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SOME REMARKS ON THE VALUE DISTRIBUTION OF ENTIRE FUNCTIONS

BY

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1. Introduction

1. We call a set E a Picard set for entire functions if every entire non-rational function f omits at most one finite value in $-E$.

Lehto [2] has proved that a countable set

$$E = \{\infty\} \cup \{a_n\}_{n=1,2,\ldots}$$

whose points converge to infinity is a Picard set for entire functions if the points a_n satisfy the condition

$$|a_n/a_{n+1}| = O(n^{-2}) .$$

Matsumoto [4] has proved the same assertion under the condition

$$\log |a_{n+1}/a_n| \geq M(n) ,$$

where $M(n)$ are positive numbers such that

$$\limsup_{n \to \infty} \frac{K^{1/M(n)}}{M(1) + M(2) + \ldots + M(n)} < \infty$$

(K a positive constant). In this paper we prove that there exist Picard sets for entire functions, which contain a sequence of discs converging to the point at infinity.

Winkler [7] has among other things proved that the entire functions

$$w(z) = \prod_{n=1}^{\infty} (1 - z/a_n)$$

with $|a_{n+1}/a_n| \geq q > 1$ take any finite value a infinitely often in the union of the discs

$$D_n = \{z : |z - a_n| < \varrho_n\}$$

with $\varrho_n = \varepsilon|a_n|^{-p}$ for any $\varepsilon > 0$ and $p > 0$, and that they take any value only finitely often in the complement of this union (See also Lehto [3], Theorem 4). Our Theorem 1 shows that the same is not true if the radii ϱ_n of D_n satisfy the condition $|a_n| = o(-\log \varrho_n)$.

2. Picard sets for entire functions

2. We begin by presenting three lemmas. We denote $\overset{+}{\log}\,\delta = \max\{0, \log \delta\}$ for $\delta \geq 0$. Our first lemma is a consequence of Schottky's theorem which is proved by Ahlfors in the following form (Dinghas [1], p. 294):

If $g(z)$ is regular in $|z| < 1$ and $g(z) \neq 0, 1$ there, then

$$\overset{+}{\log}\,|g(z)| \leq \frac{1 + |z|}{1 - |z|}\,(7 + \overset{+}{\log}\,|g(0)|)\,.$$

Lemma 1. Let f be analytic in an annulus $r < |z| < R$ $(0 < r < R < \infty)$ and omit the values 0 and 1 there. Then

$$\overset{+}{\log}\,(\max_{|z| = \sqrt{rR}} |f(z)|) \leq \{7 + \overset{+}{\log}\,(\min_{|z| = \sqrt{rR}} |f(z)|)\} \exp\left\{\frac{\pi^2}{\log\,(R/r)}\right\}.$$

Proof. We choose z_0 such that $|z_0| = \sqrt{rR}$ and $|f(z_0)| = \min\limits_{|z| = \sqrt{rR}} |f(z)|$. We denote $\mu = \log\,(R/r)$. The composite function $g(\zeta) = f(z_0 \sqrt{r/R}e^{\zeta})$ is regular, different from 0 and 1, and has the period $2\pi i$ in the strip domain

$$D = \{\zeta : 0 < \mathrm{Re}\,\zeta < \mu\,,\ -\infty < \mathrm{Im}\,\zeta < +\infty\}\,.$$

Hence any value taken by f on $|z| = \sqrt{rR}$ is taken by g on the segment

$$I = \{\zeta : \mathrm{Re}\,\zeta = \mu/2\,,\ -\pi \leq \mathrm{Im}\,\zeta \leq \pi\}\,.$$

Especially $g(\mu/2) = f(z_0)$. The function

$$w(\zeta) = \frac{e^{\pi i \zeta/\mu} - i}{e^{\pi i \zeta/\mu} + i}$$

maps D onto the unit disc $|w| < 1$ conformally and

$$w(I) = \left\{w : -\frac{e^{\pi^2/\mu} - 1}{e^{\pi^2/\mu} + 1} \leq \mathrm{Re}\,w \leq \frac{e^{\pi^2/\mu} - 1}{e^{\pi^2/\mu} + 1}\,,\ \mathrm{Im}\,w = 0\right\}.$$

Since $w(\mu/2) = 0$ and $g(\mu/2) = f(z_0)$, the lemma follows from Schottky's theorem.

