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SOME REMARKS TO MULTIVARIATE REGRESSION MODEL*

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Abstract. Some remarks to problems of point and interval estimation, testing and problems of outliers are presented in the case of multivariate regression model.

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MSC 2000: 62J05

1. Introduction

There are many papers and books on statistical problems and their solution in multivariate regression models. Nevertheless, many problems have not yet been formulated and solved.

The aim of the paper is to make some remarks to them and present solutions of several problems in the regular form of the multivariate model.

2. Notation and preliminaries

Let $\underline{\mathbf{Y}}$ mean an $n \times m$ random matrix (observation matrix) with the mean value $E(\underline{\mathbf{Y}}) = \mathbf{XB}$ and the covariance matrix $\operatorname{Var}[\operatorname{vec}(\underline{\mathbf{Y}})] = \mathbf{\Sigma} \otimes \mathbf{I}$. Here $\mathbf{X}$ is an $n \times k$ known matrix and $\mathbf{B}$ is a $k \times m$ matrix of unknown parameters. The $m \times m$ matrix $\mathbf{\Sigma}$ can be either totally known, or it is of the form $\mathbf{\Sigma} = \sigma^2 \mathbf{V}$, where $\sigma^2 \in (0, \infty)$ is an unknown parameter and the $m \times m$ positive definite matrix $\mathbf{V}$ is known, or $\mathbf{\Sigma}$ is of the form $\mathbf{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$, where $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_p)'$ is an unknown vector, $\boldsymbol{\vartheta} \in \underline{\vartheta} \subset \mathbb{R}^p$, $\underline{\vartheta}$ is an open set and the $m \times m$ symmetric matrices $\mathbf{V}_1, \ldots, \mathbf{V}_p$ are known, or $\mathbf{\Sigma}$ is

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totally unknown. The notation $\underline{\mathbf{Y}} = (\mathbf{Y}_1, \dots, \mathbf{Y}_m)$ and $\mathrm{vec}(\underline{\mathbf{Y}}) = (\mathbf{Y}_1', \mathbf{Y}_2', \dots, \mathbf{Y}_m')'$ will be used throughout the paper.

The model will be written as

$$\underline{\mathbf{Y}} \sim_{nm} (\mathbf{XB}, \boldsymbol{\Sigma} \otimes \mathbf{I}). \tag{1}$$

This model is regular if the rank $r(\mathbf{X})$ of the matrix $\mathbf{X}$ satisfies $r(\mathbf{X}) = k < n$ and the matrix $\boldsymbol{\Sigma}$ is positive definite.

Throughout the paper the model (1) is assumed to be regular.

The symbol $\mathbf{P}_X$ denotes the projection matrix on the subspace $\mathcal{M}(\mathbf{X}) = \{\mathbf{Xu}\colon \mathbf{u} \in \mathbb{R}^k\}$, i.e. $\mathbf{P}_X = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and $\mathbf{M}_X = \mathbf{I} - \mathbf{P}_X$. Further $\{\mathbf{A}\}_{i,j}$ denotes the (i,j)th entry of the matrix $\mathbf{A}$, $\{\mathbf{A}\}_{i,\cdot}$ denotes the ith row of the matrix $\mathbf{A}$ and $\{\mathbf{A}\}_{\cdot,j}$ is the jth column. Notation $\chi^2_f(0)$ means the random variable with central chi-square distribution and with the degrees of freedom equal to f, $\chi^2_f(\delta)$ means the random variable with noncentral chi-square distribution with f degrees of freedom and with parameter noncentrality equal to δ. The $(1-\alpha)$-quantile of the central chi-square distribution is denoted by $\chi^2_f(0; 1-\alpha)$. The $(1-\alpha)$-quantile of the Student distribution with f degrees of freedom is $t_f(1-\alpha)$ and the $(1-\alpha)$-quantile of the central Fisher-Snedecor distribution with (f,g) degrees of freedom is $F_{f,g}(0; 1-\alpha)$. The random variable with central Fisher-Snedecor distribution is $F_{f,g}(0)$, with noncentral distribution it is $F_{f,g}(\delta)$, where δ is the noncentrality parameter.

The two following lemmas are well known, therefore they are given without proofs.

Lemma 2.1. *The parametr matrix* $\mathbf{B}$ *is unbiasedly estimable and its* BLUE *(best linear unbiased estimator) is*

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}$$

and its covariance matrix is

$$\mathrm{Var}[\mathrm{vec}(\hat{\mathbf{B}})] = \boldsymbol{\Sigma} \otimes (\mathbf{X}'\mathbf{X})^{-1}.$$

In the case of normality $\hat{\mathbf{B}}$ *is the best unbiased estimator.*

Lemma 2.2. *The unbiased estimator of the parameter* σ^2 *is*

$$\hat{\sigma}^2 = \mathrm{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{V}^{-1})/[m(n-k)].$$

In the case of normality it is the best unbiased estimator and

$$\hat{\sigma}^2 \sim \sigma^2\chi^2_{m(n-k)}(0)/[m(n-k)], \quad \mathrm{Var}(\hat{\sigma}^2) = 2\sigma^4/[m(n-k)].$$

Lemma 2.3. *Let $\boldsymbol{\vartheta}^{(0)}$ be an approximate value of the unknown vector $\boldsymbol{\vartheta}$. If $\mathbf{g} \in \mathcal{M}(\mathbf{S}_{\Sigma_0^{-1}})$, where $\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p} \vartheta_i^{(0)} \mathbf{V}_i$ and*

$$\{\mathbf{S}_{\Sigma_0^{-1}}\}_{i,j} = \operatorname{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j), \quad i,j = 1,\dots,p,$$

*then the $\boldsymbol{\vartheta}^{(0)}$-*MINQUE *(minimum norm quadratic unbiased estimator) of the function $\mathbf{g}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$ is*

$$\widehat{\mathbf{g}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i \operatorname{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}), \quad (n-k)\mathbf{S}_{\Sigma_0^{-1}}\boldsymbol{\lambda} = \mathbf{g}.$$

If the matrix $\mathbf{S}_{\Sigma_0^{-1}}$ is regular, then

$$\hat{\boldsymbol{\vartheta}} = \frac{1}{n-k}\mathbf{S}_{\Sigma_0^{-1}}^{-1} \begin{pmatrix} \operatorname{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}) \\ \vdots \\ \operatorname{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_1\boldsymbol{\Sigma}_0^{-1}) \end{pmatrix}.$$

In the case of normality $\hat{\boldsymbol{\vartheta}}$ is the $\boldsymbol{\vartheta}_0$-locally best quadratic unbiased estimator and

$$\operatorname{Var}(\hat{\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}^{(0)}) = \frac{2}{n-k}\mathbf{S}_{\Sigma_0^{-1}}^{-1}.$$

