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SOME REMARKS ON THE STABILITY OF SIGN CHANGING SOLUTIONS

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Abstract. In this paper, we study the stability of changing sign solutions of weakly nonlinear second order elliptic equations. Here by stability, we means stability for the natural corresponding parabolic problem. We prove the instability of many sign changing solutions. On the other hand, we find a number of methods for obtaining stable changing sign solutions. Some of these methods involve singular perturbations.

1. Introduction. In this paper, we study the stability properties of the changing sign solutions of the following problems:

$$-\Delta u=h(u) \text{ in } D, \quad u=0 \text{ on } \partial D, \tag{1}$$

$$-\Delta u=h(u) \text{ in } D, \quad \frac{\partial u}{\partial n}=0 \text{ on } \partial D, \tag{2}$$

$$-\varepsilon^2\Delta u=h(u) \text{ in } D, \quad u=0 \text{ on } \partial D, \tag{3}$$

$$-\varepsilon^2\Delta u=h(u) \text{ in } D, \quad \frac{\partial u}{\partial n}=0 \text{ on } \partial D, \tag{4}$$

where D is a bounded domain in $\boldsymbol{R}^n$ $(n\geq 2)$ with regular boundary ∂D, $\varepsilon>0$ and h: $\boldsymbol{R}^1\to\boldsymbol{R}^1$ is defined by

$$h(u)=\begin{cases} au-\alpha u^2 & \text{if } u\geq 0 \\ du+u^2 & \text{if } u\leq 0\,. \end{cases}$$

Here $\alpha>0$.

Problem (1) comes as a limiting problem of the following competition species problem

$$\begin{aligned} -\Delta v&=av-v^2-cvw \\ -\Delta w&=dw-w^2-evw \quad \text{in } D \\ v&=w=0 \quad \text{on } \partial D\,, \end{aligned} \tag{5}$$

when both interaction parameters c and e go to infinity and $c/e\to\alpha$ as c, $e\to+\infty$. We have shown in [19] that if (1) has a nondegenerate solution u_0 which changes sign

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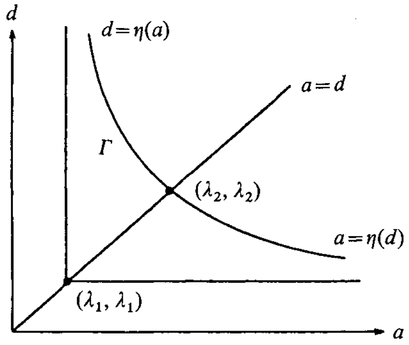

FIGURE 1.

on D, (5) has a unique positive solution (v, w) which is close to $(\alpha u_0^+, -u_0^-)$ (for c, e large and c/e close to α). Moreover (v, w) is stable if and only if u_0 is stable. Here we denote $u^+=\max\{u, 0\}$ and $u^-=\min\{u, 0\}$. Therefore, it is important to study the stability properties of the changing sign solutions of (1). By stability we mean stability as solutions of the natural corresponding parabolic equations. As we mentioned in the introduction of [19], there are analogous results for Neumann boundary conditions.

In a recent paper, Dancer and Du [17] showed that if $a, d>\lambda_2$, where λ_2 is the second eigenvalue of $-\Delta$ under the Dirichlet boundary condition, (1) has at least one changing sign solution. In a more recent paper [18], they found the exact domain for (a, d) on which (1) has at least one changing sign solution. They showed that there exists a curve Γ (cf. Figure 1) which was actually given in [8] such that (1) has a changing sign solution if and only if (a, d) is above Γ in the ad-plane.

In the present paper, we are mainly interested in the stability properties of the solutions of (1) which change sign on D. We obtain many conditions for changing sign solutions to be unstable and a number of methods of constructing stable changing sign solutions in various parameter ranges. Many of our results hold for the solutions of (1) with a much more general nonlinear term $f(u)$. Note that the stability properties of the non-constant solutions of (1) and (2) are quite different. It is well-known that in many cases, the non-constant solutions of the problem (2) are unstable. For example, Casten and Holland [6] showed that if D is a convex subset of $\boldsymbol{R}^n$, any non-constant solution of class $C^3(\bar{D})$ of (2) is unstable. It can be shown that this is not true for the problem with Dirichlet boundary condition (cf. [40]). It is also known that this is false for the case h is allowed to have explicit spatial dependence, $h=h(x, u)$ (cf. [2], [3], [29]).

We also treat the problems (3) and (4) with ε sufficiently small. Note that (3) is a special case of (1) (after a rescaling). It is well-known that the weakly stable non-constant solutions of (3), (4) correspond to the non-constant local minimizers of the functional

$$J_\varepsilon(u)=\frac{\varepsilon}{2}\int_D |\nabla u|^2dx-\frac{1}{\varepsilon}\int_D H(u)dx$$

in a suitable space. We sketch a proof of this folklore result at the end of Section 3. Therefore, our main interest is to look for the non-constant local minimizers of the functional $J_\varepsilon(u)$ in the required spaces. Such problems has been studied by many authors (cf. [5], [21], [23], [25], [26], [30], [33], [35]–[37], [39] and the references therein). It seems that Matano has some other methods for constructing local minima for some other nonlinearities but no details seem to be available.

In Section 2 we study stability properties of the changing sign solutions of (1). We find that in many cases, the changing sign solutions of (1) are unstable. In Section 3 we use domain variation techniques to construct stable changing sign solutions for problem (1). We also construct stable changing sign solutions for the problem (1) on a convex domain D. In Section 4 we study the existence of the weakly stable solutions of the problems (3) and (4) and the asymptotic behaviours of such solutions. This reduces to discussing the local minimizers of a functional involving singular perturbations. In the final section, we very briefly discuss the local minimizers of a functional such as in Section 4 with a small perturbation on the nonlinear term.

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2. Instability results for problem (1). In this section we mainly study the stability properties of the changing sign solutions of (1). We also treat the problem with a more general nonlinear term $f(u)$. Let $C_0^1(\bar{D})=\{u\in C^1(\bar{D}): u=0 \text{ on } \partial D\}$ and $W_0^{2,p}(D)=W^{2,p}(D)\cap W_0^{1,2}(D)$. By a solution, we always mean a weak solution. Let λ_1 denote the first eigenvalue of $-\Delta$ under Dirichlet boundary conditions.

THEOREM 2.1. *Suppose that $a, d>\lambda_1$ and $u_\alpha\in C_0^1(\bar{D})$ is a solution of the problem*

$$-\Delta u=h(u) \quad \text{in } D, \qquad u=0 \quad \text{on } \partial D \tag{6}$$

which changes sign on D. Then u_α is an unstable solution of (6) (*for the natural corresponding parabolic equation*) *for α sufficiently small.*

PROOF. We divided the proof into three steps.

Step 1. We prove that $\alpha u_\alpha\to 0$ in $C^1(\bar{D})$ as $\alpha\to 0$.

By an easy upper and lower solution argument, we know that for any $\alpha>0$,

$$-d\le u_\alpha\le a\alpha^{-1}. \tag{7}$$

Therefore,

$$-d\alpha\le \alpha u_\alpha\le a. \tag{8}$$

(8) implies that for any sequence $\{\alpha_n\}$ with $\alpha_n\to 0$, $\{\|\alpha_n u_n\|_\infty\}$ is uniformly bounded and

$(\alpha_n u_n)^- \to 0$ as $n \to \infty$. Here $u_n = u_{\alpha_n}$ and $u_n^- = \min\{u_n, 0\}$. Therefore, $\{\|\alpha_n u_n^+(a-\alpha_n u_n^+) + \alpha_n u_n^-(d+u_n^-)\|_\infty\}$ is uniformly bounded. Using the regularity theory for $-\Delta$, we have that $\{\|\alpha_n u_n\|_{2,p}\}$ is uniformly bounded for any $p>n$. The compactness of the embedding of $W_0^{2,p}(D)$ to $C_0^1(\bar{D})$ implies that $\alpha_n u_n \to u_0$ in $C_0^1(\bar{D})$. Moreover, $u_0 \geq 0$ in D. (This follows from (8).) For any $\phi \in C_0^2(D)$, we multiply (6) by ϕ and integrate by parts. We have that

$$(9) \qquad (\alpha_n u_n, -\Delta\phi) = \int_D [\alpha_n u_n^+(a-\alpha_n u_n^+)\phi + \alpha_n u_n^-(d+u_n^-)\phi]dx\,.$$

Passing to the limit as $n \to \infty$, we obtain that

$$(10) \qquad (u_0, -\Delta\phi) = \int_D u_0(a-u_0)\phi dx\,.$$

Here we are using that $\alpha_n u_n^- \to 0$ as $n \to \infty$. Since u_0 satisfies

$$(11) \qquad -\Delta u = u(a-u) \quad \text{in } D\,, \qquad u=0 \quad \text{on } \partial D$$

and since $a > \lambda_1$, by (ii) of Lemma 1 in [11], we see that $u_0 \equiv 0$ or $u_0 \equiv \phi_a$, where ϕ_a is the unique positive solution of (11). Now we conclude that $u_0 \equiv 0$. Suppose $u_0 \equiv \phi_a$. Let K be the natural cone of $C_0^1(\bar{D})$. It follows from the maximum principle that $u_0 \in \mathring{K}$. Then, there exists a neighborhood $B_\delta(u_0) \subset K$ of u_0 such that $u > 0$ in D for any $u \in B_\delta(u_0)$. Since $\alpha_n u_n$ changes sign on D, then for any n large, $\alpha_n u_n \notin B_\delta(u_0)$. This contradicts that $\alpha_n u_n \to u_0$ in $C_0^1(\bar{D})$. Hence, $u_0 \equiv 0$.

Step 2. We prove that $\{\|u_\alpha\|_\infty\}$ is uniformly bounded for sufficiently small α.

Suppose that there is a sequence $\{\alpha_n\}$ satisfying $\alpha_n \to 0$ as $n \to \infty$ and $\{u_n\} \equiv \{u_{\alpha_n}\}$ satisfying $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Then $w_n = u_n / \|u_n\|_\infty$ satisfies

$$(12) \qquad -\Delta w_n = w_n^+(a-\alpha_n u_n^+) + w_n^-(d+u_n^-) \quad \text{in } D\,, \qquad w_n = 0 \quad \text{on } \partial D$$

and $\|w_n\|_\infty = 1$. Since $\{\|u_n^-\|_\infty\}$ is uniformly bounded, then $w_n^- \to 0$ in $L^\infty(D)$ as $n \to \infty$. By an argument similar to that in Step 1, we have that $w_n \to \bar{w}$ in $C_0^1(\bar{D})$ where $\bar{w} \geq 0$ in D, $\bar{w} \not\equiv 0$ in D. By Step 1, we also know that $\alpha_n u_n^+ \to 0$ in D as $n \to \infty$. Then, $\bar{w}$ satisfies

$$-\Delta w = aw \quad \text{in } D\,, \qquad w = 0 \quad \text{on } \partial D\,.$$

This is a contradiction, since $a > \lambda_1$ and $\bar{w} \geq 0$ and $\bar{w} \not\equiv 0$ in D.

Step 3. We prove that u_α is an unstable solution of (6) (for the natural corresponding parabolic equation) for α sufficiently small.