Let Σ be the Riemann sphere with radius $1/2$ touching the w-plane at the origin. The chordal distance of the images on Σ of two points w and w' in the plane is denoted by $[w, w']$, and $C(w, \delta)$ is the spherical open disc with centre at the image of w and with chordal radius δ. The following lemma is proved by Matsumoto [5].

Lemma 2. Let f be analytic in an annulus $1 < |z| < e^{\mu}$ and omit the values 0 and 1. There exists a positive constant A such that the spherical diameter of the image curve of $|z| = e^{\mu/2}$ by f is not greater than $Ae^{-\mu/2}$ for all $\mu > 0$.

Let Δ be a triply connected domain with boundary components Γ_1, Γ_2 and Γ_3, and let f be analytic and omit the values 0 and 1 in $\overline{\Delta}$. We assume that the images of Γ_1, Γ_2 and Γ_3 by f are contained in the spherical discs C_1, C_2 and C_3, respectively, and give the following lemma of Matsumoto [5].

Lemma 3. Let $\delta > 0$ be so small that the spherical discs $C(0, 2\delta)$, $C(1, 2\delta)$ and $C(\infty, 2\delta)$ are mutually disjoint. If the radii of C_1, C_2 and C_3 are less than $\delta/2$, only two possibilities can occur:

(1) C_1, C_2 and C_3 contain the origin, the point $w = 1$, and the point at infinity, one by one, so that C_1, C_2 and C_3 are contained in $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$, respectively, and f takes each value outside the union of $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$ once and only once in Δ.

(2) Of C_1, C_2 and C_3 none can be disjoint from the union of the other two, so that there is a disc with radius less than $3\delta/2$ which contains the image of Δ.

3. We consider first a sequence of discs with the middle points lying in a half plane.

Theorem 1. Let D_n, $n = 1, 2, \ldots$, be a sequence of discs with centre z_n, $\operatorname{Re} z_n > 1$, and with radius ϱ_n. If

$$(1) \qquad |z_{n+1}/z_n| > \alpha > 1$$

for $n = 1, 2, \ldots$, and

$$(2) \qquad |z_n| = o(-\log \varrho_n),$$

then $E = \{\infty\} \cup \bigcup_{n=1}^{\infty} D_n$ is a Picard set for entire functions.

Proof. It is obviously sufficient to prove that the assumption of the existence of a function f, analytic and non-rational for $z \neq \infty$, and different from 0 and 1 outside E leads to a contradiction. There is no loss of generality to assume that each D_n contains at least one zero or 1-point of f, for we can delete from $\{D_n\}$ all other discs and the remaining discs also satisfy conditions (1) and (2).

We consider the function $g(z) = f(1/z)$. g is analytic and non-rational for $z \neq 0$. Since $\lim_{n\to\infty} \varrho_n = 0$, we can take $M > 0$ such that the set

$$\{z : |z| > M, \ \operatorname{Re} z < 0\}$$

contains no point of $E - \{\infty\}$. Then by (1) and (2), there exist $0 < \varrho_0 < 1/M$ and a sequence of discs B_n, $n = 1, 2, \ldots$, with centre s_n, $\operatorname{Re} s_n > 0$, and with radius σ_n satisfying the conditions

(3) $$|s_n/s_{n+1}| > \alpha > 1$$

for $n = 1, 2, \ldots,$ and

(4) $$1/(-\log \sigma_n) = o(|s_n|)\,,$$

such that $\bigcup_{n=1}^{\infty} B_n \subset \{z : \operatorname{Re} z > 0\}$, $g(z) \neq 0, 1$ outside $F = \{0\} \cup \bigcup_{n=0}^{\infty} B_n$, where $B_0 = \{z : |z| > \varrho_0\}$, and each B_n contains at least one zero or 1-point of g.

