Some remarks on weak compactness in the dual space of a JB*-triple

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Abstract

We obtain several characterizations of the relatively weakly compact subsets of the predual of a JBW*-triple. As a consequence we describe the relatively weakly compact subsets of the predual of a JBW*-algebra.

1 Introduction

The study of relatively weakly compact subsets of the predual of a von Neumann algebra is mainly due to Takesaki [24], Akemann [2], Akemann, Dodds and Gamlen [3] and Saitô [22]. Their results on characterizations of relatively weakly compact subset in the predual of a von Neumann algebras were the key tool for the description of weakly compact operators from a C*-algebra to a complex Banach space found by Jarchow [17, 18].

Every von Neumann algebra belongs to a more general class of Banach spaces known as JBW*-triples. A JB*-triple is a complex Banach space equipped with a Jordan triple product satisfying certain algebraic and geometric properties (see definition below). JB*-triples were introduced by Kaup [19] in the study of bounded symmetric domains in complex Banach spaces. The class of JB*-triples contains all C*-algebras and all JB*-algebras. A JBW*-triple is a JB*-triple which is also a dual Banach space, thus every von Neumann is a JBW*-triple.

The study of weakly compact operators from a JB*-triple to a Banach space was obtained in [9] and [21, Theorem 10 and the following remarks]. However, contrarily to the case of a C*-algebra, the characterization of weakly compact operators from a JB*-triple to a complex Banach space

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was not obtained by describing the relatively weakly compact subsets of the predual of a JBW*-triple. the objective of this paper is to describe the relatively weakly compact subsets in the predual of a JBW*-triple. Theorem 2.1 and Corollary 2.4 generalize the classical description of relatively weakly compact subsets in the predual of a von Neumann algebra to the setting of JBW*-triple preduals. The above results are particularized to JBW*-algebra preduals in Theorem 2.5.

As a consequence of our results we prove that for every norm bounded sequence (ϕ_n) in the predual of a JBW*-triple W, then for each norm-one functional $\varphi \in W_*$ and for every $\varepsilon > 0$ there exists a tripotent $e \in W$ such that $\varphi(e) > 1 - \varepsilon$ and (ϕ_n) admits a subsequence which converges weakly to a functional in $(W_2(e))_*$, where $W_2(e)$ is the Peirce 2-subspace associated to e. This result extends [7] to the setting of JBW*-triples.

Let X be a Banach space. Throughout the paper, B_X and X^* denote the closed unit ball of X and the dual space of X, respectively. If X is a dual Banach space, X_* will stand for the predual of X_* .

2 Weakly compact sets in the dual of a JB*-triple

A JB^* -triple is a complex Banach space E equipped with a continuous triple product

$$\{.,.,.\}: E \otimes E \otimes E \to E$$
$$(x,y,z) \mapsto \{x,y,z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies:

(a) (Jordan Identity)

$$L(x,y) \{a,b,c\} = \{L(x,y)a,b,c\} - \{a,L(y,x)b,c\} + \{a,b,L(x,y)c\},$$
 for all $x,y,a,b,c \in E$, where $L(x,y) : E \to E$ is the linear mapping given by $L(x,y)z = \{x,y,z\};$

- (b) The map L(x, x) is an hermitian operator with non-negative spectrum for all $x \in E$;
- (c) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$.

Every C*-algebra is a JB*-triple with respect to the triple product

$${x, y, z} = 2^{-1}(xy^*z + zy^*x).$$

Every JB*-algebra is a JB*-triple with triple product given by

$$\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

The Banach space B(H, K) of all bounded linear operators between two complex Hilbert spaces H, K is also an example of a JB*-triple with product $\{R, S, T\} = 2^{-1}(RS^*T + TS^*R)$.

A JBW*-triple is a JB*-triple which is also a dual Banach space. The bidual, E^{**} , of every JB*-triple, E, is a JBW*-triple with triple product extending the product of E (cf. [11]).