Proof. In the univariate regular model $\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sum_{i=1}^{p} \vartheta_i\mathbf{V}_i)$, $\boldsymbol{\beta} \in \mathbb{R}^k$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, the MINQUE of a function $\mathbf{g}'\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta} \in \underline{\vartheta}$, is

$$\widehat{\mathbf{g}'\boldsymbol{\vartheta}} = \sum_{i=1}^{p} \lambda_i \mathbf{Y}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{Y}, \quad \mathbf{S}_{(M_X\Sigma_0M_X)^+}\boldsymbol{\lambda} = \mathbf{g},$$

$$\{\mathbf{S}_{(M_X\Sigma_0M_X)^+}\}_{i,j} = \operatorname{Tr}[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_i(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}_j], \ i,j = 1,\dots,p.$$

If $\mathbf{Y}$ is normally distributed, then $\hat{\boldsymbol{\vartheta}}$ is the $\boldsymbol{\vartheta}_0$-locally best quadratic unbiased estimator and

$$\operatorname{Var}(\widehat{\mathbf{g}'\boldsymbol{\vartheta}}|\boldsymbol{\vartheta}_0) = 2\mathbf{g}'\mathbf{S}_{(M_X\Sigma_0M_X)^+}^{-1}\mathbf{g}.$$

(In more detail cf. [13].)

If the relations

$$[\mathbf{M}_{I\otimes X}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\mathbf{M}_{I\otimes X}]^+ = \boldsymbol{\Sigma}_0^{-1} \otimes \mathbf{M}_X,$$
$$\operatorname{Tr}\{[\mathbf{M}_{I\otimes X}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\mathbf{M}_{I\otimes X}]^+(\mathbf{V}_i \otimes \mathbf{I})[\mathbf{M}_{I\otimes X}(\boldsymbol{\Sigma}_0 \otimes \mathbf{I})\mathbf{M}_{I\otimes X}]^+(\mathbf{V}_i \otimes \mathbf{I})\}$$
$$= \operatorname{Tr}[(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j) \otimes \mathbf{M}_X] = (n-k)\operatorname{Tr}(\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_i\boldsymbol{\Sigma}_0^{-1}\mathbf{V}_j)$$

are taken into account, the statement can be easily obtained. □

Lemma 2.4. *In the case of normality,* $\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}} \sim W_m(n-k,\mathbf{\Sigma})$ *(Wishart distribution with* $n-k$ *degrees of freedom) and* $\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}$ *and* $\widehat{\mathbf{B}}$ *are stochastically independent. Thus the estimators* $\hat{\sigma}^2$ *and* $\widehat{\mathbf{B}}$ *and also the estimators* $\hat{\boldsymbol{\vartheta}}$ *and* $\widehat{\mathbf{B}}$ *are stochastically independent.*

Proof. Cf. [1]. □

3. Confidence regions

The $(1-\alpha)$-confidence ellipsoid for the matrix $\mathbf{GBH}$ ($\mathbf{G}$ is an $r \times k$ matrix with full rank in rows and $\mathbf{H}$ is an $m \times s$ matrix with full rank in columns) can be obtained in a standard way if the matrix $\mathbf{\Sigma}$ is either known or is of the form $\sigma^2\mathbf{V}$. In more detail cf. also [1] and [14]. If $\mathbf{\Sigma}$ is unknown, then the following lemma can be of some interest.

Lemma 3.1. *Let the matrix* $\mathbf{\Sigma}$ *be unknown.*

(i) *If* $\mathbf{G} = \mathbf{g}'$ *($1 \times k$ row vector), then the* $(1-\alpha)$*-confidence region for the vector* $\mathbf{g}'\mathbf{BH}$ *is*

$$\begin{aligned}\mathcal{E} = \{\mathbf{u} \in \mathbb{R}^s\colon\ &(\mathbf{u}' - \mathbf{g}'\widehat{\mathbf{B}}\mathbf{H})(\mathbf{H}'\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{H})^{-1}(\mathbf{u} - \mathbf{H}'\widehat{\mathbf{B}}'\mathbf{g})\\ &\leqslant \frac{s}{n-k-s+1}\mathbf{g}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{g}F_{s,n-k-s+1}(0;1-\alpha)\}.\end{aligned}$$

(ii) *If* $\mathbf{H} = \mathbf{h}$ *($m \times 1$ column vector), then the* $(1-\alpha)$*-confidence region for the vector* $\mathbf{GBh}$ *is*

$$\begin{aligned}\mathcal{E} = \Big\{\mathbf{u} \in \mathbb{R}^r\colon\ &\frac{(\mathbf{u} - \mathbf{G}\widehat{\mathbf{B}}\mathbf{h})'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{u} - \mathbf{G}\widehat{\mathbf{B}}\mathbf{h})}{\mathbf{h}'\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{h}}\\ &\leqslant \frac{r}{n-k}F_{r,n-k}(0;1-\alpha)\Big\}.\end{aligned}$$

Proof. (i) The random vector

$$\boldsymbol{\eta} = \frac{\mathbf{h}'(\mathbf{B} - \widehat{\mathbf{B}})\mathbf{g}}{\sqrt{\mathbf{g}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{g}}} \sim N_s(\mathbf{0}, \mathbf{H}'\mathbf{\Sigma}\mathbf{H})$$

and the Wishart matrix

$$\mathbf{W} = \mathbf{H}'\underline{\mathbf{Y}}'\mathbf{M}_\mathbf{X}\underline{\mathbf{Y}}\mathbf{H} \sim \mathbf{W}_\mathbf{s}(\mathbf{n}-\mathbf{k}, \mathbf{H}'\mathbf{\Sigma}\mathbf{H})$$

are stochastically independent. The statement (i) follows from the Hotelling theorem [11]

$$\boldsymbol{\eta}'\mathbf{W}^{-1}\boldsymbol{\eta} \sim \frac{\chi^2_s[E(\boldsymbol{\eta}')(\mathbf{H}'\boldsymbol{\Sigma}^{-1}\mathbf{H})^{-1}E(\boldsymbol{\eta})]}{\chi^2_{n-k-s+1}(0)}.$$

(ii) The random vector

$$\boldsymbol{\eta} = \mathbf{G}(\mathbf{B} - \widehat{\mathbf{B}})\mathbf{h} \sim N_r[\mathbf{0}, \mathbf{h}'\boldsymbol{\Sigma}\mathbf{h}\mathbf{G}(\mathbf{X}'\mathbf{X}^{-1}\mathbf{G}']$$

and the random variable

$$W = \mathbf{h}'\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{h} \sim \mathbf{h}'\boldsymbol{\Sigma}\mathbf{h}\chi^2_{n-k}(0)$$

are stochastically independent and thus the statement (ii) is obvious. □

Some remarks to the case $\boldsymbol{\Sigma} = \sum_{i=1}^{p} \vartheta_i \mathbf{V}_i$ is postponed to Section 5.