The proof of the instability reduces to showing that the linearized equation

$$(13) \qquad \begin{aligned} &-\Delta k = [(a-2\alpha_n u_\alpha^+)\,\mathrm{sgn}^+ u_\alpha + (d+2u_\alpha^-)\,\mathrm{sgn}^- u_\alpha]k + \lambda k \quad \text{in } D \\ &k = 0 \quad \text{on } \partial D \end{aligned}$$

has eigenvalues in $\hat{T} = \{\lambda \in C\colon \mathrm{Re}\,\lambda < 0\}$. Here

$$\mathrm{sgn}^+ u=\begin{cases}1 & \text{if } u>0\\ 0 & \text{if } u\le 0\,;\end{cases}\qquad \mathrm{sgn}^- u=\begin{cases}1 & \text{if } u<0\\ 0 & \text{if } u\ge 0\,.\end{cases}$$

Suppose that there exists a sequence $\{\alpha_n\}$ satisfying $\alpha_n\to 0$ as $n\to\infty$ such that the principal eigenvalue $\tilde{\lambda}_n$ of the problem (13) for $\alpha=\alpha_n$ satisfy $\tilde{\lambda}_n\ge 0$. Let u_n denote u_{α_n}. We know from Step 2 that $\{\|u_n\|_\infty\}$ is uniformly bounded. Hence, $\{\|h(u_n)\|_\infty\}$ is uniformly bounded. By the same arguments as those in the proof of Step 1, we know that there exists $\tilde{u}\in C_0^1(\bar{D})$ such that $u_n\to\tilde{u}$ in $C_0^1(\bar{D})$. We discuss two possibilities: $\tilde{u}\equiv 0$ and $\tilde{u}$ changes sign on D. It follows by an idea similar to that at the end of Step 1 that there are only these two possibilities.

If $\tilde{u}$ changes sign in D, then the linear operator in (13) for $\alpha=\alpha_n$ is a small perturbation (in the operator norm) of that for the problem

$$-\Delta k=[a\,\mathrm{sgn}^+\tilde{u}+d\,\mathrm{sgn}^-\tilde{u}+2\tilde{u}^-]k+\lambda k \quad \text{in } D\,, \qquad k=0 \quad \text{on } \partial D \tag{14}$$

for n large. Therefore, for n large, $\tilde{\lambda}_n$ is near the principal eigenvalue $\tilde{\lambda}$ of (14) which is negative (cf. Lemma 2.6 in [19]). Hence, $\tilde{\lambda}_n<0$ for n large. This is a contradiction.

If $\tilde{u}\equiv 0$ in D, let $\tilde{k}_n$ be the eigenfunction corresponding to $\tilde{\lambda}_n$ with $\|\tilde{k}_n\|_\infty=1$. Since u_n satisfies

$$-\Delta u_n=[(a-\alpha_n u_n^+)\,\mathrm{sgn}^+u_n+(d+u_n^-)\,\mathrm{sgn}^-u_n]u_n \quad \text{in } D\,, \qquad u_n=0 \quad \text{on } \partial D \tag{15}$$

and since the term in the bracket on the right hand side of (15) belongs to $L^\infty(D)$, it follows from Caffarelli and Friedman [4] that $\mathrm{meas}\{x\in D\colon u_n=0\}=0$ for all n. Therefore, $\mathrm{sgn}^+u_n+\mathrm{sgn}^-u_n=1$ a.e. in D for all n. This implies that there exists $\theta>0$ such that

$$-\Delta\tilde{k}_n\ge(\lambda_1+\theta)\tilde{k}_n \quad \text{a.e. in } D \tag{16}$$

for n large enough. Here we use (13) and the facts that a, $d>\lambda_1$, $\tilde{\lambda}_n$, $\tilde{k}_n\ge 0$ and $u_n\to 0$ in $C_0^1(\bar{D})$. Let ψ be a positive eigenfunction corresponding to λ_1. It follows from (16) that

$$\lambda_1\int_D \tilde{k}_n\psi\,dx\ge(\lambda_1+\theta)\int_D \tilde{k}_n\psi\,dx\,. \tag{17}$$

Since $\psi(x)>0$ in D, (17) is a contradiction. This completes the proof.

Corollary 2.2. *Suppose that $u_\alpha\in C_0^1(\bar{D})$ is a solution of* (6) *which changes sign on D. Then u_α is unstable when α is sufficiently large.*

Proof. Let $v=-\alpha u$. Then (6) becomes

$$-\Delta v=v^+(d-\alpha^{-1}v^+)+v^-(a+v^-) \quad \text{in } D\,, \qquad v=0 \quad \text{on } \partial D\,. \tag{18}$$

Therefore, if u_α is a solution of (6) for α sufficiently large, $v_\alpha=-\alpha u_\alpha$ is a solution of (18) with α^{-1} sufficiently small. Note that (18) has the same form as (6). Then using the same idea as in the proof of Theorem 2.1, we have that v_α is an unstable solution of (18) when α is sufficiently large. Since it is easy to check that the linearizations of (18) at v_α and (6) at u_α have the same principal eigenvalue, the result follows.

As mentioned before, there exists a curve Γ in *ad*-plane such that (6) has changing sign solutions if and only if (a, d) is above Γ. Now, we shall obtain the following result.

THEOREM 2.3. *Suppose* $\alpha>0$. *There exists a strip* Γ' *above* Γ *which is a one-sided neighbourhood of* Γ *in* $\mathbf{R}^2$ *such that for* $(a, d)\in\Gamma'$ *any changing sign solution of* (6) *is unstable.*

PROOF. We use a contradiction argument to prove this theorem. Suppose there exist $\{(a_n, d_n)\}$ such that (a_n, d_n) is above Γ, $a_n\to\hat{a}$, $d_n\to\hat{d}$ as $n\to\infty$, $(\hat{a}, \hat{d})\in\Gamma$ and $\{u_n\}\equiv\{u_{a_n,d_n}\}$ is a sequence of changing sign solutions of (6) for $a=a_n$, $d=d_n$. Suppose also the principal eigenvalue $\tilde{\lambda}_n$ of the problem

$$(19)\qquad \begin{aligned} &-\Delta k=[(a_n-2\alpha u_n^+)\,\mathrm{sgn}^+u_n+(d_n+2u_n^-)\,\mathrm{sgn}^-u_n]k+\lambda k \quad \text{in } D\\ &k=0 \quad \text{on } \partial D \end{aligned}$$

satisfies $\tilde{\lambda}_n\geq 0$. It follows from [18] that $\|u_n\|_\infty\to 0$ as $n\to\infty$. Since $\hat{a}, \hat{d}>\lambda_1$, there exists $\theta_1>0$ such that $a_n, d_n\geq\lambda_1+\theta_1$ for all n sufficiently large. Thus the argument in the second case of the proof of Theorem 2.1 implies that $\tilde{\lambda}_n\geq 0$ is impossible when n is sufficiently large. This is a contradiction. This completes the proof.

By Remark 3 of [18], we know that this can be improved further if there is a compact group of linear symmetries G acting orthogonally on $\mathbf{R}^n$ such that D is invariant under G. We can prove an analogous theorem for the existence of solutions of (6) which change sign and are invariant under the action of the group G. Thus, we obtain a new curve Γ_1. An argument similar to that in the proof of Theorem 2.3 implies that there is a one sided neighbourhood Γ'' of Γ_1 such that any changing sign solution for $(a, d)\in\Gamma''$ which is invariant under the natural action of G is unstable. Note that we prove a little later that if G is connected the G-invariant solutions are the only solutions which can possibly be stable.

The next theorem implies that there is no changing sign stable radial solution for the problem (6) if D is a ball or an annulus in $\mathbf{R}^n$ $(n\geq 2)$.

THEOREM 2.4. *Assume that* $D\subset\mathbf{R}^n$ *is a bounded and smooth domain,* $f\in C^1(\mathbf{R})$, $\tilde{u}\in C_0^1(\bar{D})$ *is a solution of the problem*

$$(20)\qquad -\Delta u=f(u) \quad \text{in } D, \qquad u=0 \quad \text{on } \partial D$$

and $\tilde{u}$ *satisfies* $a\cdot\nabla\tilde{u}=0$ *on* T *except for a compact set* Z *of finite* $(n-2)$*-dimensional Hausdorff measure, where* a *is any constant vector in* $\mathbf{R}^n$ *and* T *is the boundary of a component of the set* $\{x\in D: a\cdot\nabla\tilde{u}\neq 0\}$. *Then* $\tilde{u}$ *is unstable.*

PROOF. Let $\tilde{h}=a\cdot\nabla\tilde{u}$. Then $\tilde{h}$ satisfies the linearized equation of (20) at $\tilde{u}$ but does not satisfy the boundary condition. Let D_1 be a connected open subset of D such that $\partial D_1 \subset T$. Then, we can argue as in Section 1 of [10], to deduce that

$$\int_{D_1} [|\nabla \tilde{h}|^2 - f'(\tilde{u})(\tilde{h})^2] dx = \int_{\partial D_1} \tilde{h} \frac{\partial \tilde{h}}{\partial n} .$$

Note that since T is not very regular, there is a technical difficulty here to justify the integration by parts. But, we can overcome this difficulty by the same argument as in the proof of Lemma 2 of [15] (supplemented by the lemma and the remarks after it in [16]).

Since $\partial D_1 \subseteq T$ and $\tilde{h} = 0$ on $T \setminus Z$, then $\tilde{h} = 0$ a.e. on ∂D_1. Here by a.e. we mean almost everywhere with respect to surface measure. Therefore,

$$\int_{D_1} [|\nabla \tilde{h}|^2 - f'(\tilde{u})(\tilde{h})^2] dx = 0 .$$

Now, we define

$$W(v) = \frac{1}{2} \int_D [|\nabla v|^2 - f'(\tilde{u}) v^2] dx , \qquad \text{for} \quad v \in W_0^{1,2}(D) ,$$

and

$$t(x) = \begin{cases} \tilde{h} & \text{if } x \in D_1 \\ 0 & \text{otherwise} . \end{cases}$$

Then, $W(t) = 0$ and $t \in \dot{W}_0^{1,2}(D)$. In fact, by the regularity property of the solutions of (20), we know that $\tilde{u} \in C^2(D)$. Thus $\tilde{h} \in C^1(D)$. Since $\tilde{h} = 0$ a.e. on ∂D_1, by Green's theorem and the same idea as in the proof of Lemma 3.22 in [1], we have that $t \in W_0^{1,2}(D)$.

It is well-known that the smallest eigenvalue $\tilde{\lambda}$ of the problem

$$(21) \qquad -\Delta h = f'(\tilde{u}) h + \lambda h \quad \text{in } D , \qquad h = 0 \quad \text{on } \partial D$$

is equal to $\inf\{W(v) : v \in W_0^{1,2}(D), \|v\|_2 = 1\}$ and the only minimizers are ± 1 times the first eigenfunction corresponding to $\tilde{\lambda}$ (and hence are non-zero on all of D). Hence, $\pm \mu t(x)$ cannot minimize our problem (where $\|\mu t\|_2 = 1$). Hence, the minimum is negative and thus $\tilde{\lambda} < 0$. Hnece $\tilde{u}$ is unstable.

REMARK. If f is Lipschitz continuous on $\boldsymbol{R}$, the result is still true (since $f(u) \in W^{1,2}$). Moreover, we can allow many domains with corners.