Let E be a JB*-triple. An element $e \in E$ is said to be a tripotent if $\{e, e, e\} = e$. The set of all tripotents of E is denoted by Tri(E). Given a tripotent $e \in E$ there exist a decomposition of E in terms of the eigenspaces of L(e, e) given by

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e), \tag{1}$$

where $E_k(e) := \{x \in \mathcal{E} : L(e,e)x = \frac{k}{2}x\}$ is a subtriple of E(k:0,1,2). The natural projection of E onto $E_k(e)$ will be denoted by $P_k(e)$. The following rules are also satisfied

$${E_k(e), E_l(e), E_m(e)} \subseteq E_{k-l+m}(e),$$

$${E_0(e), E_2(e), E} = {E_2(e), E_0(e), E} = 0,$$

where $E_{k-l+m}(e) = 0$ whenever k-l+m is not in $\{0,1,2\}$. It is also known that $E_2(e)$ is a unital JB*-algebra with respect to the product and involution given by $x \circ y = \{x, e, y\}$ and $x^* = \{e, x, e\}$, respectively. When E is a JBW*-triple then $E_2(e)$ is a JBW*-algebra.

For background about JB- and JBW-algebras the reader is referred to [14]. We recall that JB-algebras (respectively, JBW-algebras) are nothing but the self-adjoint parts of JB*-algebras (respectively, JBW*-algebras) [26] (respectively, [12]).

Two tripotents e, f in a JB*-triple E are said to be orthogonal if e belongs to $E_0(f)$ and f belongs to $E_0(e)$. Let $e, f \in E$. Following [20, §5], we say that $e \leq f$ if and only if f - e is a tripotent which is orthogonal to e. It is also known that $e \leq f$ if and only if e is a symmetric projection in $E_2(f)$.

Let W be a JBW*-triple and let φ a norm-one element in W_* . Let z be a norm-one element in W such that $\varphi(z) = 1$. By [4] the mapping

 $(x,y)\mapsto \varphi\left\{x,y,z\right\}$ defines a positive sesquilinear form on W which does not depend on the element z. Thus the law $x\mapsto \|x\|_{\varphi}:=(\varphi\left\{x,x,z\right\})^{\frac{1}{2}}\ (x\in W)$ defines a prehilbert seminorm on W. If E is a JB*-triple and φ is a norm-one element in E^* then $\|.\|_{\varphi}$ is a prehilbertian seminorm on E^{**} and hence on E. The strong*-topology of W, introduced by Barton and Friedman in [5], is the topology on W generated by the family of seminorms $\{\|.\|_{\varphi}:\varphi\in S_{W_*}\}$. We use the symbol $S^*(W,W_*)$ to denote the strong*-topology of W. When φ_1,φ_2 are two norm-one functionals in W_* , then we write $\|.\|_{\varphi_1,\varphi_2}$ for the hilbertian semi-norm defined by

$$||x||_{\varphi_1,\varphi_2}^2 := ||x||_{\varphi_1}^2 + ||x||_{\varphi_2}^2.$$

If A is a JBW*-algebra regarded as a JB*-triple, then the $S^*(A, A_*)$ coincides with the algebra strong*-topology of A generated by all the seminorms of the form $x \mapsto \sqrt{\phi(x \circ x^*)}$, where ϕ is any normal state in A. Consequently, when a von Neumann algebra M is regarded as a JBW*-triple then the $S^*(M, M_*)$ coincides with the strong*-topology on M (see [23, Definition 1.8.7]).

A JB*-triple E is said to be abelian if for every $x, y, a, b \in E$, the operators L(x, y) and L(a, b) commute. Every abelian JBW*-triple is triple isomorphic (and hence isometric) to a von Neumann algebra.

Let W be a JBW*-triple with predual W_* . Since the triple product of W is separately weak*-continuous (compare [6]), then every maximal abelian subtriple is weak*-closed and hence a JBW*-subtriple of W.