4. Testing linear hypotheses

Let in this section the matrix $\underline{\mathbf{Y}}$ be normally distributed and the matrices $\mathbf{G}$ and $\mathbf{H}$ be the same as in the preceding section.

In many situations the null and the alternative hypotheses are of the form H_0: $\mathbf{GB} + \mathbf{H}_0 = \mathbf{0}$ and H_a: $\mathbf{GB} + \mathbf{H}_0 \neq \mathbf{0}$, respectively. In such a case C. R. Rao (in more detail cf. [11], Chapt. 8) proposed to modify the problem in the following way. Let $\mathbf{l} \in \mathbb{R}^m$ be an arbitrary vector, $\boldsymbol{\beta}_l = \mathbf{Bl}$, $\mathbf{h}_{0,l} = \mathbf{H}_0\mathbf{l}$, $\sigma_l^2 = \mathbf{l}'\boldsymbol{\Sigma}\mathbf{l}$, $\hat{\sigma}_l^2 = \mathbf{l}'\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{l}/(n-k) \sim \sigma^2\chi^2_{n-k}(0)/(n-k)$. Then the $\mathbf{l}$-class of test statistics for the $\mathbf{l}$-class of hypotheses $H_{0,l}$: $\mathbf{G}\boldsymbol{\beta}_l + \mathbf{h}_{0,l} = \mathbf{0}$ versus $\mathbf{G}\boldsymbol{\beta}_l + \mathbf{h}_{0,l} \neq \mathbf{0}$ is

$$(\mathbf{G}\hat{\boldsymbol{\beta}}_l + \mathbf{h}_{0,l})'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}]^{-1}(\mathbf{G}\hat{\boldsymbol{\beta}}_l + \mathbf{h}_{0,l}) \sim \hat{\sigma}_l^2 r F_{r,n-k}(\delta),$$
$$\delta = (\mathbf{G}\boldsymbol{\beta}_l + \mathbf{h}_{0,l})'[\mathbf{G}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\boldsymbol{\beta}_l + \mathbf{h}_{0,l})/\sigma_l^2.$$

If $\mathbf{l}$ belongs to the minimum eigenvalue $\lambda_{\min}$ of the matrix $\boldsymbol{\Sigma}$, then this test is the most sensitive test of all statistics from the $\mathbf{l}$-class.

Analogously the $\mathbf{k}$-class of test statistics can be constructed for the hypotheses H_0: $\mathbf{BH} + \mathbf{H}_0 = \mathbf{0}$ versus H_0: $\mathbf{BH} + \mathbf{H}_0 \neq \mathbf{0}$ even in the case that $\boldsymbol{\Sigma}$ is totally unknown.

Lemma 4.1. *Let* $\mathbf{\Sigma}$ *be unknown and* $\mathbf{G} = \mathbf{I}$. *Then a* $\mathbf{k}$*-class of test statistics is*

$$\frac{(\mathbf{H}'\widehat{\mathbf{B}}'\mathbf{k} + \mathbf{H}_0'\mathbf{k})'[\mathbf{H}'\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{H}]^{-1}(\mathbf{H}'\widehat{\mathbf{B}}'\mathbf{k} + \mathbf{H}_0'\mathbf{k})}{\mathbf{k}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{k}}$$
$$\sim \frac{s}{n-k-s+1} F_{s,n-k-s+1}(\delta),$$
$$\delta = \frac{(\mathbf{H}'\widehat{\mathbf{B}}'\mathbf{k} + \mathbf{H}_0'\mathbf{k})'[\mathbf{H}'\mathbf{\Sigma}\mathbf{H}]^{-1}(\mathbf{H}'\widehat{\mathbf{B}}'\mathbf{k} + \mathbf{H}_0'\mathbf{k})}{\mathbf{k}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{k}},$$

where $\mathbf{k}$ *is any vector from* $\mathbb{R}^k$.

P r o o f follows from the relations

$$(\mathbf{H}' \otimes \mathbf{k}') \operatorname{vec}(\widehat{\mathbf{B}}) \sim N_s[(\mathbf{H}' \otimes \mathbf{k}') \operatorname{vec}(\mathbf{B}), \mathbf{k}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{k}\mathbf{H}'\mathbf{\Sigma}\mathbf{H}],$$
$$\mathbf{H}'\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{H} \sim W_s(n-k, \mathbf{H}'\mathbf{\Sigma}\mathbf{H})$$

and from the stochastical independence of $\widehat{\mathbf{B}}$ and $\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}$. Now the Hotelling theorem is used in order to complete the proof. □

5. Outliers

To reveal an outlier observation in the multivariate model is in general a more complicated problem than in the univariate one. Several approaches to solution have been studied; in more detail cf., e.g. [3, p. 292–317]. One approach is described in the following text. It is based on the idea that the observation $Y_{i,j} + \Delta_{i,j}$ is made instead of the suspicious observation $Y_{i,j}$. To verify this assumption the test of the hypothesis $\Delta_{i,j} = 0$ is performed.

The residual matrix $\underline{\mathbf{v}}$ is given by the relation $\underline{\mathbf{v}} = \underline{\mathbf{Y}} - \mathbf{X}\widehat{\mathbf{B}}$ and the covariance matrix $\operatorname{Var}[\operatorname{vec}(\underline{\mathbf{v}})]$ of the vector $\operatorname{vec}(\underline{\mathbf{v}})$ is $\operatorname{Var}[\operatorname{vec}(\underline{\mathbf{v}})] = \mathbf{\Sigma} \otimes \mathbf{M}_X$.

A value $v_{i,j} = \{\mathbf{Y}_i\}_j - \{\mathbf{X}\}_{j,\cdot}\{\widehat{\mathbf{B}}\}_{\cdot,i}$, $i = 1, \dots, m$; $j = 1, \dots, n$, is suspicious if

$$|v_{i,j}|/\sqrt{\sigma_{i,i}\{\mathbf{M}_X\}_{j,j}} > u(1 - \tfrac{1}{2}\alpha), \tag{2}$$

where $u\big(1 - \frac{1}{2}\alpha\big)$ is the $(1 - \frac{1}{2}\alpha)$-quantile of the normal distribution $N(0,1)$ for a sufficiently small value α.

If $\mathbf{\Sigma} = \sigma^2\mathbf{V}$, then instead of (2) the inequality

$$|v_{i,j}|/\sqrt{\hat{\sigma}^2 V_{i,i}\{\mathbf{M}_X\}_{j,j}} > t_{m(n-k)}(1 - \tfrac{1}{2}\alpha)$$

is used. Here $\hat{\sigma}^2 = \operatorname{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{V}^{-1})/[m(n-k)]$ and $t_{m(n-k)}(1-\frac{1}{2}\alpha)$ is the $(1-\frac{1}{2}\alpha)$-quantile of the Student distribution with $m(n-k)$ degrees of freedom.