The following result was known to Lin and Ni [32] and Sweers [40]. We obtain it as a corollary of Theorem 2.4.

COROLLARY 2.5. *Assume D is a ball or an annulus in $\boldsymbol{R}^n$ $(n \geq 2)$ and $\tilde{u} \in C_0^1(\bar{D})$ is a radial solution of* (20) *which changes sign on D. Then $\tilde{u}$ is unstable.*

PROOF. We consider the case of a ball. The other case is similar. Now $\tilde{u}$ attains a positive maximum and a negative minimum on D and only one of them can be attained at $r=0$. Thus there exists at least one $r>0$ such that $\tilde{u}'(r)=0$. Let r_0 be the first such r, $T=\{r=0\}\cup\{r=r_0\}$ and $D_1=\{r: 0<r<r_0\}$. Theorem 2.4 implies that $\tilde{u}$ is unstable.

Using the same idea as in the proof of Theorem 2.4, we also obtain the following result.

THEOREM 2.6. *Assume that* $f\in C^1(\boldsymbol{R})$, $f(0)=0$, *and* $D\subset \boldsymbol{R}^n$ $(n\geq 2)$ *is a connected smooth domain which is invariant under reflection in the hyperplane* $x_1=0$ *and* $e_1\cdot\hat{n}>0$ *for* $x\in\partial D$ *with* $x_1>0$. *Here* $\hat{n}$ *is the outward normal vector at* x, e_1 *is the direction of* x_1. *If* $\tilde{u}\in C_0^1(\bar{D})$ *is a solution of* (20) *which is odd in* x_1 *and is positive for* $x\in D$ *with* $x_1>0$, *then* $\tilde{u}$ *is unstable.*

PROOF. Since $\tilde{u}>0$ for $x\in D$ with $x_1>0$ and $\tilde{u}$ is a solution of (20), by the strong maximum principle, $\partial\tilde{u}/\partial\hat{n}<0$ on ∂D_1, where $D_1=\{x\in D: x_1>0\}$. Thus, $\partial\tilde{u}/\partial x_1<0$ for $x\in\partial D$ and $x_1>0$ since $\hat{n}\cdot e_1>0$ for $x\in\partial D$ and $x_1>0$. Let $\tilde{h}(x)=\partial\tilde{u}/\partial x_1=(1,0,\dots,0)\cdot\nabla\tilde{u}$. Then $\tilde{h}(x)$ satisfies the equation

$$\Delta h+f'(\tilde{u}(x))h=0 \quad \text{in } D \tag{22}$$

and $\tilde{h}(x)<0$ on $\partial D_1\cap\{x_1>0\}$. The conditions on $\tilde{u}$ imply that $\tilde{h}(x)>0$ on $(\partial D_1\cap\{x_1=0\})\setminus(\partial D\cap\{x_1=0\})$. Hence if we define D_2 to be the component of $\{x\in D: \tilde{h}(x)>0\}$ containing $D\cap\{x\in\boldsymbol{R}^n: x_1=0\}$, then $\partial D_2\cap\partial D\subseteq\partial D\cap\{x_1=0\}$, (since $\tilde{h}<0$ close to any part of ∂D not in $x_1=0$ and $\tilde{h}$ is even in x_1). Since $\partial D\cap\{x_1=0\}$ has finite $(n-2)$-dimensional Hausdorff measure, the result follows from Theorem 2.4.

Now, we give a simple way of obtaining changing sign solutions which are unstable solutions for domains of the type in Theorem 2.6.

THEOREM 2.7. *Assume that* f *is odd,* $f'(s)\leq f(s)/s$ *for* $s>0$, $f'(0)>0$, $f(s)>0$ *for* $0<s<\theta$ *and* $f(\theta)=0$. *Assume that* $D\subset\boldsymbol{R}^n$ $(n\geq 2)$ *is as in Theorem* 2.6. *Then there exists a* $\hat{\lambda}>0$ *such that when* $\lambda>\hat{\lambda}$, *then problem*

$$-\Delta u=\lambda f(u) \quad \text{in } D, \qquad u=0 \quad \text{on } \partial D \tag{23}$$

has a unique solution $\tilde{u}$, $-\theta\leq\tilde{u}\leq\theta$, *which is odd in* x_1 *and positive for* $x\in D$, $x_1>0$ *and* $\tilde{u}$ *is unstable.*

PROOF. Without loss of generality, we assume $f'(0)=1$. We first show that such solutions exist. Let $D_1=\{x\in D: x_1>0\}$, $D_2=\{x\in D: x_1<0\}$ and let $\hat{\lambda}$ denote the first eigenvalue for the Dirichlet problem on D_1. We consider the problem

$$-\Delta u=\lambda f(u) \quad \text{in } D_1, \qquad u=0 \quad \text{on } \partial D_1 . \tag{24}$$

By standard bifurcation theorems (cf. [13, Theorem 2]), a branch $\tilde{C}$ of positive solutions of (24) will branch off at $(0,\hat{\lambda})$ and $\tilde{C}$ is unbounded in $C_0(\bar{D}_1)\times[0,\infty)$. Hence $C_0(\bar{D}_1)$

denotes the set of continuous functions on D_1 vanishing on ∂D_1. By the maximum principle, $0<u_\lambda\leq\theta$ since $f(s)\leq 0$ if $s\geq\theta$ by the assumption on f. Thus, for any $\lambda>\hat{\lambda}$, there exists a positive solution of (24). Since $f'(s)\leq f(s)/s$ for $s>0$, there is only one such solution u_λ for any λ. (See [11, p. 739–740].) By the facts that $f(s)$ is odd and D is connected and invariant under reflection in the hyperplane $x_1=0$, we have that

$$\tilde{u}_\lambda=\begin{cases} u_\lambda & \text{in } D_1 \\ -u_\lambda & \text{in } D_2 \end{cases}$$

is a solution of (23) for $\lambda>\lambda_1$ and clearly $\tilde{u}_\lambda$ is odd in x_1 and positive for $x\in D_1$. For any $\lambda>\hat{\lambda}$, if there exists another solution $\hat{u}_\lambda$ of (23) which is odd in x_1 and positive for $x\in D_1$, then $\hat{u}_\lambda|_{D_1}$ is a positive solution of (24) and thus, $u_\lambda\equiv\hat{u}_\lambda|_{D_1}$. This implies $\tilde{u}_\lambda\equiv\hat{u}_\lambda$ in D. The instability of $\tilde{u}_\lambda$ can be obtained by Theorem 2.6. This completes the proof of Theorem 2.7.

REMARKS 1. The same arguments as in Theorems 2.6 and 2.7 work for a number of other cases if we look for solutions odd in both x_1 and x_2 and we assume $u>0$ if $x\in D$ with $x_1, x_2>0$ (with a suitable condition on the normal $\hat{n}$). Thus many of the simple ways of constructing changing sign solutions tend to yield unstable solutions.

2. Note that the conditions on the normal n in Theorems 2.6 and 2.7 are necessary. One can give counterexamples by domain variation arguments (even with the dimension n equal to two). For example, let D_m be smooth symmetric domains approximating $B_1\cup B_2$, where B_1, B_2 are disjoint balls with the same radius 1 such that ∂B_1 and ∂B_2 intersect at a single point. We easily see from Section 3 below that for large m there exists a stable solution for (6) on D_m approximating a function on $B_1\cup B_2$ which is positive in B_2 and negative in B_1. If we choose $a=d$ and $\alpha=1$ in $h(u)$, this stable solution is an odd function of x_1. (See Section 3 below.) With some care one can show the positivity condition of Theorems 2.6 and 2.7 is also true in this example.

3. The conclusions of Theorems 2.6 and 2.7 can be applied to the problem (6) when $a=d$ and $\alpha=1$ since if $\tilde{u}$ is a changing sign solution of (6), $\tilde{u}$ satisfies $-d\leq\tilde{u}\leq a\alpha^{-1}$. It is clear that $h(s)$ satisfies a condition similar to that of $f(s)$ in Theorems 2.6 and 2.7 in $(-d, a\alpha^{-1})$.

Now, we prove the following theorem which implies that the changing sign solutions of the problem (6) in a 2-dimensional connected smooth strictly convex domain are unstable if they have some special properties. The proof uses an unpublished idea of Matano (who used it for a slightly different situation).

THEOREM 2.8. *Assume that f is locally Lipschitz continuous and $f(0)=0$. Assume that $D\subset \mathbf{R}^2$ is a connected smooth strictly convex domain. If $\tilde{u}\in C_0^1(\bar{D})$ is a solution of*

$$-\Delta u=f(u) \quad \text{in } D\,, \qquad u=0 \quad \text{on } \partial D \tag{25}$$

which does not change sign near ∂D, but $\tilde{u}$ changes sign on D, then $\tilde{u}$ is unstable.

Proof. Let $a=(a_1, a_2)\in \boldsymbol{R}^2$, $\nabla\tilde{u}=(\partial\tilde{u}/\partial x_1, \partial\tilde{u}/\partial x_2)$. Then, $a\cdot\nabla\tilde{u}$ satisfies the equation

$$-\Delta h=f'(\tilde{u})h \quad \text{in } D.$$

Let $\tilde{h}=a\cdot\nabla\tilde{u}$. We know $\tilde{h}\in C^1(D)\cap C_0(\bar{D})$ (since $u\in C^2(D)$ by standard regularity theory). Moreover, there exist at least two points $\hat{x}_i\in D$ $(i=1, 2)$ such that $\tilde{h}(\hat{x}_i)=0$. (We know this from the fact that $\tilde{u}$ attains a positive maximum and a negative minimum in D.) Without loss of generality, we assume $\hat{x}_1=0$. (Otherwise, we use a linear transformation to shift $\hat{x}_1$ to 0.) Thus, $\tilde{h}(x)=o(|x|^m)$ as $|x|\to 0$ for some $m\geq 0$ and $m\in N_0$. If this holds for every integer m, then $\tilde{h}\equiv 0$ in D by unique continuation. This is impossible. Thus, by Hartman and Wintner [27], in a neighbourhood of $x=(0, 0)$, we have the asymptotic relations

$$\tilde{h}(x)=p_{m+1}(x)+o(|x|^{m+1}), \tag{26}$$

$$\nabla\tilde{h}(x)=\nabla p_{m+1}(x)+o(|x|^m), \tag{27}$$

as $|x|\to 0$. Here $p_{m+1}(x)$ is a nonvanishing, homogeneous, harmonic polynomials of degree $m+1$. Note that we assume a little less regularity than that in [27] but an examination of their proof shows that the result is still true under our assumptions. It is easy to show that for every such polynomial in $\boldsymbol{R}^2$, there is a constant $c>0$, such that $|\nabla p_{m+1}(x)|\geq c|x|^m$ on $\boldsymbol{R}^2$. It then follows easily (cf. Pagani-Masciadri [38]) that there is an open neighbourhood V of $\hat{x}_1=0$ such that $\{\tilde{h}=0\}\cap V$ consists of Jordan arcs γ_j, $j=1, 2, \ldots, 2k+2$, which, emanating from $\hat{x}_1=0$, locally divide V into $2k+2$ disjoint subdomains Ω_j such that $\tilde{h}>0$ in Ω_j, $j=1, 3, \ldots, 2k+1$ and $\tilde{h}<0$ elsewhere in $\{x\in V: \tilde{h}(x)\neq 0\}$.