By (3) and (4), we can choose an n_1 so large that the annulus

$$S_n = \{z : \sigma_n < |z - s_n| < |s_n|(\alpha - 1)/2\alpha\}$$

contains no point of F for any $n \geq n_1$. Applying Lemma 2 to S_n we conclude that the spherical diameter of the image of

$$\gamma_n = \{z : |z - s_n| = \sqrt{\sigma_n |s_n|(\alpha - 1)/2\alpha}\}$$

by g is dominated by

(5) $$\delta_n = A\sqrt{2\alpha\sigma_n/(\alpha - 1)|s_n|}$$

for $n \geq n_1$. Hence there exists a spherical disc C_n with radius less than δ_n which contains this image.

We take $\delta > 0$ so small that the discs $C(0, 2\delta)$, $C(1, 2\delta)$ and $C(\infty, 2\delta)$ are mutually disjoint. Since the origin is an essential singularity of g, we have

(6) $$\lim_{r \to 0} M(r) = \infty\,,$$

where $M(r) = \max\{|g(z)| : |z| = r\}$.

By (3) and (4), we can take an $n_2 \geq n_1$ such that the annulus

$$R_n = \{z : |s_n|(2\alpha/(3\alpha - 1))^2 < |z| < 2\alpha|s_n|/(3\alpha - 1)\}$$

contains no point of F for $n \geq n_2$. The modulus of each R_n is $\log((3\alpha - 1)/2\alpha) > 0$. Applying Lemma 1 to R_n, $n \geq n_2$, we see by (6) that the image of

$$\lambda_n = \{z : |z| = |s_n|(2\alpha/(3\alpha - 1))^{3/2}\}$$

by g is contained in $C(\infty, \delta/2)$ for sufficiently large n, say for $n \geq n_3$. We may assume $n_3 \geq n_2$.

By (4) there exists $n_4 \geq n_3$ such that $\delta_n < \delta/4$ for $n \geq n_4$. Applying Lemma 3 to the triply connected domain with λ_n, λ_{n-1} and γ_n as boundary, we see that C_n is contained in $C(\infty, \delta)$ for $n > n_4$.

We choose $n_5 > n_4$ so large that $|s_n| < \varrho_0/2$ for $n \geq n_5$. We apply Schottky's theorem to the disc

$$\{z : |z + \varrho_0/2| < \varrho_0/2\}$$

and get

$$\overset{+}{\log} |g(-|s_n|\{2\alpha/(3\alpha-1)\}^{3/2})| \leq \{7 + \overset{+}{\log} |g(-\varrho_0/2)|\} \varrho_0|s_n|^{-1}\{(3\alpha-1)/2\alpha\}^{3/2}$$

for $n \geq n_5$. We use K_1 and K_2 to denote positive constants depending only on ϱ_0, $\overset{+}{\log} |g(-\varrho_0/2)|$ and α. Applying Lemma 1 to the annulus R_n we get

(7) $$\overset{+}{\log} |g(z)| < K_1 + K_2|s_n|^{-1} = M_n$$

for $z \in \lambda_n$.

We denote by Λ_n the unbounded component of the complement of λ_n. The maximum principle applied to Λ_n yields $\overset{+}{\log} |g(z)| < M_n$ in Λ_n. We take $n_6 \geq n_5$ so large that

$$|s_n| - \sqrt{\sigma_n|s_n|(\alpha-1)/2\alpha} > |s_n|(2\alpha/(3\alpha-1))^{3/2}$$

for any $n \geq n_6$. Then we have $\gamma_n \subset \Lambda_n$, and $g(\gamma_n) \subset T_n$ with

$$T_n = \{w : [w\,,\,\infty] \geq (1 + e^{2M_n})^{-1/2}\}\,.$$

Since $g(\gamma_n) \subset C_n$, we get $C_n \cap T_n \neq \emptyset$.

Instead of (4) we can write

$$|s_n|^{-1} = o(-\log \sigma_n)\,,$$

and this implies by (5) and (7) that there exists $n_7 \geq n_6$ such that

(8) $$\delta_n < 1/4(1 + e^{2M_n})^{1/2}$$

for any $n \geq n_7$.