If $\mathbf{\Sigma}$ is unknown, it can be estimated as $\widehat{\mathbf{\Sigma}} = (n-k)^{-1}\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}$, $(n-k)\widehat{\mathbf{\Sigma}} \sim W_m(n-k, \mathbf{\Sigma})$, and the inequality

$$|v_{i,j}|/\sqrt{\{\widehat{\mathbf{\Sigma}}\}_{i,i}\{\mathbf{M}_X\}_{j,j}} > t_{n-k}(1-\tfrac{1}{2}\alpha)$$

indicates the possibility that an outlier occurs in the measurement $\{\mathbf{Y}_i\}_j$.

Let residuals $v_{1,j_1^{(1)}}, \ldots, v_{1,j_{s_1}^{(1)}}, v_{2,j_1^{(2)}}, \ldots, v_{2,j_{s_2}^{(2)}}, \ldots, v_{m,j_1^{(m)}}, \ldots, v_{m,j_{s_m}^{(m)}}$ be suspicious. Now instead of the model

$$\operatorname{vec}(\underline{\mathbf{Y}}) \sim N_{nm}[(\mathbf{I}\otimes\mathbf{X})\operatorname{vec}(\mathbf{B}), \mathbf{\Sigma}\otimes\mathbf{I}],$$

the model

$$\operatorname{vec}(\underline{\mathbf{Y}}) \sim N_{nm}\left[(\mathbf{I}\otimes\mathbf{X}, \mathbf{E})\begin{pmatrix}\operatorname{vec}(\mathbf{B})\\ \mathbf{\Delta}\end{pmatrix}, \mathbf{\Sigma}\otimes\mathbf{I}\right]$$

is considered. Here

$$\mathbf{E} = \begin{pmatrix}\mathbf{E}_1, & \mathbf{0}, & \ldots, & \mathbf{0}\\ \mathbf{0}, & \mathbf{E}_2, & \ldots, & \mathbf{0}\\ \ldots & \ldots & \ldots & \ldots\\ \mathbf{0}, & \mathbf{0}, & \ldots, & \mathbf{E}_m\end{pmatrix}, \quad \mathbf{E}_i = \begin{pmatrix}\left(\mathbf{e}_{j_1^{(i)}}^{(s_i)}\right)'\\ \vdots\\ \left(\mathbf{e}_{j_{s_i}^{(i)}}^{(s_i)}\right)'\end{pmatrix}, \tag{3}$$

$$r(\mathbf{E}_i) = s_i, \quad i = 1, \ldots, m,$$

and $\mathbf{\Delta} = (\mathbf{\Delta}_1', \ldots, \mathbf{\Delta}_m')'$, $\mathbf{\Delta}_i \in \mathbb{R}^{s_i}$, $i = 1, \ldots, m$, $s = s_1 + \ldots + s_m$. The symbol $\mathbf{e}_i^{(n)}$ means the n-dimensional vector with 1 at the ith position and with other components equal to 0.

Let the null and alternative hypotheses be

$$H_0\colon\ \mathbf{\Delta} = \mathbf{0} \quad \text{and} \quad H_a\colon\ \mathbf{\Delta} \neq \mathbf{0}.$$

The hypothesis can be tested iff $\mathcal{M}\begin{pmatrix}\mathbf{0}_{(km,s)}\\ \mathbf{I}_{s,s}\end{pmatrix} \subset \mathcal{M}\begin{pmatrix}\mathbf{I}\otimes\mathbf{X}'\\ \mathbf{E}'\end{pmatrix}$. This follows by the following consideration. Let a univariate model $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \mathbf{\Sigma})$, $\boldsymbol{\beta} \in \mathbb{R}^k$, and the hypothesis $H_0\colon\ \mathbf{H}\boldsymbol{\beta} + \mathbf{h} = \mathbf{0}$ versus $H_a\colon\ \mathbf{H}\boldsymbol{\beta} + \mathbf{h} \neq \mathbf{0}$, be assumed. If $\mathbf{\Sigma}$ is p.d. and

$$R_0^2 = \min\{(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})\colon\ \boldsymbol{\beta} \in \mathbb{R}^k\},$$
$$R_1^2 = \min\{(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})'\mathbf{\Sigma}^{-1}(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})\colon\ \mathbf{H}\boldsymbol{\beta} + \mathbf{h} = \mathbf{0}\},$$

then the test statistic is $R_1^2 - R_0^2$. If $\mathcal{M}(\mathbf{X}') \cap \mathcal{M}(\mathbf{H}') = \{\mathbf{0}\}$, then $\mathcal{M}(\mathbf{X}) = \mathcal{M}(\mathbf{X}\mathbf{M}_{H'})$, which is a consequence of the relations $r\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix} = r(\mathbf{X}) + r(\mathbf{H})$ (assumption) and $r\begin{pmatrix}\mathbf{X}\\ \mathbf{H}\end{pmatrix} = r(\mathbf{X}\mathbf{M}_{H'}) + r(\mathbf{H})$ (in general; cf. [12, p. 137). Thus $r(\mathbf{X}) = r(\mathbf{X}\mathbf{M}_{H'}) \Leftrightarrow \mathcal{M}(\mathbf{X}) = \mathcal{M}(\mathbf{X}\mathbf{M}_{H'})$. The projection matrix $\mathbf{P}_X^{\Sigma^{-1}}$ (in the norm $\|\mathbf{u}\|_{\Sigma^{-1}} = \sqrt{\mathbf{u}'\mathbf{\Sigma}^{-1}\mathbf{u}}$, $\mathbf{u} \in \mathbb{R}^n$) is the same as the projection matrix $\mathbf{P}_{XM_{H'}}^{\Sigma^{-1}}$, and since $R_1^2 = (\mathbf{Y} - \mathbf{P}_{XM_{H'}}^{\Sigma^{-1}})'\mathbf{\Sigma}^{-1}(\mathbf{Y} - \mathbf{P}_{XM_{H'}}^{\Sigma^{-1}})$ and $R_0^2 = (\mathbf{Y} - \mathbf{P}_X^{\Sigma^{-1}})'\mathbf{\Sigma}^{-1}(\mathbf{Y} - \mathbf{P}_X^{\Sigma^{-1}})$, the test statistic is zero in this case. It is obvious how to proceed in the case of multivariate models.