Theorem 2.1. Let W be a JBW^* -triple with predual W_* and let K be a subset in W_* . Then the following are equivalent:

- (a) K is relatively weakly compact.
- (b) There exist norm-one elements $\varphi_1, \varphi_2 \in W_*$ having the following property: given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in W$ with $||x|| \le 1$ and $||x||_{\varphi_1,\varphi_2} < \delta$, then $|\phi(x)| < \varepsilon$ for each $\phi \in K$.
- (c) The restriction $K|_C$ of K to each maximal abelian subtriple C of W is relatively $\sigma(C_*, C)$ -compact.

Proof. $(a) \Rightarrow (b)$ We assume that $K \subset W_*$ is relatively weakly compact. We may also assume that $K \subseteq B_{W_*}$. Let us fix $\varepsilon > 0$. Let $D = \overline{|\operatorname{co}|}^w(K)$, be the weakly closed absolutely convex hull of K in W_* . Then D is an absolutely

convex weakly compact subset of W_* . Let Y denote the Banach space $\ell_1(D)$ and F the bounded linear operator from Y to W_* given by

$$F(\{\lambda_{\varphi}\}_{\varphi \in D}) := \sum_{\varphi \in D} \lambda_{\varphi} \varphi.$$

Clearly $F(B_Y) = D$. Since D is weakly compact then F (and hence F^*) is a weakly compact operator. By [21, Theorem 10] there exist norm-one elements $\varphi_1, \varphi_2 \in W_*$ and a function $N: (0, +\infty) \to (0, +\infty)$ such that

$$||F^*(x)|| \le N(\varepsilon)||x||_{\varphi_1,\varphi_2} + \varepsilon||x||,$$

for all $x \in W$ and $\varepsilon > 0$.

Let x be an element in W. It is clear that

$$\sup_{\phi \in D} |\phi(x)| = \sup_{y \in B_Y} |F(y)(x)| = \sup_{y \in B_Y} |F^*(x)(y)| \le ||F^*(x)||$$

$$\leq N\left(\frac{\varepsilon}{2}\right) \ \|x\|_{\varphi_1,\varphi_2} + \frac{\varepsilon}{2} \|x\|.$$

Finally, taking $\delta = N\left(\frac{\varepsilon}{2}\right)^{-1} \frac{\varepsilon}{2}$, we conclude that for every $x \in W$ with $||x|| \le 1$ and $||x||_{\varphi_1,\varphi_2} \le \delta$ we have $|\phi(x)| \le \varepsilon$ for each $\phi \in K$.

 $(b) \Rightarrow (c)$ Suppose that there exists a maximal abelian subtriple C of W such that $K|_C$ is not relatively $\sigma(C_*, C)$ -compact. Since C is a maximal abelian subtriple then C is weak*-closed and thus C is isomorphic (and hence isometric) to a von Neumann algebra when the latter is considered as a JB*-triple. By [2, Theorem II.2] (see also [25, Theorem 5.4]) there exists an orthogonal sequence (p_n) of symmetric projections in C and a sequence $(\varphi_n) \subseteq K$ satisfying

$$|\varphi_n(p_n)| \ge \Theta > 0. \tag{2}$$

By hypothesis, there are norm-one elements φ_1, φ_2 in W_* and $\delta > 0$ such that for every $x \in W$ with $||x|| \le 1$ and $||x||_{\varphi_1,\varphi_2} < \delta$, then $|\phi(x)| < \frac{\Theta}{2}$ for each $\phi \in K$.

Let ψ be a normal state of C. Since $\psi(p_np_n^*+p_n^*p_n)=2$ $\psi(p_n)$ tends to zero, it follows that (p_n) is a strong*-null sequence in C. By [8, Corollary] we conclude that $(p_n) \to 0$ in the S* (W, W_*) -topology of W. In particular $||p_n||_{\varphi_1,\varphi_2} \to 0$. Therefore, there exists $N \in \mathbb{N}$ such that for every $n \in NN$, $n \geq N$ we have

$$||p_n||_{\varphi_1,\varphi_2} < \delta.$$

As consequence, $|\phi(p_n)| < \frac{\Theta}{2}$, for each $\phi \in K$, which contradicts (2).