We consider two possibilities here:

(i) each of the subdomain Ω_j can be extended to the boundary ∂D;

(ii) there exists a subdomain which cannot be extended to the boundary.

If the second case occurs, we easily prove that $\tilde{u}$ is unstable by the same arguments as in the proof of Theorem 2.4.

Now we only consider the first case. If $k\geq 1$, there are at least four subdomains which can be extended to the boundary of D. This implies that there are at least four Jordan arcs which emanate from $\hat{x}_1=0$ to the boundary of D.

On the other hand, $\nabla\tilde{u}$ is prarallel to n on ∂D and $\nabla\tilde{u}\cdot n\neq 0$ on ∂D, where n is the outward normal at $x\in\partial D$. (This follows from the maximum principle since $\tilde{u}$ does not change sign near ∂D and $f(0)=0$.) Hence $a\cdot\nabla\tilde{u}$ can only be 0 on ∂D at the two points where $a\cdot n=0$. (That there are only two points follows from the strict convexity of D.) Thus, if $k\geq 1$, we can find at least one closed Jordan curve in D which connects $\hat{x}=0$ and one (or two) of the two points on ∂D where $a\cdot n=0$ such that $a\cdot\nabla\tilde{u}=0$ on it. Then, Theorem 2.4 implies that $\tilde{u}$ is unstable.

The argument shows that we have finished the proof unless $k=0$ and (i) holds for every Ω_j. If $k=0$ and (i) holds for every Ω_j, it is easy to see that the zero set of $a\cdot\nabla\tilde{u}$ is a single Jordan curve joining the two points on ∂D where $a\cdot n=0$. If there exists $a_0\in \boldsymbol{R}^2$ with $|a_0|=1$ such that the zero set of $a_0\cdot\nabla\tilde{u}$ is not a single Jordan curve Γ_{a_0} joining the two points on ∂D where $a_0\cdot n=0$, then by the arguments above, we conclude that $\tilde{u}$ is unstable. Now, we assume that for each $a\in\boldsymbol{R}^2$ with $|a|=1$, the zero set of $a\cdot\nabla\tilde{u}$ is a single Jordan curve Γ_a joining the two points c_1^a, c_2^a on ∂D wehre $a\cdot n=0$. The curve will change continuously with a. As a varies around the circle $|a|=1$, the two end points will move around ∂D. Moreover, all the curves must pass through at least two fixed points $p_1, p_2\in D$ (the maximum point and the minimum point of $\tilde{u}$). We shall prove that this is impossible. In fact, we see that the order of the four points c_1^a, p_1, p_2, c_2^a on Γ_a will be unchanged as we vary a continuously around the circle $|a|=1$. We also know that c_1^a, c_2^a are also the end points of Γ_{-a}, since Γ_{-a} is the same as Γ_a. Thus, as we move a continuously to $-a$, c_1^a will be moved to c_2^a and c_2^a will be moved to c_1^a, but the order of c_1^a, p_1, p_2, c_2^a is clearly changed. This contradicts the fact that the order of the four points is unchanged as we vary a continuously. This completes the proof.

Remark. These ideas can be used to restrict the behaviour of stable solutions on convex domains in $\boldsymbol{R}^2$ whose zero sets intersect ∂D in exactly two points.

New, we shall obtain the following result which implies that when D is a ball or an annulus in $\boldsymbol{R}^n$ $(n\geq 2)$, if $\tilde{u}$ is a non-radial solution of (20), then $\tilde{u}$ is unstable. There have been several claims of results of this type but the proofs have been incomplete (sometimes in the case of an annulus when $n\geq 3$).

Theorem 2.9. *Let D be invariant under the action of a connected closed subgroup G of SO(n), whre SO(n) is the real special orthogonal group consisting of matrices on* $\boldsymbol{R}^n$ *with determinant* $+1$. *Assume that* $\tilde{u}\in C_0^1(\bar{D})$ *is a solution of* (20) *where* f *is locally Lipschitz. If* $\tilde{u}$ *is stable, then* $\tilde{u}(x)=\tilde{u}(gx)$ *for* $x\in D$ *and* $g\in G$.

Remark. Simple examples show that the result may be false if G is not connected. Note that the component containing the identity of the symmetry group of a domain is always closed.

Proof. Suppose that there exist $x\in D$ and $g_0\in G$ such that $\tilde{u}(x)\neq\tilde{u}(g_0x)$. We shall prove $\tilde{u}$ is unstable.

Note that G is a compact connected Lie group. Thus, $g_0=\exp A$, where A is an element of Lie algebra of G (cf. [22, p. 113]). Moreover, $S=\{\exp tA : t\in\boldsymbol{R}\}$ is a subgroup of G. Hence, $\bar{S}$ is a connected compact commutative subgroup of G and thus a torus group (cf. [28, p. 78]), that is, a product of circle groups. Therefore, a dense set of elements of $\bar{S}$ have finite order and hence we can find an element $\tilde{g}_0$ of $\bar{S}$ of finite order which is arbitrarily close to g_0. Therefore, there exists $m>0$ such that $\tilde{g}_0^m=e$, where e is the identity of $SO(n)$ and $\tilde{u}(x)\neq\tilde{u}(\tilde{g}_0x)$ (by continuity).

By [22] again, $\tilde{g}_0=\exp\tilde{A}$, where $\tilde{A}$ is an element of the Lie algebra of $\bar{S}$. Let

$\alpha(t)=\exp t\tilde{A}$. Then $\{\alpha(t): t\in \boldsymbol{R}\}\subset \bar{S}\subseteq G$ and $\alpha(m)=e$. If we let $\tilde{w}(t)(x)=d/dt(\tilde{u}(\alpha(t)x))$, then $\tilde{w}(t)(x)$ satisfies

$$-\Delta(\tilde{w}(t)(x))=f'(\tilde{u}(\alpha(t))x)\tilde{w}(t)x \qquad \text{for} \quad t\in[0,m]$$
$$\tilde{w}(t)(x)=0 \quad \text{on } \partial D\,,$$

since $\tilde{u}(\alpha(t)x)$ is a solution of (20) for all t. Since there exists $x\in D$ such that $\tilde{u}(x)\neq\tilde{u}(\tilde{g}_0x)$, there exists $t_0\in(0,m)$ such that $\tilde{w}(t_0)(x)\not\equiv 0$ in D. Now, we shall prove that $\tilde{w}(0)(x)$ changes sign on D. Suppose $\tilde{w}(0)(x)\geq 0$ (≤ 0) on D. We have that $\tilde{w}(0)(x)=\nabla\tilde{u}(x)\cdot\tilde{A}x\geq 0$ (≤ 0) on D. Here we use that $\alpha'(t)=\exp(t\tilde{A})\cdot\tilde{A}$ and $\alpha'(0)=\tilde{A}$. Therefore,

$$\tilde{w}(t)(x)=\nabla_{\alpha(t)x}\tilde{u}(\alpha(t)x)\cdot\exp(t\tilde{A})\tilde{A}x=\nabla_y\tilde{u}(y)\cdot\tilde{A}y\geq 0 \ \ (\leq 0) \quad \text{on } D\,, \tag{28}$$

where $y=\alpha(t)x$. (Note that $y\in D$.)

Since $\tilde{u}(\alpha(0)x)\equiv\tilde{u}(\alpha(m)x)\equiv\tilde{u}(x)$ in D, then if $\tilde{w}(t)\not\equiv 0$ for some $t\in(0,m)$, by a well-known analysis theorem, we have that there exist t_1, t_2 and $x_0\in D$ such that $\tilde{w}(t_1)(x_0)<0$ and $\tilde{w}(t_2)(x_0)>0$. This contradicts (28). Thus, $\tilde{w}(0)(x)$ changes sign on D. Since $\tilde{w}(0)(x)$ is an eigenfunction of the problem

$$-\Delta h=f'(\tilde{u}(x))h+\lambda h \quad \text{in } D\,, \qquad h=0 \quad \text{on } \partial D \tag{29}$$

corresponding to $\lambda=0$, it follows that the principal eigenvalue of (29) is negative. This implies that $\tilde{u}$ is unstable. This completes the proof.

The following result was also known from [32] and [40]. We obtain it as an immediate corollary of Theorem 2.9.

COROLLARY 2.10. *Assume that D is a ball or an annulus in $\boldsymbol{R}^n$ $(n\geq 2)$. Assume $\tilde{u}\in C_0^1(\bar{D})$ is a non-radial solution of* (20). *Then $\tilde{u}$ is unstable.*

REMARK. This result and an earlier result imply that on an annulus and a ball there are no stable changing sign solutions of (20).

There is an analogue to Corollary 2.10 and a partial analogue to Corollary 2.5 for the problem (20) when D is a cylinder.

PROPOSITION 2.11. *Assume that f is locally Lipschitz continuous and $D=B_R(0)\times(0,L)$ where $B_R(0)\subset\boldsymbol{R}^p$ $(p\geq 2)$ is the ball $\{x: 0\leq\|x\|<R\}$. Let $\tilde{u}(x,z)=\tilde{u}(r,z)\in C_0^1(\bar{D})$ be a changing sign solution of* (20) *which satisfies that $\tilde{u}'_r(R,z)\neq 0$ for every $z\in(0,L)$. Then, $\tilde{u}$ is unstable.*

PROOF. We write (20) as

$$\begin{aligned} &u_{rr}+\frac{(p-1)}{r}u_r+u_{zz}+f(u)=0\\ &u_r(0,z)=u(R,z)=u(r,0)=u(r,L)=0\,. \end{aligned} \tag{30}$$

Since $\tilde{u}$ changes sign on D, there exists $z_0 \in (0, L)$ such that $\tilde{u}(r, z_0)$ changes sign. (The only other possibility is that there exists $z_1 \in (0, L)$ such that $\tilde{u}(r, z_1) \equiv 0$, but this is impossible by our assumption on $\tilde{u}_r$.) Note that $\tilde{u}_r(0, z) = 0$ because by regularity theory $\tilde{u}(x, z) \in C^2(D)$. Thus, there exists $\hat{r} > 0$ which depends on z_0 such that $\tilde{u}_r(\hat{r}, z_0) = 0$. Let $\tilde{h}(r, z) = \partial \tilde{u}(r, z)/\partial r$. Then $\tilde{h}(r, z)$ satisfies the equation

$$h_{rr} + h_{zz} + f'(\tilde{u})h + \left(\frac{p-1}{r} h\right)_r = 0 \tag{31}$$

and $\tilde{h}(\hat{r}, z_0) = 0$.

To prove $\tilde{u}$ is unstable, we only need to find a piecewise C^1 closed Jordan curve Γ on the rz-plane such that $\tilde{h} = 0$ on Γ. Let $\hat{\Gamma}$ be the domain bounded by Γ. If we can find such at Γ, then $\tilde{h}(r, z)p(\omega)$ (where $p(\omega)$ is a degree one spherical harmonic in the x variable) will be a solution of the linearization of (20) on $Y = \{(x, z) : (\|x\|, z) \in \hat{\Gamma}\}$ vanishing on the boundary. Theorem 2.4 then implies that $\tilde{u}$ is unstable.