Further,

$$\mathcal{M}\begin{pmatrix}\mathbf{0}_{(km,s)}\\ \mathbf{I}_{s,s}\end{pmatrix} \subset \mathcal{M}\begin{pmatrix}\mathbf{I}\otimes\mathbf{X}'\\ \mathbf{E}'\end{pmatrix} \Leftrightarrow \forall\{i=1,\dots,m\}\ \mathcal{M}\begin{pmatrix}\mathbf{0}_{k,s_i}\\ \mathbf{I}_{s_i,s_i}\end{pmatrix} \subset \mathcal{M}\begin{pmatrix}\mathbf{X}'\\ \mathbf{E}_i'\end{pmatrix}.$$

Since it is assumed that $r(\mathbf{I}\otimes\mathbf{X}, \mathbf{E}) = (mk) + s$ (regularity), we have

$$\mathcal{M}\begin{pmatrix}\mathbf{0}_{(km,s)}\\ \mathbf{I}_{s,s}\end{pmatrix} \subset \mathcal{M}\begin{pmatrix}\mathbf{I}\otimes\mathbf{X}'\\ \mathbf{E}'\end{pmatrix}.$$

Lemma 5.1. *In the regular model*

$$\operatorname{vec}(\underline{\mathbf{Y}}) \sim N_{nm}\left[(\mathbf{I}\otimes\mathbf{X}, \mathbf{E})\begin{pmatrix}\operatorname{vec}(\mathbf{B})\\ \mathbf{\Delta}\end{pmatrix}, \mathbf{\Sigma}\otimes\mathbf{I}\right]$$

the BLUE *of* $\mathbf{\Delta}$ *is*

$$\hat{\mathbf{\Delta}} = [\mathbf{E}'(\mathbf{\Sigma}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]^{-1}\mathbf{E}'(\mathbf{\Sigma}^{-1}\otimes\mathbf{M}_X)\operatorname{vec}(\underline{\mathbf{Y}}) \sim N_s\{\mathbf{\Delta}, [\mathbf{E}'(\mathbf{\Sigma}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]^{-1}\}.$$

The test statistic of the hypothesis $H_0\colon\ \mathbf{\Delta} = \mathbf{0}$ *versus* $H_a\colon\ \mathbf{\Delta} \neq \mathbf{0}$ *is*

$$\hat{\mathbf{\Delta}}'[\mathbf{E}'(\mathbf{\Sigma}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]\hat{\mathbf{\Delta}} \sim \chi_s^2(\delta), \quad \delta = (\mathbf{\Delta}^*)'[\mathbf{E}'(\mathbf{\Sigma}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]\mathbf{\Delta}^*,$$

where $\mathbf{\Delta}^*$ *is the actual value of the vector* $\mathbf{\Delta}$.

P r o o f is elementar. □

R e m a r k 5.2. If $\mathbf{E}_{mn,n} = \mathbf{e}_j^{(m)} \otimes \mathbf{I}_{n,n}$, then the hypothesis $\mathbf{\Delta} = \mathbf{0}$ cannot be tested, since

$$\mathcal{M}\begin{pmatrix}\mathbf{0}\\ \mathbf{I}\end{pmatrix} \not\subset \mathcal{M}\begin{pmatrix}\mathbf{I}\otimes\mathbf{X}'\\ (\mathbf{e}_j^{(m)})'\otimes\mathbf{I}_{n,n}\end{pmatrix}.$$

Lemma 5.3. *The best estimator of* σ^2 *in the model*

$$\text{vec}(\underline{\mathbf{Y}}) \sim N_{nm}\left\{[(\mathbf{I}\otimes\mathbf{X}),\mathbf{E}]\begin{pmatrix}\text{vec}(\mathbf{B})\\ \boldsymbol{\Delta}\end{pmatrix}, \sigma^2(\mathbf{V}\otimes\mathbf{I})\right\} \tag{4}$$

is

$$\begin{aligned}\hat{\sigma}^2_{\text{cor}} &= [\text{vec}(\mathbf{Y})]'\{\mathbf{V}^{-1}\otimes\mathbf{M}_X - (\mathbf{V}^{-1}\otimes\mathbf{M}_X)\mathbf{E}\\ &\times[\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]^{-1}\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\}\,\text{vec}(\mathbf{Y})/[m(n-k)-s]\\ &= \frac{1}{m(n-k)-s}\{m(n-k)\hat{\sigma}^2 - [\text{vec}(\mathbf{Y})]'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\mathbf{E}[\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]^{-1}\\ &\times\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\,\text{vec}(\mathbf{Y})\} \sim \sigma^2\chi^2_{m(n-k)-s}(0)/[m(n-k)-s]\end{aligned}$$

where

$$\hat{\sigma}^2 = \frac{1}{m(n-k)}\,\text{Tr}(\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\mathbf{V}^{-1})$$

(the estimator in the model (1)*).*

Proof. In the regular model $\mathbf{Y}\sim N_n(\mathbf{X}\boldsymbol{\beta},\sigma^2\mathbf{V})$, $\boldsymbol{\beta}\in\mathbb{R}^k$, the best estimator $\hat{\sigma}^2$ is given by the relation

$$\hat{\sigma}^2 = \frac{1}{n-k}\mathbf{Y}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{Y}.$$

Analogously the model (4) can be considered. Here

$$\begin{aligned}&[\mathbf{M}_{(I\otimes X,E)}(\mathbf{V}\otimes\mathbf{I})\mathbf{M}_{(I\otimes X,E)}]^+ = \mathbf{V}^{-1}\otimes\mathbf{I} - (\mathbf{V}^{-1}\otimes\mathbf{I})(\mathbf{I}\otimes\mathbf{X},\mathbf{E})\\ &\times\begin{pmatrix}\mathbf{V}^{-1}\otimes\mathbf{X}'\mathbf{X}, & (\mathbf{V}^{-1}\otimes\mathbf{X})\mathbf{E}\\ \mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{X}), & \mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{I})\mathbf{E}\end{pmatrix}\begin{pmatrix}\mathbf{I}\otimes\mathbf{X}'\\ \mathbf{E}'\end{pmatrix}(\mathbf{V}^{-1}\otimes\mathbf{I})\\ &= \mathbf{V}^{-1}\otimes\mathbf{X} - [\mathbf{V}^{-1}\otimes\mathbf{X},(\mathbf{V}^{-1}\otimes\mathbf{I})\mathbf{E}]\begin{pmatrix}\boxed{11}, & \boxed{12}\\ \boxed{21}, & \boxed{22}\end{pmatrix}\begin{pmatrix}\mathbf{V}^{-1}\otimes\mathbf{X}'\\ \mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{I})\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\boxed{11} &= \mathbf{V}\otimes(\mathbf{X}'\mathbf{X})^{-1} + [\mathbf{I}\otimes(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{E}[\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]^{-1}\mathbf{E}'\\ &\quad\times[\mathbf{I}\otimes\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}],\\ \boxed{12} &= -[\mathbf{I}\otimes(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{E}[\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]^{-1},\\ \boxed{21} &= -[\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]^{-1}\mathbf{E}'[\mathbf{I}\otimes\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}],\\ \boxed{22} &= [\mathbf{E}'(\mathbf{V}^{-1}\otimes\mathbf{M}_X)\mathbf{E}]^{-1}.\end{aligned}$$

Now it is easy to complete the proof. □

Let the matrix $\boldsymbol{\Sigma}$ be unknown but let a matrix $\mathbf{S}$ be at our disposal and $f\mathbf{S}\sim W_n(f,\boldsymbol{\Sigma})$. Let the matrix $\mathbf{S}$ be stochastically independent of the vector $\hat{\boldsymbol{\Delta}}$.