 $(c) \Rightarrow (a)$ Suppose that the restriction $K|_C$ of K to each maximal abelian subtriple C of W is relatively $\sigma(C_*, C)$ -compact. Let $x \in W$. The JBW*-subtriple of W generated by x is abelian, by Zorn's Lemma there exists a maximal abelian subtriple C of W containing x. By hypothesis, $K|_C$ is relatively $\sigma(C_*, C)$ -compact, thus $\{\phi(x) : \phi \in K\}$ is bounded. By the uniform boundedness theorem K is bounded. Let \widetilde{K} denote the $\sigma(W^*, W)$ -closure of K in W^* . Since K is bounded then \widetilde{K} is $\sigma(W^*, W)$ -compact.

We claim $\widetilde{K} \subset W_*$. Indeed, let $\phi \in \widetilde{K}$. Let C be any maximal abelian subtriple of W. Then $\phi|_C$ is in the $\sigma(C^*, C)$ -closure of $K|_C$. By assumptions, $K|_C$ is relatively $\sigma(C_*, C)$ -compact and thus $\phi|_C \in C_*$. Now, by [15, Theorem 3.23], it follows that $\phi \in W_*$ as we claimed.

Since $\widetilde{K} \subset W_*$ then \widetilde{K} coincides with the $\sigma(W_*, W)$ -closure of K in W and hence K is relatively $\sigma(W_*, W)$ -compact.

The following corollary extends [10, Lemma 4] (see also [1, Lemma 1]) to general JBW*-triples and it is in fact the natural extension of [25, Lemma III.5.5] to the setting of JBW*-triples.

Corollary 2.2. Let W be a JBW*-triple, let (φ_k) be a weakly convergent sequence in W_* and let (x_n) be a strong-*-null sequence in W. Then

$$\lim_{n \to +\infty} \sup_{k \in \mathbb{N}} |\varphi_k(x_n)| = 0.$$

Proof. Suppose $(\varphi_k) \to \varphi$ weakly in W_* . The set $K = \{\varphi_k : k \in \mathbb{N}\}$ is a relative weakly compact subset of W_* by the Eberlein-Smulian theorem. Let $\varepsilon > 0$. By Theorem 2.1 there are norm-one elements $\varphi_1, \varphi_2 \in W_*$ and $\delta > 0$ such that for every $x \in W$ with $||x|| \le 1$ and $||x||_{\varphi_1, \varphi_2} < \delta$, we have $|\phi(x)| < \varepsilon$ for each $\phi \in K$. Since (x_n) is strong*-null, there exists $N \in \mathbb{N}$ such that for every $n \ge N$ it follows that $||x_n|| \le \delta$. Thus, for every $n \ge N$ we have $|\varphi(x_n)| \le \varepsilon$, for all $f \in K$.

Remark 2.3. Let W be a JBW*-triple. Suppose that $K \in W_*$ is a relatively weakly compact set. Then similar arguments to those given in the proof of Corollary 2.2 show that for each strong*-null sequence (x_n) in W we have

$$\lim_{n \to +\infty} \varphi(x_n) = 0,$$

uniformly for $\varphi \in K$.

Using Theorem 2.1 we generalize to the setting of JBW*-triples some known characterizations of weak compactness in the predual of a W*-algebra (compare [25, Theorem 5.4]).

Corollary 2.4. Let K be a bounded subset in the predual of a JBW^* -triple W. The following assertions are equivalent:

