k_1 + \ldots + k_d$ and $D^{\boldsymbol{k}}$ is the differential operator corresponding to $\boldsymbol{k}$, i.e.

$$D^{\boldsymbol{k}} = \frac{\partial^{|\boldsymbol{k}|}}{\partial t_1^{k_1} \ldots \partial t_p^{k_d}}.$$

The Sobolev space $W_p^m(I^d)$ with $m \geqslant 0$, $d \geqslant 1$ and $1 \leqslant p < \infty$, which is the set of all $f \in L_p(I^d)$ such that the generalized derivatives $D^{\boldsymbol{k}} f$ are functions and belong to $L_p(I^d)$ for each $\boldsymbol{k}$, $|\boldsymbol{k}| \leqslant m$. The norm is defined as follows

$$\|f\|_p^{(m)} = \Big(\sum_{|\boldsymbol{k}| \leqslant m} \|D^{\boldsymbol{k}} f\|_p\Big)^{1/p},$$

where $\| \ \|_p$ is the usual L_p-norm over I^d.