Lemma 5.4. *Let* $\mathbf{E} = \mathbf{I}_{m,m} \otimes \mathbf{e}_i^{(n)}$. *Then*

$$\hat{\boldsymbol{\Delta}} = [\mathbf{I} \otimes (\{\mathbf{M}_X\}_{i,i})^{-1}\{\mathbf{M}_X\}_{i,\cdot}]\,\mathrm{vec}(\underline{\mathbf{Y}}) \sim N_m\Big(\boldsymbol{\Delta}, \frac{1}{\{\mathbf{M}_X\}_{i,i}}\boldsymbol{\Sigma}\Big)$$

and

$$\hat{\boldsymbol{\Delta}}'\mathbf{S}^{-1}\hat{\boldsymbol{\Delta}}\{\mathbf{M}_X\}_{i,i} \sim \frac{fm}{f-m+1}F_{m,f-m+1}(\delta), \quad \delta = \{\mathbf{M}_X\}_{i,i}(\boldsymbol{\Delta}^*)'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}^*.$$

P r o o f. It is sufficient to use the Hotelling theorem (cf. proof of Lemma 2.2). □

R e m a r k 5.5. In Lemma 5.4 the degrees of freedom of the Wishart matrix must be larger than $m-1$. Since $\underline{\mathbf{Y}}'\mathbf{M}_{(X,e_i^{(n)})}\underline{\mathbf{Y}}$ is the Wishart matrix with $n-k-1$ degrees of freedom and it is stochastically independent of $\hat{\boldsymbol{\Delta}}$, it can be used in the test from Lemma 5.4, however, $n > m+k$ must be valid.

Lemma 5.6. *Let* $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, $\boldsymbol{\beta} \in \mathbb{R}^k$, *be the regular model and let* $f\mathbf{S} \sim W_n(f, \boldsymbol{\Sigma})$ $(f > n)$ *be stochastically independent of* $\mathbf{Y}$. *Let*

$$\hat{\hat{\boldsymbol{\beta}}} = (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\mathbf{Y}$$

(plug-in estimator) and $\hat{\mathbf{v}} = \mathbf{Y} - \mathbf{X}\hat{\hat{\boldsymbol{\beta}}}$. *If* $\mathbf{G}$ *is an* $r \times k$ *matrix with full rank in rows, i.e.* $r(\mathbf{G}) = r \leqslant k$, *then*

$$\frac{(\mathbf{G}\boldsymbol{\beta} - \mathbf{G}\hat{\hat{\boldsymbol{\beta}}})'[\mathbf{G}(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{G}']^{-1}(\mathbf{G}\boldsymbol{\beta} - \mathbf{G}\hat{\hat{\boldsymbol{\beta}}})}{1 + (1/f)\hat{\mathbf{v}}'\mathbf{S}^{-1}\hat{\mathbf{v}}} \sim \frac{fr}{f-n+k-r+1}F_{r,f-n+k-r+1}(0).$$

P r o o f. It is a consequence of [10], [5, Theorem 7.3.8] and [4]. □

Lemma 5.7. *Let in the model* (2) *the matrix* $\mathbf{E}$ *be of the form* $\mathbf{E} = \mathbf{e}_j^{(m)} \otimes \mathbf{e}_i^{(n)}$. *If we have at our disposal the matrix* $\mathbf{S}$, $f\mathbf{S} \sim W_m(f, \boldsymbol{\Sigma})$, *independent of the observation matrix* $\underline{\mathbf{Y}}$, *then the plug-in estimator of* $\Delta_{i,j}$ *is*

$$\hat{\hat{\Delta}}_{i,j} = \frac{1}{\{\mathbf{M}_X\}_{i,i}\{\mathbf{S}^{-1}\}_{j,j}}\{\mathbf{S}^{-1}\}_{j,\cdot}\begin{pmatrix}\{\mathbf{M}_X\}_{i,\cdot}\mathbf{Y}_1\\ \vdots\\ \{\mathbf{M}_X\}_{i,\cdot}\mathbf{Y}_m\end{pmatrix}$$

and

$$(5)\qquad \frac{1}{\{\mathbf{M}_X\}_{i,i}\{\mathbf{S}^{-1}\}_{j,j}}\frac{\big(\Delta_{i,j}-\hat{\hat{\Delta}}_{i,j}\big)^2}{1+(1/f)\hat{\mathbf{v}}'\mathbf{S}^{-1}\hat{\mathbf{v}}}\sim\frac{f}{f-m+1}F_{1,f-m+1}(0),$$

$$(6)\qquad \hat{\mathbf{v}}=\frac{1}{\sqrt{\{\mathbf{M}_X\}_{i,i}}}\left(\mathbf{I}-\frac{\mathbf{e}_j^{(m)}\{\mathbf{S}^{-1}\}_{j,\cdot}}{\{\mathbf{S}^{-1}\}_{j,j}}\right)\begin{pmatrix}\{\mathbf{M}_X\}_{i,\cdot}\mathbf{Y}_1\\ \vdots\\ \{\mathbf{M}_X\}_{i,\cdot}\mathbf{Y}_m\end{pmatrix}.$$

P r o o f. Let in Lemma 5.6 the vector

$$(7)\qquad \boldsymbol{\xi}=\frac{1}{\sqrt{\{\mathbf{M}_X\}_{i,i}}}\begin{pmatrix}\{\mathbf{M}_X\}_{i,\cdot}\mathbf{Y}_1\\ \vdots\\ \{\mathbf{M}_X\}_{i,\cdot}\mathbf{Y}_m\end{pmatrix}\sim N_m\Big(\sqrt{\{\mathbf{M}_X\}_{i,i}}\mathbf{e}_j^{(m)}\Delta_{i,j},\boldsymbol{\Sigma}\Big)$$

be considered instead of $\mathbf{Y}$. Further, $\mathbf{G}=1, f\mathbf{S}\sim W_m(f,\boldsymbol{\Sigma})$. The estimator $\hat{\Delta}_{i,j}$ is the same as the estimator $\hat{\Delta}_{i,j}$ from the model (7). Thus the relations (5) and (6), based on Lemma 5.6, can be obtained. □

R e m a r k 5.8. In Lemma 5.7 the matrix $\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}$ cannot be used, since the assumption of stochastical independence is not satisfied. We have to have another Wishart matrix $f\mathbf{S}$ at our disposal, e.g. from a former experiment.