Assume $\tilde{h}(\hat{r}, z_0) = 0$ where $\hat{r} > 0$. Let $y_1 = r - \hat{r}$, $y_2 = z - z_0$ and $\hat{h}(y) = \tilde{h}(r, z)$. Then $\hat{h}(y)$ satisfies the equation

$$\Delta \hat{h}(y) + f'(\tilde{u}(y_1 + \hat{r}, y_2 + z_0))\hat{h}(y) + \left(\frac{p-1}{y_1 + \hat{r}} \hat{h}\right)_{y_1} = 0 \quad \text{in } \hat{D}$$

and $\hat{h}(0) = 0$, where $\hat{D} = \{(y_1, y_2) : y_1 = r - \hat{r}, y_2 = z - z_0 \text{ for } (r, z) \in (0, R) \times (0, L)\}$. Using the same arguments as in the proof of Theorem 2.8, we obtain that there is an open neighbourhood $\hat{V}$ of $y = 0$ such that $\{\hat{h} = 0\} \cap \hat{V}$ consists of quite smooth Jordan arcs $\hat{\gamma}_j$, $j = 1, 2, \ldots, 2m+2$, which, emanating from $y = 0$, loally divide $\hat{V}$ into $2m+2$ disjoint subdomains $\hat{\Omega}_j$ such that $\hat{h}$ does not change sign on $\hat{\Omega}_j$. We are using essentially that $\hat{r} \neq 0$. This implies that there is an open neighbourhood V of $(\hat{r}, z_0)$ such that $\{\tilde{h} = 0\} \cap V$ consists of Jordan arcs $\gamma_j, j = 1, 2, \ldots, 2m+2$, which, emanating from $(\hat{r}, z_0)$, divide V into $2m+2$ disjoint subdomains Ω_j. We also consider two cases here:

(i) all Ω_j can be extended to the boundary of the rectangle $(0, R) \times (0, L)$;

(ii) there is a subdomain Ω_j which cannot be extended to the boundary of the rectangle.

If (ii) occurs, we know that $\tilde{h} \neq 0$ on Ω_j and $\tilde{h} = 0$ on $\partial \Omega_j$ and $\partial \Omega_j \subset (0, R) \times (0, L)$. Thus, $\partial \Omega_j$ is a closed Jordan curve satisfying our requirement. Now, we only need to treat the case (i). We have that $\tilde{h} = 0$ on the set $\{0\} \times (0, L)$ by the regularity of $\tilde{u}$. Since each Jordan arc γ_j can be extended to the boundary of the rectangle $(0, R) \times (0, L)$ and $\tilde{h} \neq 0$ on the set $\{R\} \times (0, L)$, there exists at least one Jordan curve Γ passing through $(\hat{r}, z_0)$, which ends at two points on the boundary of the rectangle. Let e_1, e_2 denote its two end points. There are four cases:

(i) both of them belong to $[0, R] \times \{0\}$ or $[0, R] \times \{L\}$,

(ii) one of them belongs to $\{0\} \times [0, L]$, while the other belongs to $(0, R) \times \{0\}$ or $(0, R) \times \{L\}$,

(iii) both of them belong to $\{0\} \times [0, L]$,

(iv) one of them belongs to $(0, R) \times \{0\}$, while the other belongs to $(0, R) \times \{L\}$.

Since $\partial\tilde{u}/\partial r \equiv 0$ on $[0, R] \times \{0\}$, $[0, R] \times \{L\}$ and $\{0\} \times [0, L]$, for each of the four cases above, we can find a closed curve Γ^* (a part of it may be a part of the boundary of the rectangle) such that $\partial\tilde{u}/\partial r = 0$ on Γ^*. We can then construct Y as before and obtain the instability of $\tilde{u}$. We do not know the smoothness of Y near $r=0$ but since this has zero $(n-1)$-dimensional Hausdorff measure, the argument in [15] shows that this does not cause difficulties.

REMARK. Arguments similar to those in the proof of Proposition 2.11 still work for the case where $D = \Omega \times (0, L)$ and Ω is an annulus if $f(0) = 0$. Here we also need to assume that $\tilde{u}(\cdot, z)$ changes sign for z in $(0, L)$ arbitrarily close to 0 or to L. We use reflection tricks (since $f(0) = 0$) and Hartman-Wintner's results on $\tilde{u}$ and $\partial\tilde{u}/\partial r$. The details of the proof are rather more tedious. We omit the proof here.

COROLLARY 2.12. *Assume that* $f(0) = 0$ *and* $D = \Omega \times E$, *where* $\Omega \subset \boldsymbol{R}^p$ $(p \geq 2)$ *is a ball or an annulus and* $E \subset \boldsymbol{R}^{n-p}$ *is a smooth bounded connected manifold. Let* $\tilde{u}(x, z) \in C_0^1(\bar{D})$ *be a solution of* (20) *which is not a radial solution of* x. *Then* $\tilde{u}$ *is unstable.*

PROOF. This follows easily from Theorem 2.9 by setting $G = SO(p)$ acting on the first variable. There is a slight technical trouble with smoothness of ∂D at the corners but this can be overcome with reflection tricks. (Here we are using the fact that $f(0) = 0$.)

3. Stable changing sign solutions of (1).

In this section we shall use domain variation techniques as in [9], [10] and an idea of Sweers [40] to construct stable changing sign solutions of the problem (1) in various situations. Some of these examples are rather symmetric solutions on rather symmetric convex domains.

Let

$$I(u) = \frac{1}{2}\int_D |\nabla u|^2 - \int_D H(u) \qquad \text{for} \quad u \in W_0^{1,2}(D),$$

where

$$H(u) = \begin{cases} \dfrac{1}{2}au^2 - \dfrac{1}{3}\alpha u^3 & \text{if} \quad u \geq 0 \\[2mm] \dfrac{1}{2}du^2 + \dfrac{1}{3}u^3 & \text{if} \quad u \leq 0\,. \end{cases}$$

Then the critical points of $I(u)$ in $W_0^{1,2}(D)$ are solutions of (1). It is well-known (cf. [18]) that there are two strictly local minimizers of $I(u)$, $u_1 = \alpha^{-1}\phi_a$ and $u_2 = -\phi_d$ where ϕ_r is the unique positive solution of the problem

$$(32) \qquad -\Delta u = u(r-u) \quad \text{in } D\,, \qquad u = 0 \quad \text{on } \partial D$$

for $r > \lambda_1$. Moreover, these solutions are non-degenerate and stable.

Let B_1 and B_2 be two disjoint balls in $\boldsymbol{R}^n$ such that ∂B_1, ∂B_2 intersect at a single point. Let $\hat{\Omega} = B_1 \cup B_2$. Then, we easily see that the function

$$\tilde{u} = \begin{cases} u_1 & \text{in } B_1 \\ u_2 & \text{in } B_2 \end{cases}$$

is a non-degenerate solution of (1) in $\tilde{\Omega}$ and is stable, since the principal eigenvalue $\tilde{\lambda}$ of the problem

$$-\Delta k = h'(\tilde{u})k + \lambda k \quad \text{in } \tilde{\Omega}, \qquad k = 0 \quad \text{on } \partial\tilde{\Omega} \tag{33}$$

satisfies $\tilde{\lambda} \geq \min\{\tilde{\lambda}_1, \tilde{\lambda}_2\} > 0$, where $\tilde{\lambda}_i$ $(i = 1, 2)$ is the principal eigenvalue of the linearization of (1) at u_i on B_i, respectively.

It is clear that $\tilde{u}$ changes sign on $\tilde{\Omega}$. Moreover, if $a = d$, $\alpha = 1$, then, $u_1 = -u_2$. Therefore, $\tilde{u}$ is an odd function of x_1.

Now, we shall construct the changing sign stable solutions for (1) with smooth domains approximating $\tilde{\Omega}$.

Choose Ω_m star-shaped for $m \geq 4$ such that Ω_m decreases to $\tilde{\Omega}$ in the sense of [14]. As in the proof of Theorem 1 in [9], one easily obtains that the problem

$$-\Delta u = h(u) \quad \text{in } \Omega_m, \qquad u = 0 \quad \text{on } \partial\Omega_m \tag{34}$$

has a locally unique solution u_m in $W_0^{1,2}(\Omega_m)$, $u_m \to \tilde{u}$ in $L^p(\tilde{B})$ for all $p > 1$. Moreover, for large m, the eigenvalue problems (33) and

$$-\Delta k = h'(u_m)k + \lambda k \quad \text{in } \Omega_m, \qquad k = 0 \quad \text{on } \partial\Omega_m \tag{35}$$

have the same number of negative eigenvalues counting multiplicity as (33) and 0 is not an eigenvalue for (35) for large m since 0 is not an eigenvalue for (33). Thus, u_m is stable. Moreover, since $\tilde{u}$ changes sign in $\tilde{\Omega}$, u_m changes sign in Ω_m when m is sufficiently large. This is the required example. Note that if $a = d$ and $\alpha = 1$ and if B_1 and B_2 have the same radius, local uniqueness shows that u_m is odd in x_1. The same construction can still be used if B_1 and B_2 are not balls.

Now, we shall construct a stable changing sign solution of the problem (1) in convex domains with our special nonlinearity. The domains here are rather different from the previous example and the parameter range is quite different. This example is an interesting contrast to some of the results in the previous section. Our construction is a modification of the one in Sweers [40].

We assume that

$$h_\varepsilon(s) = \begin{cases} as - \alpha s^2, & s \geq 0 \\ \varepsilon^{-2} as + 2\varepsilon^{-3} \alpha s^2, & s < 0 \end{cases}$$

and

$$D=\left\{(x_1,\ldots,x_n)\in \boldsymbol{R}^n\,(n\geq 2),\left(\frac{a^2+2\alpha}{2\alpha(n-1)}\right)^{1/2}(x_2^2+\cdots+x_n^2)^{1/2}<x_1<1\right\}.$$

Note that $h_\varepsilon(s)$ satisfies

$$h_\varepsilon(s)=-2\varepsilon h_\varepsilon\left(-\frac{1}{2}\varepsilon s\right)\quad \text{for}\quad s>0\,. \tag{36}$$

By the maximum principle, we know that if u is a changing sign solution of the problem (1) with $h=h_\varepsilon$,

$$-a\varepsilon(2\alpha)^{-1}\leq u(x)\leq a\alpha^{-1}\quad \text{in } D\,.$$

As earlier, the problem

$$-\Delta u=\lambda h_\varepsilon(u)\quad \text{in } D\,,\qquad u=0\quad \text{on } \partial D \tag{37}$$

has a unique positive solution for every $\lambda>a^{-1}\lambda_1$, since $s^{-1}h_\varepsilon(s)$ is decreasing on $[0, a\alpha^{-1}]$.

Similar arguments hold for negative solutions. Let U_λ and V_λ^ε denote the positive, respectively the negative solution of (37) for $\lambda>a^{-1}\lambda_1$. Note that this is a different notation from earlier in the section. Let $J_\varepsilon(\lambda, u)$ denote the energy functional for (37), that is,

$$J_\varepsilon(\lambda, u)=\int_D\left(\frac{1}{2}|\nabla u|^2-\lambda\int_0^u h_\varepsilon(s)ds\right)dx\,.$$

We have the following lemmas.