- (a) K is relatively weakly compact.
- (b) The restriction of K to $W_2(e)$ is relatively $\sigma((W_2(e))_*, W_2(e))$ -compact in $(W_2(e))_*$, for every tripotent $e \in W$.
- (c) For any monotone decreasing sequence of tripotents (e_n) in W with $(e_n) \to 0$ in the weak*-topology, we have $\lim_{n \to +\infty} \phi(e_n) = 0$ uniformly for $\phi \in K$.
- *Proof.* $(a) \Rightarrow (b)$ Suppose K is relatively weakly compact in W_* . Let e be a tripotent in W. Since the map: $\phi \to \phi|_{W_2(e)}$ is a weakly continuous operator from W_* to $(W_2(e))_*$, it follows that $K|_{W_2(e)}$ is relatively $\sigma((W_2(e))_*, W_2(e))$ -compact in $(W_2(e))_*$.
- $(b) \Rightarrow (c)$ Let (e_n) be a monotone decreasing sequence in W with $(e_n) \to 0$ in the $\sigma(W, W_*)$ -topology. Since for each natural n, we have $e_1 \geq e_n$, it follows that (e_n) is a monotone decreasing sequence of projections in $W_2(e_1)$ with $(e_n) \to 0$ in the $\sigma(W_2(e_1), (W_2(e_1))_*)$ -topology. It is not hard to see that $(e_n) \to 0$ in the strong-* topology of $W_2(e_1)$. Since, by assumptions, $K|_{W_2(e_1)}$ is relatively $\sigma((W_2(e_1))_*, W_2(e_1))$ -compact, we conclude from Remark 2.3 that $\lim_{n \to +\infty} \phi(e_n) = 0$ uniformly for $\phi \in K$.
- $(c) \Rightarrow (a)$ To obtain a contradiction, suppose that K is not relatively weakly compact. By Theorem 2.1 there exists a maximal abelian JBW*-subtriple C of W such that $K|_C$ is not relatively $\sigma(C_*, C)$ -compact. As we have commented above, C is triple isomorphic to an abelian von Neumann algebra when the latter is regarded as a JBW*-triple. By [25, Theorem 5.4] there exists a monotone decreasing sequence (p_n) of projections in C with $(p_n) \to 0$ in the $\sigma(C, C_*)$ -topology and $\lim_{n \to +\infty} \phi(p_n) \neq 0$ uniformly for $\phi \in K|_C$. Therefore there exists a monotone decreasing sequence (p_n) of tripotents in W with $(p_n) \to 0$ in the weak*-topology of W and $\lim_{n \to +\infty} \phi(p_n) \neq 0$ uniformly for $\phi \in K$, which is a contradiction.

We do not know if the semi-norm $\|.\|_{\varphi_1,\varphi_2}$ appearing in Theorem 2.1 (b) could be replace by a semi-norm of the form $\|.\|_{\varphi}$ for a suitable normone functional $\varphi \in W_*$. This problem is connected with the problem on Grothendieck's inequalities for JB*-triples (compare [21, Remark 3]). We

next show a positive answer to the above problem in the particular case of a JBW*-algebra.

Let M be a JBW*-algebra with predual M_* . Let φ_1, φ_2 be two normone functionals in M_* . For each $i \in \{1, 2\}$ we take a tripotent $e_i \in M$ such that $\varphi_i(e_i) = 1$. Let ψ_i denote the norm-one functional in M_* given by $\psi_i(x) := \varphi_i(x \circ e_i)$ ($\forall x \in M$). From the expression

$$\{x, x, e_i\} + \{x^*, x^*, e_i\} = 2e_i \circ (x \circ x^*),$$

we conclude that ψ_i is a positive normal state of M. Moreover, the identity

$$||x||_{\varphi_i}^2 + ||x^*||_{\varphi_i}^2 = 2\psi_i(x \circ x^*) = 2||x||_{\psi_i}^2$$

holds for all $x \in M$. Set $\psi = \frac{1}{2}(\psi_1 + \psi_2)$. Then ψ is a normal state of M satisfying

$$||x||_{\varphi_1,\varphi_2} \le 2 ||x||_{\psi},$$

for all $x \in M$. We can now reformulate Theorem 2.1 to the setting of JBW*-algebras.