6. The matrix $\boldsymbol{\Sigma}$ is of the form $\sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$

Let the regular model

$$\operatorname{vec}(\underline{\mathbf{Y}})\sim N_{nm}\left[(\mathbf{I}\otimes\mathbf{X},\mathbf{E})\begin{pmatrix}\operatorname{vec}(\mathbf{B})\\ \boldsymbol{\Delta}\end{pmatrix},\boldsymbol{\Sigma}\otimes\mathbf{I}\right],$$

where $\boldsymbol{\Sigma}=\sum_{i=1}^{p}\vartheta_i\mathbf{V}_i$, be considered in this section. The matrix $\mathbf{E}$ is of the form (3). Let

$$h(\boldsymbol{\vartheta})=\hat{\boldsymbol{\Delta}}'(\boldsymbol{\vartheta})\{\mathbf{E}'([\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\otimes\mathbf{M}_X]\mathbf{E}\}\hat{\boldsymbol{\Delta}}(\boldsymbol{\vartheta}),$$

where

$$\hat{\boldsymbol{\Delta}}(\boldsymbol{\vartheta})=\{\mathbf{E}'[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\otimes\mathbf{M}_X]\mathbf{E}\}^{-1}\mathbf{E}'[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})\otimes\mathbf{M}_X]\operatorname{vec}(\underline{\mathbf{Y}}).$$

If the hypothesis $H_0:\boldsymbol{\Delta}=\mathbf{0}$ is true and $\boldsymbol{\vartheta}=\boldsymbol{\vartheta}^*$ (actual value of the vector $\boldsymbol{\vartheta}$), then $h(\boldsymbol{\vartheta}^*)\sim\chi^2_s(0)$.

Lemma 6.1. *Let*

$$\left.\frac{\partial h(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}'}\right|_{\vartheta=\vartheta^*} = \boldsymbol{\xi}', \quad \boldsymbol{\xi}' = (\xi_1, \ldots, \xi_p).$$

Then

$$\begin{aligned}\xi_i = & -\hat{\boldsymbol{\Delta}}'(\boldsymbol{\vartheta}^*)(\mathbf{E}'\{[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)] \otimes \mathbf{M}_X\}\mathbf{E})\hat{\boldsymbol{\Delta}}(\boldsymbol{\vartheta}^*) \\ & - 2\hat{\boldsymbol{\Delta}}'(\boldsymbol{\vartheta}^*)\mathbf{E}'\{[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)] \otimes \mathbf{M}_X\}\operatorname{vec}(\underline{\mathbf{v}}),\end{aligned}$$

where

$$\operatorname{vec}(\underline{\mathbf{v}}) = \operatorname{vec}(\underline{\mathbf{Y}}) - (\mathbf{I} \otimes \mathbf{X}, \mathbf{E})\begin{pmatrix}\operatorname{vec}(\widehat{\mathbf{B}}) \\ \hat{\boldsymbol{\Delta}}(\boldsymbol{\vartheta}^*)\end{pmatrix},$$

$$\begin{aligned}\operatorname{vec}(\widehat{\mathbf{B}}) = & [\mathbf{I} \otimes (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\operatorname{vec}(\underline{\mathbf{Y}}) - \{\mathbf{I} \otimes [(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\}\mathbf{E}[\mathbf{E}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \otimes \mathbf{M}_X)\mathbf{E}]^{-1} \\ & \times \mathbf{E}'(\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \otimes \mathbf{I})\{\mathbf{I} \otimes \mathbf{I} - \mathbf{I} \otimes [(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\}\operatorname{vec}(\underline{\mathbf{Y}}).\end{aligned}$$

P r o o f. Since

$$\begin{aligned}\xi_i = \left.\frac{\partial h(\boldsymbol{\vartheta})}{\partial \vartheta_i}\right|_{\vartheta=\vartheta^*} & = 2\hat{\boldsymbol{\Delta}}'(\boldsymbol{\vartheta}^*)\{\mathbf{E}'[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \otimes \mathbf{M}_X]\mathbf{E}\}\left.\frac{\partial \hat{\boldsymbol{\Delta}}(\boldsymbol{\vartheta})}{\partial \vartheta_i}\right|_{\vartheta=\vartheta^*} \\ & + \hat{\boldsymbol{\Delta}}'(\boldsymbol{\vartheta}^*)\left\{\mathbf{E}'\left[\left.\frac{\partial \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})}{\partial \vartheta_i}\right|_{\vartheta=\vartheta^*} \otimes \mathbf{M}_X\right]\mathbf{E}\right\}\hat{\boldsymbol{\Delta}}(\boldsymbol{\vartheta}^*)\end{aligned}$$

and

$$\left.\frac{\partial \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta})}{\partial \vartheta_i}\right|_{\vartheta=\vartheta^*} = -\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*),$$

we have

$$\begin{aligned}\left.\frac{\partial \hat{\boldsymbol{\Delta}}(\boldsymbol{\vartheta})}{\partial \vartheta_i}\right|_{\vartheta=\vartheta^*} & = -\left\{\mathbf{E}'[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \otimes \mathbf{M}_X]\mathbf{E}\right\}^{-1} \\ \times \mathbf{E}'\{[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)] & \otimes \mathbf{M}_X\}[\operatorname{vec}(\underline{\mathbf{Y}}) - \mathbf{E}\hat{\boldsymbol{\Delta}}(\boldsymbol{\vartheta}^*) - (\mathbf{I} \otimes \mathbf{X})\operatorname{vec}(\widehat{\mathbf{B}})].\end{aligned}$$

Now we can easily obtain the expression for ξ_i. The estimator $\operatorname{vec}(\widehat{\mathbf{B}})$ can be obtained in a standard way for the model considered. □

Lemma 6.2. *Let*

$$\mathbf{A}_i = \mathbf{E}'\{[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)] \otimes \mathbf{M}_X\}\mathbf{E}, \quad i = 1, \dots, p,$$
$$\mathbf{B}_i = \mathbf{E}'\{[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)] \otimes \mathbf{M}_X\Big\}, \quad i = 1, \dots, p.$$

Then

$$E(\xi_i) = -\operatorname{Tr}[\mathbf{A}_i \operatorname{Var}(\hat{\boldsymbol{\Delta}})], \quad i = 1, \dots, p,$$
$$\operatorname{Var}(\boldsymbol{\xi}) = 2\mathbf{S}_{\operatorname{Var}(\hat{\Delta})} + 4\mathbf{C}_{\operatorname{Var}[\operatorname{vec}(v)],\operatorname{Var}(\hat{\Delta})},$$

where

$$\operatorname{Var}(\hat{\boldsymbol{\Delta}}) = \{\mathbf{E}'[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \otimes \mathbf{M}_X]\mathbf{E}\}^{-1},$$
$$\operatorname{Var}[\operatorname{vec}(\underline{\mathbf{v}})] = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \otimes \mathbf{M}_X - (\mathbf{I} \otimes \mathbf{M}_X)\mathbf{E} \times \{\mathbf{E}'[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*) \otimes \mathbf{M}_X]\mathbf{E}\}^{-1}\mathbf{E}'(\mathbf{I} \otimes \mathbf{M}_X),$$
$$\{\mathbf{S}_{\operatorname{Var}(\hat{\Delta})}\}_{i,j} = \operatorname{Tr}[\mathbf{A}_i \operatorname{Var}(\hat{\boldsymbol{\Delta}})\mathbf{A}_j \operatorname{Var}(\hat{\boldsymbol{\Delta}})], \quad i, j = 1, \dots, p,$$
$$\{\mathbf{C}_{\operatorname{Var}[\operatorname{vec}(v)],\operatorname{Var}(\hat{\Delta})}\}_{i,j} = \operatorname{Tr}\{\mathbf{B}_i \operatorname{Var}[\operatorname{vec}(\underline{\mathbf{v}})]\mathbf{B}_j' \operatorname{Var}(\hat{\boldsymbol{\Delta}})\}, \quad i, j = 1, \dots, p.$$