LEMMA 3.1.

$$\lim_{\lambda\to\infty}\lambda^{-1}J_\varepsilon(\lambda, U_\lambda)=-\frac{1}{6}a^3\alpha^{-2}|D|\,,\qquad \lim_{\lambda\to\infty}\lambda^{-1}J_\varepsilon(\lambda, V_\lambda^\varepsilon)=-\frac{1}{24}a^3\alpha^{-2}|D|\,,$$

uniformly for $\varepsilon\in(0, 1]$, *where* $|D|$ *is the Lebesgue measure of* D.

LEMMA 3.2.

$$U_{\tilde\lambda}(x_1, x_2,\ldots,x_n)<\tilde\lambda\left(x_1^2-\frac{(a^2+2\alpha)}{2\alpha(n-1)}(x_2^2+\cdots+x_n^2)\right)\quad \textit{in } D\,,$$

for $\tilde\lambda$ *sufficiently large.*

PROOF OF LEMMA 3.1. We will show the second statement. The basic idea of the proof is the same as that of the proof of Lemma 3 in [40]. Since V_λ^ε is the only stable solution of

$$-\Delta u=\lambda\min\{h_\varepsilon(u), 0\}\quad \text{in } D\,,\qquad u=0\quad \text{on } \partial D\,, \tag{38}$$

the function minimizes

$$J_\varepsilon^-(\lambda, u)=\int_D\left(\frac{1}{2}|\nabla u|^2-\lambda\int_0^u \min\{h_\varepsilon(s), 0\}ds\right)dx$$

for $\lambda>a^{-1}\lambda_1$. We can estimate $J_\varepsilon^-(\lambda, u)$ from below by $-a^3\alpha^{-2}|D|/24$ since

$$J_\varepsilon^-(\lambda, u)\geq -\lambda\int_D\int_0^u \min\{h_\varepsilon(s), 0\}dsdx\geq -\frac{\lambda}{24}a^3\alpha^{-2}|D|\,.$$

It is sufficient to show that for all $\sigma>0$ there is $\phi_\varepsilon\in W_0^{1,2}(D)$ such that uniformly for $\varepsilon\in(0, 1]$,

$$\lim_{\lambda\to\infty}\lambda^{-1}J_\varepsilon^-(\lambda, \phi_\varepsilon)<-\frac{1}{24}a^3\alpha^{-2}|D|+\sigma\,.$$

Take $\phi\in C_0^\infty(D)$ with $-a(2\alpha)^{-1}\leq\phi\leq 0$ and $\phi=-a(2\alpha)^{-1}$ on a closed subset of D with measure larger than $|D|-12a^{-3}\alpha^2\sigma$. The result follows for λ large since

$$\begin{aligned}\lambda^{-1}J_\varepsilon^-(\lambda, \varepsilon\phi)&<\lambda^{-1}\varepsilon^2\int_D\frac{1}{2}|\nabla\phi|^2dx-\frac{1}{24}a^3\alpha^{-2}|D|+\frac{1}{2}\sigma\\&\leq\lambda^{-1}\int_D\frac{1}{2}|\nabla\phi|^2dx-\frac{1}{24}a^3\alpha^{-2}|D|+\frac{1}{2}\sigma\,.\end{aligned}$$

This completes the proof.

Because of Lemma 3.1, there is $\tilde{\lambda}>a^{-1}\lambda_1$ such that

$$J_\varepsilon^-(\lambda, U_\lambda)<J_\varepsilon(\lambda, V_\lambda^\varepsilon)<-\frac{1}{48}a^3\alpha^{-2}|D|\lambda$$

for all $\lambda\geq\tilde{\lambda}$ and $\varepsilon\in(0, 1]$.

PROOF OF LEMMA 3.2. For t large enough, we easily see that

$$U_{\tilde{\lambda}}(x_1, \ldots, x_n)<\tilde{\lambda}\left[(x_1+t)^2-\frac{(a^2+2\alpha)}{2\alpha(n-1)}(x_2^2+\cdots+x_n^2)\right]\quad\text{in } D\,. \tag{39}$$

Indeed, since

$$-\Delta\left[\tilde{\lambda}\left((x_1+t)^2-\frac{(a^2+2\alpha)}{2\alpha(n-1)}(x_2^2+\cdots+x_n^2)\right)\right]=\frac{a^2}{\alpha}\tilde{\lambda}>(\max h_\varepsilon)\tilde{\lambda}$$

and since $\tilde{\lambda}((x_1+t)^2-(a^2+2\alpha)(2\alpha(n-1))^{-1}(x_2^2+\cdots+x_n^2))>0$ in $\bar{D}$ for $t>0$, the function on the right hand side of (39) is a supersolution of (37) for $t\geq 0$. By the sweeping principle [7], one finds that (39) holds for all $t\geq 0$. Hence the lemma follows.

Finally we will show, for $\varepsilon>0$ but small enough, that $U_{\tilde{\lambda}}$ does not minimize $J_\varepsilon(\tilde{\lambda}, \cdot)$. We shall modify $U_{\tilde{\lambda}}$ near (0, 0) to obtain a $W_0^{1,2}(D)$-function with lower energy. Hence the solution of (37) for $\lambda=\tilde{\lambda}$ that minimizes $J_\varepsilon(\tilde{\lambda},\ \cdot)$ is not $V_{\tilde{\lambda}}^{\varepsilon}$ or $U_{\tilde{\lambda}}$, which are the only stable solutions with fixed sign. (Note that $V_{\tilde{\lambda}}^{\varepsilon}$ has higher energy than $U_{\tilde{\lambda}}$.)

Set

$$D_\delta^1=\{(x_1, x_2, \dots, x_n)\in D;\ x_1<\delta\}\,,$$

$$D_\delta^2=\{(x_1, x_2, \dots, x_n)\in D;\ \delta<x_1<2\delta\}\,.$$

Then $|D_{2\delta}^1|=C\delta^n$. Moreover, define z on $\boldsymbol{R}$ by

$$z(s)=0 \quad \text{for } s\le 1\,, \quad z(s)=s-1 \quad \text{for } 1<s\le 2\,, \quad z(s)=1 \quad \text{for } s>2\,,$$

and set

$$u_\delta(x_1, x_2, \dots, x_n)=z(\delta^{-1}x_1)U_{\tilde{\lambda}}(x_1, \dots, x_n)\,.$$

Then, $u_\delta\in W_0^{1,2}(D)$ and

$$\nabla u_\delta(x_1, \dots, x_n)=\delta^{-1}U_{\tilde{\lambda}}(x_1, \dots, x_n)(1, 0, \dots, 0)+z(\delta^{-1}x_1)\nabla U_{\tilde{\lambda}}(x_1, \dots, x_n) \text{ in } D_\delta^2\,.$$

By Lemma 3.2 we can estimate the difference in energy as follows:

$$\begin{aligned}
J_\varepsilon(\tilde{\lambda}, u_\delta)-J_\varepsilon(\tilde{\lambda}, U_{\tilde{\lambda}})&\le\frac{1}{2}\int_{D_{2\delta}^1}(|\nabla u_\delta|^2-|\nabla U_{\tilde{\lambda}}|^2)dx+\tilde{\lambda}\int_{D_{2\delta}^1}\frac{1}{2}aU_{\tilde{\lambda}}^2dx+\tilde{\lambda}\int_{D_{2\delta}^1}\frac{1}{3}\alpha u_\delta^3dx\\
&\le\int_{D_\delta^2}\left(\frac{1}{2}\delta^{-2}U_{\tilde{\lambda}}^2+\delta^{-1}U_{\tilde{\lambda}}z(\delta^{-1}x_1)\frac{\partial}{\partial x_1}U_{\tilde{\lambda}}\right)dx+\tilde{\lambda}\int_{D_{2\delta}^1}\frac{1}{2}aU_{\tilde{\lambda}}^2dx+\tilde{\lambda}\int_{D_{2\delta}^1}\frac{1}{3}\alpha U_{\tilde{\lambda}}^3dx\\
&\le|D_{2\delta}^1|\left(\frac{1}{2}\delta^{-2}(\tilde{\lambda}4\delta^2)^2\right)+\delta^{-1}(\tilde{\lambda}4\delta^2)\int_{D_\delta^2}|\nabla U_{\tilde{\lambda}}|dx+\frac{1}{2}a\tilde{\lambda}(4\tilde{\lambda}\delta^2)^2|D_{2\delta}^1|\\
&\quad+\frac{1}{3}\alpha\tilde{\lambda}(4\tilde{\lambda}\delta^2)^3|D_{2\delta}^1|\\
&\le C_1(\tilde{\lambda})\delta^{n+2}+4\tilde{\lambda}C^{1/2}\delta^{1+n/2}\left(\int_{D_{2\delta}^1}|\nabla U_{\tilde{\lambda}}|^2dx\right)^{1/2}.
\end{aligned}$$

Since δ is sufficiently small, we can choose $M>0$ (M is independent of δ) such that $D_{2\delta}^1\subset K_{M\delta}\cap D$, where $K_{M\delta}$ is an n-dimensional ball with center 0 and radius $M\delta$. By the facts that ∂D satisfies the *condition* (A) of [31, p. 6] and that $U_{\tilde{\lambda}}\in W_0^{1,2}(D)\cap L^\infty(D)$, we obtain by using the remarks before Lemma 1.2′ of [31, p. 253] that

$$\int_{D_{2\delta}^1}|\nabla U_{\tilde{\lambda}}|^2dx\le\int_{K_{M\delta}\cap D}|\nabla U_{\tilde{\lambda}}|^2dx\le C_2(M\delta)^{n-2+2\beta}\,,$$

where $C_2>0$, $0<\beta<1$. Therefore,

$$\begin{aligned}
J_\varepsilon(\tilde{\lambda}, u_\delta)-J_\varepsilon(\tilde{\lambda}, U_{\tilde{\lambda}})&\le C_1(\tilde{\lambda})\delta^{n+2}+4\tilde{\lambda}C^{1/2}\delta^{1+n/2}C_2^{1/2}M^{n/2-1+\beta}\delta^{\beta+n/2-1}\\
&\le C_3(\tilde{\lambda})\delta^{n+\beta}\,, \qquad \text{for} \quad \delta<1\,.
\end{aligned}$$

Let

$$v_\delta(x_1,\dots,x_n)=-\frac{1}{2}\delta U_{\tilde\lambda}(\delta^{-1}x_1,\dots,\delta^{-1}x_n)\,.$$

Then

$$-\Delta v_\delta(x)=\frac{1}{2}\,\delta^{-1}(\Delta U_{\tilde\lambda})(\delta^{-1}x)=-\frac{1}{2}\,\delta^{-1}\tilde\lambda h_\delta(U_{\tilde\lambda}(\delta^{-1}x))=\tilde\lambda h_\delta(v_\delta(x))\,.$$

Here we use (36). Hence, v_δ is a solution of (37) with $\varepsilon=\delta$ and D replaced by D_δ^1. After extending v_δ by 0 outside of D_δ^1 we obtain

$$\begin{aligned}J_\delta(\tilde\lambda, v_\delta)&=\int_{D_\delta^1}\left(\frac{1}{2}\,|\nabla v_\delta|^2-\tilde\lambda\int_0^{v_\delta}h_\delta(s)ds\right)dx\\&=\frac{1}{4}\int_{D_\delta^1}\left(\frac{1}{2}\,|\nabla U_{\tilde\lambda}(\delta^{-1}x)|^2-\tilde\lambda\int_0^{U_{\tilde\lambda}(\delta^{-1}x)}h_\delta(s)ds\right)dx\\&=\frac{1}{4}\,\delta^n J_\delta(\tilde\lambda, U_{\tilde\lambda})\,.\end{aligned}$$

Finally, we set $w_\delta=u_\delta+v_\delta$. Since $\operatorname{supp}u_\delta\subset\bar D\setminus D_\delta^1$ and since $\operatorname{supp}v_\delta\subset\bar D_\delta^1$, we find that, by the estimates above

$$\begin{aligned}J_\delta(\tilde\lambda, w_\delta)=J_\delta(\tilde\lambda, u_\delta)+J_\delta(\tilde\lambda, v_\delta)&\le\left(1+\frac{1}{4}\,\delta^n\right)J_\delta(\tilde\lambda, U_{\tilde\lambda})+C_3(\tilde\lambda)\delta^{n+\beta}\\&<J_\delta(\tilde\lambda, U_{\tilde\lambda})\,,\end{aligned}$$

for δ sufficiently small. This implies that the global minimizer is not $U_{\tilde\lambda}$ and hence the global minimizer must change sign as required.