Since $C_n \cap T_n \neq \emptyset$, we see by (8) that C_n cannot contain the point at infinity for $n \geq n_7$. Then the maximum principle applied to the bounded disc G_n with γ_n as boundary yields $g(G_n) \subset C_n$. Since $B_n \subset G_n$, we get $g(B_n) \subset C_n$. This is a contradiction, for C_n contains no zero or 1-point of g, and the theorem is proved.

4. If we assume that the middle points of the discs D_n need not lie in a half plane, we must replace the condition (2) by a stronger one.

Theorem 2. Let D_n, $n = 1, 2, \ldots$, be a sequence of discs with centre z_n and with radius ϱ_n. If

(a) $$|z_{n+1}/z_n| > \alpha > 1$$

for $n = 1, 2, \ldots$, and

(b) $$|z_n|^n = O(-\log \varrho_n)\,,$$

the $E = \{\infty\} \cup \bigcup_{n=1}^{\infty} D_n$ is a Picard set for entire functions.

Proof. As in the proof of Theorem 1, it is sufficient to prove that the assumption of the existence of a function f, analytic and non-rational for $z \neq \infty$, and different from 0 and 1 outside E, leads to a contradiction.

We consider the function $g(z) = f(1/z)$. g is analytic and non-rational for $z \neq 0$. By (a) and (b), there exist $\varrho_0 > 0$ and a sequence of discs B_n, $n = 1, 2, \ldots,$ with centre t_n and with radius σ_n satisfying the conditions

(c) $$|t_n/t_{n+1}| > \alpha > 1$$

for $n = 1, 2, \ldots,$ and

(d) $$1/(-\log \sigma_n) = O(|t_n|^n)\,,$$

such that $g(z) \neq 0, 1$ outside $F = \{0\} \cup \bigcup_{n=0}^{\infty} B_n$, where $B_0 = \{z : |z| > \varrho_0\}$, and each B_n contains at least one zero or 1-point of g.

We take $\delta > 0$ so small that the spherical disc $C(0, 2\delta)$, $C(1, 2\delta)$ and $C(\infty, 2\delta)$ are mutually disjoint. As in the proof of Theorem 1, we can take n_1 so large that for any $n \geq n_1$ the image of

$$\gamma_n = \{z : |z - t_n| = \sqrt{\sigma_n |t_n| (\alpha - 1)/2\alpha}\}$$

by g is contained in a spherical disc C_n with radius less than

(e) $$\delta_n = A\sqrt{2\alpha\sigma_n/(\alpha - 1)|t_n|}\,,$$

where A is the constant of Lemma 2, and $C_n \subset C(\infty, \delta)$.

We choose $n_2 \geq n_1$ so large that

(f) $$2d = |t_{n_2}| < \min\{\varrho_0\,,\,(2\alpha/(3\alpha - 1))^{3/2}\}\,,$$

(g) $$\sum_{n=n_2}^{\infty} \sigma_n/|t_n| < 1/8\,,$$

and that for any $n \geq n_2$, the annulus

$$R_n = \{z : |t_n|(2\alpha/(3\alpha - 1))^2 < |z| < 2\alpha|t_n|/(3\alpha - 1)\}$$

contains no point of F. We denote $L = \max\{\overset{+}{\log}\,|g(z)| : |z| = d\}$. We take $n_3 \geq n_2$ such that $|t_{n_3}| < d$. Then we see by (g) that there exists for each $n \geq n_3$ a φ_n such that the set

$$\{z : |t_n|(2\alpha/(3\alpha - 1))^2 < |z| < 2d\,,\ |\arg z - \varphi_n| < \pi/2n\}$$

contains no point of F. Considering the function $h(\zeta) = g(\sqrt[n]{\zeta})$ on the disc

$$\{\zeta : |\zeta - d^n e^{in\varphi_n}| < d^n - |t_n|^n (2\alpha/(3\alpha - 1))^{2n}\}$$

we get by Schottky's theorem

$$N_n = \overset{+}{\log} |g(|t_n|\{2\alpha/(3\alpha - 1)\}^{3/2} e^{i\varphi_n})|$$
$$\leq \frac{2d^n(7 + L)}{(|t_n|\{2\alpha/(3\alpha - 1)\}^{3/2})^n - (|t_n|\{2\alpha/(3\alpha - 1)\}^2)^n} .$$