Theorem 2.5. Let M be a JBW^* -algebra. Let K be a norm bounded subset in M_* . The following assertions are equivalent:

- (a) K is relatively weakly compact,
- (b) The restriction $K|_C$ of K to each maximal associative subalgebra C of M is relatively $\sigma(C_*, C)$ -compact.
- (c) there exists a normal state $\psi \in M_*$ having the following property: given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in W$ with $\|x\| \le 1$ and $\|x\|_{\psi} < \delta$, then $|\phi(x)| < \varepsilon$ for each $\phi \in K$.
- (d) For any monotone decreasing sequence of projections (e_n) in W with $(e_n) \to 0$ in the weak*-topology, we have $\lim_{n \to +\infty} \phi(e_n) = 0$ uniformly for $\phi \in K$.

3 Applications

Let (ϕ_n) be a bounded sequence in the predual of a JBW*-triple W. It is known that, in general, (ϕ_n) needs not admit a weakly convergent subsequence. In the setting of von Neumann algebras we can say more about bounded sequences of normal functionals. Indeed, in a recent paper, Brooks,

Saitô and Wright [7] have shown that each bounded sequence in the predual of a von Neumann algebra has a subsequence which is nearly weakly convergent. More concretely, for each bounded sequence (ϕ_n) in the predual of a von Neumann algebra M, for each normal state ψ and for each $\varepsilon > 0$ there exists a projection $e \in M$ such that $\psi(1-e) \le \varepsilon$ and the restriction of (ϕ_n) to eMe has a subsequence which converges weakly to a normal functional on eMe. The aim of this section is to obtain an analogue of the above fact in the setting of JBW*-triples.

The following lemma provides sufficient conditions to assure relative weak compactness in the predual of a JBW*-triple. It is also the natural extension of [7, Lemma 2] to the setting of JBW*-triples.

Lemma 3.1. Let (ϕ_n) be a bounded sequence in the predual of a JBW*-triple W. Let φ be a norm-one element in W_* such that the following property holds: for each c > 0 there exists $\eta > 0$ such that for every tripotent $e \in W$ with $\|e\|_{\varphi} < \eta$ the set

$$\{m \in \mathbb{N} : \exists u \in Tri(W) \text{ with } u \leq e \text{ and } |\phi_m(u)| \geq c\}$$

is finite. Then $\{\phi_n : n \in \mathbb{N}\}\$ is relatively weakly compact in W_* .

Proof. Let (e_n) be a weak*-null, monotone decreasing sequence of tripotents in W. Let c > 0 and let $\eta > 0$ be the positive given by the property.

Since for each $n \in \mathbb{N}$, $e_1 \geq e_n$ we conclude that (e_n) is a weak*-null, monotone decreasing sequence of projections in $W_2(e_1)$. As we have commented in the above section, it is not hard to see that (e_n) is strong*-null in $W_2(e)$ and from [8, Corollary] (e_n) is strong*-null in W. In particular $||e_n||_{\varphi} \to 0$. Then there exists $m_1 \in \mathbb{N}$ such that for each $n \geq m_1$ we have $||e_n||_{\varphi} < \eta$. Since the set

$$\{m \in \mathbb{N} : |\phi_m(e_n)| \ge c \text{ for some } n \ge m_1\}$$

is finite by hypothesis, we conclude that there exists $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$ we have $|\phi_m(e_n)| < c$ for every natural $n \geq m_1$.

Since for each $j:1...m_0$ the sequence $(\phi_j(e_n))_{n\in\mathbb{N}}$ tends to zero we deduce that there exists $m_2\in\mathbb{N}$ such that for each $n\geq m_2$ and $j:1,\ldots,m_0$ we have $|\phi_j(e_n)|< c$. Therefore, for each $n\geq \max\{m_1,m_2\}$, we have $|\phi_m(e_n)|< c$ for all $m\in\mathbb{N}$. Corollary 2.4 gives the desired statement. \square

When in the proof of Lemma 3.1, Theorem 2.5 replaces Theorem 2.1 we obtain the following.