Now we are in a position to prove the following theorem.

THEOREM 3.3. *Let D be as above and $\Omega=D\cup D^*\cup S$, where*

$$D^*=\{(x_1,\dots,x_n)\in \boldsymbol{R}^n(n\ge2),\left(\frac{a^2+2\alpha}{2\alpha(n-1)}\right)^{1/2}(x_2^2+\cdots+x_n^2)^{1/2}<2-x_1,\ 1\le x_1<2\}\,,$$

$$S=\left\{(x_1,\dots,x_n)\in \boldsymbol{R}^n(n\ge2),\left(\frac{a^2+2\alpha}{2\alpha(n-1)}\right)^{1/2}(x_2^2+\cdots+x_n^2)^{1/2}=x_1=1\right\}.$$

Then, there exist $\tilde\lambda$, $\delta>0$ with $\tilde\lambda$ and δ^{-1} sufficiently large such that the problem

$$(40)\qquad -\Delta u=\tilde\lambda h_\delta(u)\quad in\ \Omega\,,\qquad u=0\quad on\ \partial\Omega$$

has at least one stable changing sign solution which is symmetric in x_1 about $x_1=1$.

PROOF. We first modify the argument above to show that there exist $\tilde\lambda$, $\delta>0$

such that $J_\delta(\tilde{\lambda}, u)$ has a changing sign global minimizer in the set of functions in $W_0^{1,2}(\Omega)$ which are symmetric for reflections in S. Now U_λ and V_λ^ε denote the solutions on Ω (with Dirichlet boundary conditions on $\partial\Omega$). By the uniqueness of the positive and negative solutions of (37) on Ω, U_λ and V_λ^ε are symmetric for reflections in S. We now repeat the same argument as above except that we modify $U_{\tilde{\lambda}}$ by symmetrically modifying it near the corner at $x_1=0$ and the one at $x_1=2$. The argument is much the same as before except that we have two v_δ terms (one at either end). Thus we find a changing sign solution of (20) which is symmetric for reflections in S and which is a global minimizer of $J_\delta(\tilde{\lambda}, u)$ for the functions symmetric for reflections in S. We need only to prove that it is a local minimizer of $J_\delta(\tilde{\lambda}, u)$ in $W_0^{1,2}(\Omega)$. We shall give a general proof.

Let $g\in C^1(\boldsymbol{R})$ be sublinear and $T\subseteq W_0^{1,2}(\Omega)$ be the subspace of symmetric functions. If u is a local minimizer of

$$E(u)=\frac{1}{2}\int_\Omega (|\nabla u|^2-G(u))dx \qquad u\in T,$$

where $G(u)=\int_0^u g(s)ds$, then, by standard arguments, every eigenvalue of the problem

$$(41) \qquad -\Delta h-g'(u)h=\lambda h \quad \text{in } \Omega, \qquad h=0 \quad \text{on } \partial\Omega, \qquad h\in T$$

is nonnegative. However, the eigenfunction $\phi(x)>0$ corresponding to the smallest eigenvalue of the problem

$$(42) \qquad -\Delta h-g'(u)h=\lambda h \quad \text{in } \Omega, \qquad h=0 \quad \text{on } \partial\Omega, \qquad h\in W_0^{1,2}(\Omega)$$

must have the symmetries of the domain Ω. Thus it belongs to T. This follows because if A is the Lie group of symmetries of Ω, then, since the equation and $W_0^{1,2}(\Omega)$ are invariant under the usual orthogonal action of the Lie group, $\tilde{\phi}=\int_A T_a\phi d\mu$ must also be a positive eigenfunction, where μ is the invariant Haar measure and $T_a f$ denotes the naturally induced action of A on the function f. Note that $\tilde{\phi}$ is non-trivial since $\phi(x)>0$ on Ω implies $\int_A(T_a\phi)(x)d\mu>0$ for every $x\in\Omega$. Thus, the smallest eigenvalue of (42) is also an eigenvalue of (41). This implies that every eigenvalue of (42) is nonnegative.

If every eigenvalue of (42) is positive, then it is easy to see that u is a local minimizer of E on $W_0^{1,2}(\Omega)$. If not, 0 is a simple eigenvalue of (42) with simple eigenfunction $\phi\in T$. Hence, we see that if we do a Lyapunov-Schmidt reduction of our equation near u on T or $W_0^{1,2}(\Omega)$, we have the same bifurcation equation $B(0)=0$ on $\boldsymbol{R}$ (defined near zero). Now $B=\nabla b$ where $b:\boldsymbol{R}\to\boldsymbol{R}$. Then, it follows easily from the generalized Morse Lemma (cf. [34]) that u is a local minimizer of $E(u)$ on $W_0^{1,2}(\Omega)$ if and only if 0 is a local minimizer of b on $\boldsymbol{R}$ and the same result is true on T. Hence, our claim follows. There is one more technical point. For the above argument, we need E to be C^2. This is true if g is C^1 and $g'(s)\to 0$ as $|s|\to\infty$ by standard arguments. But in our case $h'_\delta(s)$ has one jump discontinuity at 0. With care, one can show that the above conclusion is still true by proving that in this case E is still C^1 and $E'(u)$ is strictly differentiable at a solution u if $u\not\equiv 0$ (cf. [18]) and then by proving a slight variant of the generalized Morse Lemma.

By a truncation argument on h_δ (since the minimizer of J_δ is bounded from above and below), we can meet the requirement of sublinearity. This completes the proof.

REMARKS. 1. By domain variation arguments as in Theorem 1 of [9], we can obtain a stable changing sign solution of (40) on a smooth convex domain also with some symmetry.

2. We could obtain even more symmetric local minimizers by working in the subspace of functions symmetric under rotations in $x_2, \ldots, x_n$. By making minor changes in our Sweers type construction and using our argument above, we obtain changing sign local minimizers which have all the symmetries of rather symmetric convex domains (doubly symmetrized if $n=2$). As above, we can smooth the domains if we wish. It would be interesting to know if the global minimizers have the symmetries of the domain in these cases.

3. It is implicit in our work that for smooth enough nonlinearities the local minimizers and the stable solutions are the same. This follows by a slight variant of the argument at the end of the proof of the last theorem.

4. Local minimizers and singular perturbations. In this section we consider a third method for constructing stable changing sign solutions. This is only valid for a, d both large, but works for more general domains than domain variation arguments. One can show that some of the examples we construct here can only occur for large a and d because if for fixed a, d our problem has no weakly stable changing sign solution on D, then it is not difficult to show that the same is true for domains close to D in the sense of [14].

Consider the problem

$$-\varepsilon^2 \Delta u = h(u) \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D. \tag{43}$$

It can be shown that the stable solutions of (43) are the local minimizers of the functional

$$J_\varepsilon(u) = \frac{\varepsilon}{2} \int_D |\nabla u|^2 dx - \frac{1}{\varepsilon} \int_D H(u) dx \qquad \text{for} \quad u \in W^{1,2}(D).$$

Here $H(u)$ is as in Section 3. We know that $H(s)$ has two local maximizers $s_1 = a\alpha^{-1}$, $s_2 = -d$; $H(s_1) = a^3\alpha^{-2}/6$ and $H(s_2) = d^3/6$. Let $a^3\alpha^{-2} = d^3$ for a, $d > \lambda_1$. We assume this for the rest of this paper. Then $F(s) = -H(s) + d^3/6$ satisfies $F(s_1) = F(s_2) = 0$ and $F(s) \geq 0$ for $s \in \boldsymbol{R}$.

Define

$$\hat{J}_\varepsilon(u) = \frac{\varepsilon}{2} \int_D |\nabla u|^2 dx + \frac{1}{\varepsilon} \int_D F(u) dx \qquad \text{for} \quad u \in W^{1,2}(D).$$

Then the stable solutions of (43) correspond to the local minimizers of $\hat{J}_\varepsilon(u)$. Thus, we are interested in looking for the local minimizers of $\hat{J}_\varepsilon(u)$.

Define $\hat{J}_0 : L^1(D) \to \boldsymbol{R}$ by

$$\hat{J}_0(u) = \begin{cases} \left(2\int_{-d}^{a\alpha^{-1}} \sqrt{F(s)}ds\right) \mathrm{Per}_D\{x : u = a\alpha^{-1}\}, \; u \in BV(D), \; F(u(x)) = 0 \text{ a.e. in } D, \\ +\infty \quad \text{if the perimeter is infinite}. \end{cases}$$

Here $\mathrm{Per}_D A$ is the perimeter of A in D, which is well-defined (but possibly infinite) for any measurable set A. See, e.g. [24]. For the sake of completeness we give the definition here: if $\chi_A(x)$ is the characteristic function of A, equal to 1 on A and 0 on $D \setminus A$, then

$$\mathrm{Per}_D A = \sup\left\{ \int_D \chi_A \cdot \operatorname{div} \sigma dx : \sigma \in C_0^\infty(D, \boldsymbol{R}^n) : |\sigma(x)| \leq 1 \text{ for } x \text{ in } D \right\}.$$

BV(D) is the space of functions with bounded variation (cf. [24]). Then we have the following theorem, which, together with Remark 3 below answer an open problem in [30]. (See Remark 2.2 there.)

THEOREM 4.1. *Let D be a bounded domain in $\boldsymbol{R}^n$ with Lipschitz boundary. Assume that u_0 is a L^1-local minimizer of $\hat{J}_0$ and there exists a bounded open set $Q \subset L^1(D)$ such that $u_0 \in Q$ and $\hat{J}_0(u_0) \leq \hat{J}_0(u)$ for $u \in Q$, $\hat{J}_0(u_0) < \hat{J}_0(u)$ for $u \in \bar{Q} \setminus Q$. Here $\bar{Q}$ is the closure of Q in $L^1(D)$. Then there exists $\varepsilon_0 > 0$ and a family $\{u_\varepsilon\}_{\varepsilon < \varepsilon_0}$ such satisfies $\hat{J}_0(u^*) = \hat{J}_0(u_0)$ and $u^* \equiv u_0$ if u_0 is the only minimizer of $\hat{J}_0$ in Q.*

PROOF. The proof is due to Kohn and Sternberg [30] for the case where u_0 is isolated. The general case follows by a very similar argument. (The details appear in [20].)