We take $n_4 \geq n_3$ such that

(h) $$N_n \leq 4d^n(7 + L)(|t_n|\{2\alpha/(3\alpha - 1)\}^{3/2})^{-n}$$

for $n \geq n_4$. We apply Lemma 1 to R_n and get

(i) $$\overset{+}{\log} |g(z)| \leq (7 + N_n) \exp\{\pi^2/\log((3\alpha - 1)/2\alpha)\}$$

for $z \in \lambda_n$ with

$$\lambda_n = \{z : |z| = |t_n|(2\alpha/(3\alpha - 1))^{3/2}\} .$$

By the conditions (f), (h) and (i), we get for $z \in \lambda_n$ the estimate

(j) $$\overset{+}{\log} |g(z)| \leq K_1 + K_2(2|t_n|)^{-n} = M_n ,$$

where K_1 and K_2 are positive constants depending only on L and α.

We denote by Λ_n the unbounded component of the complement of λ_n. The maximum principle applied to Λ_n yields $\overset{+}{\log} |g(z)| \leq M_n$ in Λ_n. We take $n_5 \geq n_4$ so large that $\gamma_n \subset \Lambda_n$ and $\gamma_n \cap F = \emptyset$ for $n \geq n_5$. Then we have $g(\gamma_n) \subset T_n$ with

$$T_n = \{w : [w , \infty] \geq (1 + e^{2M_n})^{-1/2}\} ,$$

and $C_n \cap T_n \neq \emptyset$. We get by (d)

$$|t_n|^{-n} = O(- \log \sigma_n) ,$$

and this implies by (e) and (j) that there exists $n_6 \geq n_5$ such that

(k) $$\delta_n < 1/4(1 + e^{2M_n})^{1/2}$$

For any $n \geq n_6$.

Since $C_n \cap T_n \neq \emptyset$, we see by (k) that C_n cannot contain the point at infinity for $n \geq n_6$. Then the maximum principle applied to the bounded disc G_n with γ_n as boundary yields $g(G_n) \subset C_n$. Since $B_n \subset G_n$, we get $g(B_n) \subset C_n$. This is a contradiction, for C_n contains no zero or 1-point of g, and the theorem is proved.

3. Meromorphic functions

5. No theorem like the theorems 1 and 2 is valid for meromorphic functions. We can in fact prove that given any sequence of discs D_n, $n = 1, 2, \ldots$, which converge to the point at infinity, then there exists a function f, meromorphic and non-rational for $z \neq \infty$, and bounded outside

$$E = \{\infty\} \cup \bigcup_{n=1}^{\infty} D_n .$$

We construct a sequence B_n, $n = 1, 2, \ldots$, of discs with centre z_n and with radius ϱ_n satisfying the conditions $|z_1| > 1$, and $B_n \subset \bigcup_{p=1}^{\infty} D_p$,

(1) $$|z_{n+1}/z_n| > e^n ,$$

and

(2) $$\varrho_n/|z_n| < e^{-n}$$

for $n = 1, 2, \ldots$ We denote $r_n = \varrho_n e^{-n}$, and define

$$g(z) = \prod_{n=1}^{\infty} \frac{1 - z/(z_n - r_n)}{1 - z/(z_n + r_n)} .$$

For $z \notin B_n$ we get by (1) and (2)

$$\left| \frac{1 - z/(z_n - r_n)}{1 - z/(z_n + r_n)} \right| \leq 1 + 16 r_n/\varrho_n = 1 + 16 e^{-n} .$$

Then we have for $z \notin \{\infty\} \cup \bigcup_{n=1}^{\infty} B_n$ the estimate

$$|g(z)| \leq \prod_{n=1}^{\infty} (1 + 16 e^{-n}) < \infty .$$

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