Lemma 3.2. Let M be a JBW*-algebra and let (ϕ_n) be a bounded sequence in M_* . Let φ be a normal state of M such that the following property holds: for each c > 0 there exists $\eta > 0$ such that for every projection $e \in M$ with $\|e\|_{\varphi} < \eta$ the set

$$\left\{m \in \mathbb{N} : \text{ there exists a projection } p \in M \text{ with } p \leq e \text{ and } |\phi_m(p)| \geq c\right\}$$
 is finite. Then $\{\phi_n : n \in \mathbb{N}\}$ is relatively weakly compact in M_* .

Let M be a JBW*-algebra. Let φ be a positive normal functional on M and let (ϕ_n) be a norm-bounded sequence in M_* . We shall denote by Δ the set of all $c \in \mathbb{R}^+$ such that for each $\eta > 0$ there exists a projection $e_{\eta} \in W$ such that $||e_{\eta}||_{\varphi} < \eta$ and the set

$$\left\{m\in\mathbb{N}: \text{ there exists a projection } p\in M \text{ with } p\leq e_{\eta} \text{ and } |\phi_m(p)|\geq c\right\}$$

is infinite. Following [7, Definition in page 162], we call Δ the *anti-compactness set* of (ϕ_n) with respect to the functional φ . It is clear that Δ is bounded.

Remark 3.3. Let M be a JBW*-algebra. Let φ be a positive functional in M_* , (ϕ_n) a norm-bounded sequence in M_* , and Δ the anti-compactness set of (ϕ_n) with respect to φ . We claim that (ϕ_n) is relatively weakly compact in M_* whenever $\Delta = \emptyset$. Indeed, let $c \in \mathbb{R}^+$. Since $c \notin \Delta$ there exists $\eta > 0$ such that for every projection $e \in M$ with $\|e\|_{\varphi} < \eta$ the set

$$\left\{ m \in \mathbb{N} : \text{ there exists a projection } p \in M \text{ with } p \leq e \text{ and } |\phi_m(p)| \geq c \right\}$$

is finite. We conclude from Lemma 3.2 that (ϕ_n) is relatively weakly compact in W_* .

We recall that a positive functional ψ of a JB*-algebra A is said to be faithful if and only if $\psi(x) > 0$ for every positive element $x \in A \setminus \{0\}$. Suppose that a JBW*-algebra M has a faithful normal state ψ . Then the strong*-topology in the closed unit ball of M is metrized by the distance

$$d_{\psi}(a,b) := (\psi((a-b) \circ (a-b)^*))^{\frac{1}{2}}.$$

More precisely, a bounded net $(x_i)_{i\in I}$ in M converges in the strong*-topology of M to an element $x \in M$ if and only if $d_{\psi}(x_i, x) \to 0$ (compare [16, page 200]). When M is regarded as a JBW*-triple we have $d_{\psi}(a, b) = ||a - b||_{\psi}$.

The following lemma is a verbatim extension of [7, Lemmma 3] to the setting of JBW*-algebras.

Lemma 3.4. Let M be a JBW*-algebra having a faithful positive normal functional ψ . Let (ϕ_n) be a norm bounded sequence in M_* and let Δ be the anti-compactness set of (ϕ_n) with respect to ψ , considering M as a JBW*-triple. Then (ϕ_n) is relatively weakly compact in M_* if and only if $\Delta = \emptyset$.

We sketch the main ideas of the proof for completeness. We have already shown that $\Delta = \emptyset$ implies (ϕ_n) is relatively weakly compact in M_* (compare Remark 3.3).

To prove the if-implication we suppose, contrary to our claim, that $\Delta \neq \emptyset$. There is no loss of generality in assuming $\|\psi\| = 1$. Let $c \in \Delta$. Then for each $k \in \mathbb{N}$ there exists a tripotent $e_k \in M$ satisfying $\|e_k\|_{\psi} < 2^{-k}$ and the set

$$\left\{ m \in \mathbb{N} : \exists u \in \operatorname{Tri}(M) \text{ with } u \le e_k \text{ and } |\phi_m(u)| \ge c \right\}$$
 (3)

is infinite. Thus (e_k) is a bounded sequence in M satisfying

$$d_{\psi}(e_k,0) = ||e_k||_{\psi} \to 0.$$

Since ψ is a faithful normal state of M, and the strong*-topology of M is determined by the metric d_{ψ} , we deduce that (e_k) tends to zero in the strong*-topology of M.