REMARKS. 1. If $u_0 \in L^\infty(D)$, we can choose $\{u_\varepsilon\}$ such that $\|u_\varepsilon - u^*\|_p \to 0$ as $\varepsilon \to 0$ for all $p > 1$. Here u_ε is a minimizer of $\hat{J}_\varepsilon$ and u^* is as in Theorem 4.1.

2. When D is as in Figure 2,

$$u_0(x) = \begin{cases} -d & \text{on the left hand side of } \Gamma_0 \\ a\alpha^{-1} & \text{on the right hand side of } \Gamma_0 \end{cases}$$

is an isolated local minimizer of $\hat{J}_0$. The proof is due to Kohn and Sternberg [30] for the case $n = 2$. The proofs are easily generalized to the case $n > 2$ by the same idea.

3. When D is as in Figure 3,

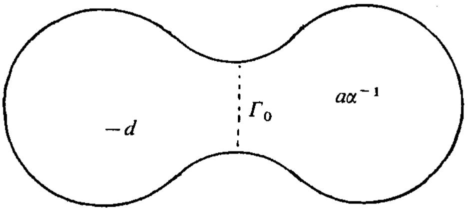

FIGURE 2.

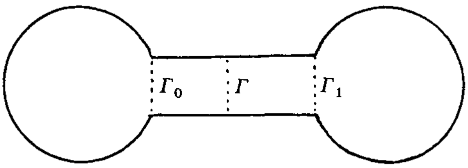

FIGURE 3.

$$u_i(x)=\begin{cases} -d & \text{on the left hand side of } \Gamma_i \\ a\alpha^{-1} & \text{on the right hand side of } \Gamma_i , \end{cases} \qquad i=0, 1 .$$

Then, u_i is a non-isolated local minimizer of $\hat{J}_0$. Moreover, there exists an open set $Q\subset L^1(D)$ such that $u_0\in Q$ and $\hat{J}_0(u_0)\leq\hat{J}_0(u)$ for $u\in Q$ and $\hat{J}_0(u_0)<\hat{J}_0(u)$ for $u\in\bar{Q}\setminus Q$. A detailed proof for $n=2$ appears in [20]. The proof is also true for $n>2$ by the same idea.

4. It is easy to see that if D is convex and $n=2$, $\hat{J}_0$ can have no local minima.

We can also obtain a result analogous to Theorem 4.1 for the problem (3). We define the sequence of functionals I_ε: $L^1(D)\to\boldsymbol{R}$, by

$$I_\varepsilon(u)=\frac{\varepsilon}{2}\int_D |\nabla u|^2+\frac{1}{\varepsilon}\int_D F(u) , \qquad u\in W_0^{1,2}(D)$$

and I_0: $L^1(D)\to\boldsymbol{R}$ by

$$I_0(u)=\begin{cases} 2(\int_{-d}^{a\alpha^{-1}}\sqrt{F(s)}ds)\operatorname{Per}_D\{x: u=a\alpha^{-1}\}+\int_{\partial D}|\Phi(0)-\Phi(\tilde{u}(x))|\,dH_{n-1}(x) , \\ +\infty \quad \text{if the perimeter is infinite} . \end{cases}$$

Here $u\in\mathrm{BV}(D)$, $u(x)\in\{-d, a\alpha^{-1}\}$, a.e. in D, $F(u)$ is as before, H_{n-1} is $(n-1)$-dimensional Hausdorff measure, $\Phi(s)=2\int_{-d}^{s}\sqrt{F(t)}dt$ and $\tilde{u}$ equals the trace of u on ∂D. It follows from [37] that $\{I_\varepsilon\}$ Γ-converges to I_0, in the sense of Theorem 2.1 of [37]. Now we discuss the local minimizers of I_0. We assume u_0 is of the form

$$u_0(x)=\begin{cases} a\alpha^{-1} & x\in A \\ -d & x\in B , \end{cases}$$

where $A\cup B=D$. Let $\Gamma=\partial A\cap\partial B$ be the interface between the regions A and B (cf. Figure 4). By Proposition 5.2 in [37], the interface Γ has zero (mean) curvature if the interface is sufficiently smooth.

For u_Γ as shown in Figure 4, $I_0(u_\Gamma)$ can be written to be of the form

$$\begin{aligned} I_0(u_\Gamma)&=\Phi(a\alpha^{-1})\operatorname{Per}_D\{x: u=a\alpha^{-1}\}+\Phi(0)H_{n-1}(B_\Gamma)+(\Phi(a\alpha^{-1})-\Phi(0))H_{n-1}(A_\Gamma) \\ &=\Phi(a\alpha^{-1})\operatorname{Per}_D\{x: u=a\alpha^{-1}\}+(\Phi(a\alpha^{-1})-2\Phi(0))H_{n-1}(A_\Gamma)+\Phi(0)H_{n-1}(\partial D) , \end{aligned}$$

where A_Γ, B_Γ are as in Figure 4.

When $2\Phi(0)=\Phi(a\alpha^{-1})$, i.e., when $\alpha=1$ and $a=d$, it follows easily from the formula above that

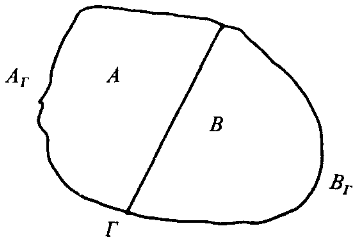

FIGURE 4.

$$I_0(u)=\Phi(a\alpha^{-1})\,\mathrm{Per}_D\{x: u=a\alpha^{-1}\}+\Phi(0)H_{n-1}(\partial D).$$

Thus, we can obtain the minimizer of I_0 in the same way as we did for $\hat{J}_0$. Hence, I_0 has an isolated local minimizer when D is as in Figure 2 and has non-isolated local minimizers when D is as in Figure 3.

When $2\Phi(0)\neq\Phi(a\alpha^{-1})$, the problem seems difficult to analyse. In the case of an ellipse D in $\boldsymbol{R}^2$, it can be shown that I_0 never has a local minimizer of the type we are looking for (with a smooth interface). Details appear in [20]. This strongly suggests that one layer solutions do not exist in this case. If $2\Phi(0)$ is close to $\Phi(a\alpha^{-1})$, one can obtain local minima of I_0 by treating I_0 as a perturbation of $\hat{J}_0$.

5. Perturbations on the nonlinear terms. In this section we shall discuss very briefly the singular perturbation problems (3) and (4) with a small perturbation on the nonlinear term $h(u)$. We shall consider a case more general than (3) and (4) though the special case of (3) and (4) is our main problem of interest.

Let $W\in C^1(\boldsymbol{R})$ satisfy $W(s)\to+\infty$ as $|s|\to+\infty$, $W(s)=0$ has only two roots τ, μ, $\tau<\mu$ and $W(s)\geq 0$ on $\boldsymbol{R}$. Moreover, assume that $W(s)$ satisfies

$$C_1|s|^p\leq W(s)\leq C_2|s|^p$$

for $|s|\geq s_0$, where C_1, C_2, s_0 are positive constants and $p\geq 2$. For convenience, we assume that $\tau<0$ and $\mu>0$. We are interested in the local minimizers of the functional

$$\hat{I}_\varepsilon=\int_D\left[\frac{\varepsilon}{2}|\nabla u|^2+\frac{1}{\varepsilon}(W(u)+\varepsilon g(u))\right],\qquad u\in W^{1,2}(D).$$

Here $\varepsilon>0$ is sufficiently small; $g\in C^1(\boldsymbol{R})$ for $s\in\boldsymbol{R}$ and $|g(s)|\leq B_1+B_2|s|^q$, $0<q<p$, where B_1, $B_2\geq 0$. It is clear that a local minimizer of $\hat{I}_\varepsilon$ is a stable solution of the problem

$$(44)\qquad -\varepsilon^2\Delta u=W'(u)+\varepsilon g'(u)\ \text{ in } D,\qquad \frac{\partial u}{\partial n}=0\ \text{ on } \partial D.$$

Since $F(s)$ has the same form as $W(s)$, the results obtained in the previous section apply to this problem when g vanishes identically.

Define

$$\hat{I}_0(u)=\begin{cases}(2\int_\tau^\mu \sqrt{W(s)}ds)\,\mathrm{Per}_D\{u=\tau\}+g(\tau)\,\mathrm{meas}(B_\Gamma)+g(\mu)\,\mathrm{meas}(A_\Gamma)\\ +\infty \quad \text{if the perimeter is infinite}\,,\end{cases}$$

for $u\in \mathrm{BV}(D)$, $u(x)\in\{\tau,\mu\}$ a.e. in D, B_Γ and A_Γ are as in Figure 4 and $\mathrm{meas}(B_\Gamma)+\mathrm{meas}(A_\Gamma)=\mathrm{meas}(D)$. Then, we have the following theorem.

THEOREM 5.1. *Let D be a bounded domain in R^n $(n\geq 2)$ with Lipschitz boundary, and suppose that u_0 is a L^1-local minimizer of $\hat{I}_0$. Moreover, assume that there exists an open set $Q\subset L^1(D)$ such that $u_0\in Q$, $\hat{I}_0(u)\geq \hat{I}_0(u_0)$ for $u\in Q$ and $\hat{I}_0(u)>\hat{I}_0(u_0)$ for $u\in\bar{Q}\setminus Q$. Then there exists $\varepsilon_0>0$ and a family $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ such that*

$$u_\varepsilon \text{ is an } L^1\text{-local minimizer of } \hat{I}_\varepsilon\,, \text{ and } \|u_\varepsilon-u^*\|_{L^1(D)}\to 0 \text{ as } \varepsilon\to 0\,.$$

Here $u^*\in Q$ satisfies $\hat{I}_0(u^*)=\hat{I}_0(u_0)$. If u_0 is the only minimizer of $\hat{I}_0$ in Q, then $u^*\equiv u_0$.

PROOF. The proof of this theorem is very similar to the proof of Theorem 1 of [39] and of Theorem 4.1 above.

It is easily seen that when $g(\tau)=g(\mu)$,

$$\hat{I}_0(u)=\left(2\int_\tau^\mu \sqrt{W(s)}ds\right)H_{n-1}(\Gamma)+g(\tau)\,\mathrm{meas}(D)\,.$$

This implies that the behaviour of $\hat{I}_0$ is the same as that of $\hat{J}_0$ before. When $g(\tau)\neq g(\mu)$, the problem seems difficult to analyse.

REMARK. We can also obtain a result analogous to Theorem 5.1 for the problem with Dirichlet boundary conditions. Here we assume $\tau<0<\mu$. The limit problem here is

$$\tilde{I}_0(u)=\begin{cases}(2\int_\tau^\mu \sqrt{W(s)}ds)\,\mathrm{Per}_D\{u=\tau\}+\int_D g(u)dx+\int_{\partial D}|\Phi(0)-\Phi(\tilde{u}(x))|dH_{n-1}(x)\\ +\infty, \quad \text{if the perimeter is infinite}\,,\end{cases}$$

for $u\in\mathrm{BV}(D)$, $u(x)\in\{\tau,\mu\}$, a.e. in D; $\Phi(s)=2\int_\tau^s\sqrt{W(s)}ds$ and $\tilde{u}$ equals the trace of u on ∂D. As before, we can simplify this formula. In various special cases, $\tilde{I}_0$ reduces to one of our earlier problems.

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