Since, by assumptions, (ϕ_n) is relatively weakly compact, then by Theorem 2.1 there exist norm-one functionals $\varphi_1, \varphi_2 \in M_*$ and $\delta > 0$ satisfying that for every $x \in M$ with $||x|| \le 1$ and $||x|||_{\varphi_1,\varphi_2} < \delta$ we have $||\phi_n(x)|| \le \frac{c}{2}$ for all $n \in \mathbb{N}$. Since $(e_k) \to 0$ in the strong*-topology, there exists $k_0 \in \mathbb{N}$ such that $\forall k \ge k_0$ we have $||e_k||_{\varphi_1,\varphi_2} < \delta$. Let $k \ge k_0$. It is not hard to see that (from the orthogonality of u and $e_k - u$) for each tripotent $u \le e_k$ we have $||u||_{\varphi_1,\varphi_2} \le ||e_k||_{\varphi_1,\varphi_2} < \delta$. Consequently, $||\phi_n(u)|| \le \frac{c}{2}$ for all $n \in \mathbb{N}$, which contradicts (3).

Having the above facts in mind, the proof of [7, Proposition 4] can be lightly adapted to prove the following result.

Proposition 3.5. Let M be a JBW*-algebra having a faithful positive functional ψ . Let (ϕ_n) be a norm bounded sequence in M_* . Then for every $\varepsilon > 0$ there exists a projection $p \in M$ such that $\psi(p) < \varepsilon$ and there is a subsequence of ϕ_n , (β_n) , such that the sequence (β_n) restricted to $P_2(1-p)(M)$ is relatively weakly compact.

Let φ be a norm-one functional in the predual of a JBW*-triple W. By [13, Proposition 2], there exists a unique tripotent $e = e(\varphi) \in W$ such that $\varphi = \varphi P_2(e)$ and $\varphi|_{W_2(e)}$ is a faithful normal state of the JBW*-algebra $W_2(e)$. This unique tripotent $e = e(\varphi)$ is called the *support tripotent* of φ .

We can now state the analogue of [7, Theorem 8] in the setting of JBW*-triples.

Theorem 3.6. Let W be a JBW^* -triple. Let φ be a norm-one element in W_* and let (ϕ_n) be a norm bounded sequence in W_* . Then for each $1 > \varepsilon > 0$ there exists a tripotent $e \in W$ such that $\|e\|_{\varphi} > 1 - \varepsilon$ and there is a subsequence $(\phi_{\sigma(n)})$ such that $(\phi_{\sigma(n)}|_{W_2(e)})$ is relatively $\sigma((W_2(e))_*, W_2(e))$ -compact in $(W_2(e))_*$.

Proof. Let $s = s(\varphi)$ be the support tripotent of φ . Let $\varepsilon > 0$. By Proposition 3.5 there exists a projection $p \in W_2(s)$ such that $\varphi(p) < \varepsilon$ and there is a subsequence $(\phi_{\sigma(n)})$ such that $(\phi_{\sigma(n)})$ restricted to $P_2(s-p)(W)$ is relatively weakly compact. We take e = s - p to obtain the desired statement. \square

When in the proof of [7, Corollaries 9,10] we replace [7, Theorem 8] by Theorem 3.6 we obtain:

Corollary 3.7. Let φ be a norm-one functional in the predual of a JBW*-triple W. Let (ϕ_n) be a norm bounded sequence in W_* and let $s = s(\varphi)$ be the support tripotent of φ . Then there exists a sequence of tripotents (e_k) (with $e_k \leq s$ for each $k \in \mathbb{N}$) which converges in the strong*-topology to s and there is a subsequence of (ϕ_n) , $(\phi_{\sigma(n)})$, such that

$$\lim_{n \to +\infty} \phi_{\sigma(n)} P_2(e_k)(x)$$

exists for each $x \in W$.

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