Proof. The vectors $\hat{\boldsymbol{\Delta}}$ and $\operatorname{vec}(\underline{\mathbf{v}})$ are stochastically idependent. Under the null hypothesis ($\boldsymbol{\Delta} = \mathbf{0}$) and by virtue of Lemma 6.1 we have $E(\xi_i) = -\operatorname{Tr}[\mathbf{A}_i \operatorname{Var}(\hat{\boldsymbol{\Delta}})]$, $i = 1, \dots, p$, and

$$\operatorname{cov}(\xi_i, \xi_j) = \operatorname{cov}(\hat{\boldsymbol{\Delta}}'\mathbf{A}_i\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\Delta}}'\mathbf{A}_j\hat{\boldsymbol{\Delta}}) - 2\operatorname{cov}[\hat{\boldsymbol{\Delta}}'\mathbf{B}_i \operatorname{vec}(\underline{\mathbf{v}}), \hat{\boldsymbol{\Delta}}'\mathbf{A}_j\hat{\boldsymbol{\Delta}})$$
$$-2\operatorname{cov}[\hat{\boldsymbol{\Delta}}'\mathbf{A}_i\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\Delta}}'\mathbf{B}_j \operatorname{vec}(\underline{\mathbf{v}}) + 4\operatorname{cov}[\hat{\boldsymbol{\Delta}}'\mathbf{B}_i \operatorname{vec}(\underline{\mathbf{v}}), \hat{\boldsymbol{\Delta}}'\mathbf{B}_j \operatorname{vec}(\underline{\mathbf{v}})],$$

moreover,

$$\operatorname{cov}(\hat{\boldsymbol{\Delta}}'\mathbf{A}_i\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\Delta}}'\mathbf{A}_j\hat{\boldsymbol{\Delta}}) = 2\operatorname{Tr}[\mathbf{A}_i \operatorname{Var}(\hat{\boldsymbol{\Delta}})\mathbf{A}_j \operatorname{Var}(\hat{\boldsymbol{\Delta}})],$$
$$\operatorname{cov}[\hat{\boldsymbol{\Delta}}'\mathbf{B}_i \operatorname{vec}(\underline{\mathbf{v}}), \hat{\boldsymbol{\Delta}}'\mathbf{A}_j\hat{\boldsymbol{\Delta}})] = \operatorname{cov}[\hat{\boldsymbol{\Delta}}'\mathbf{A}_i\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\Delta}}'\mathbf{B}_j \operatorname{vec}(\underline{\mathbf{v}})] = 0,$$
$$\operatorname{cov}[\hat{\boldsymbol{\Delta}}'\mathbf{B}_i \operatorname{vec}(\underline{\mathbf{v}}), \hat{\boldsymbol{\Delta}}'\mathbf{B}_j \operatorname{vec}(\underline{\mathbf{v}})] = \operatorname{Tr}[\mathbf{B}_i \operatorname{Var}[\operatorname{vec}(\mathbf{v})]\mathbf{B}_j' \operatorname{Var}(\hat{\boldsymbol{\Delta}})],$$

and consequently

$$\operatorname{Var}(\boldsymbol{\xi}) = 2\mathbf{S}_{\operatorname{Var}(\hat{\Delta})} + 4\mathbf{C}_{\operatorname{Var}[\operatorname{vec}(v)],\operatorname{Var}(\hat{\Delta})}.$$

Further,

$$\operatorname{vec}(\underline{\mathbf{v}}) = \left[\mathbf{I} \otimes \mathbf{I} - (\mathbf{I} \otimes \mathbf{X}, \mathbf{E}) \begin{pmatrix} \boldsymbol{\Sigma}^{-1} \otimes (\mathbf{X}'\mathbf{X}), & \boldsymbol{\Sigma}^{-1} \otimes \mathbf{I})\mathbf{E} \\ \mathbf{E}'(\boldsymbol{\Sigma}^{-1} \otimes \mathbf{I}), & \mathbf{E}'(\boldsymbol{\Sigma}^{-1} \otimes \mathbf{I})\mathbf{E} \end{pmatrix} \times \begin{pmatrix} \boldsymbol{\Sigma}^{-1} \otimes \mathbf{X} \\ \mathbf{E}'(\boldsymbol{\Sigma}^{-1} \otimes \mathbf{I}) \end{pmatrix}\right] \operatorname{vec}(\underline{\mathbf{Y}})$$
$$= \{(\mathbf{I} \otimes \mathbf{M}_X) - (\mathbf{I} \otimes \mathbf{M}_X)\mathbf{E}[\mathbf{E}'(\boldsymbol{\Sigma}^{-1} \otimes \mathbf{M}_X)\mathbf{E}]^{-1}\mathbf{E}'(\boldsymbol{\Sigma}^{-1} \otimes \mathbf{M}_X)\} \operatorname{vec}(\underline{\mathbf{Y}}).$$

Now the expression for $\operatorname{Var}[\operatorname{vec}(\mathbf{v})]$ can be easily obtained. □

Definition 6.3. The set

$$\{\delta\boldsymbol{\vartheta}\colon\ P\{h(\boldsymbol{\vartheta}^*)+\boldsymbol{\xi}'\delta\boldsymbol{\vartheta}\geqslant\chi^2_s(0;1-\alpha)\}\leqslant\alpha+\varepsilon\}$$

is the nonsensitiveness region for the level α of the test for the hypothesis $\boldsymbol{\Delta}=\mathbf{0}$.

Lemma 6.4. *Let*

$$\begin{aligned}\mathbf{U}&=\{\mathbf{E}'[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}^*)\otimes\mathbf{M}_X]\mathbf{E}\}^{-1},\\ \mathbf{V}&=\mathrm{Var}[\mathrm{vec}(\underline{\mathbf{v}})],\\ \{\mathbf{S}_U\}_{i,j}&=\mathrm{Tr}(\mathbf{U}\mathbf{A}_i\mathbf{U}\mathbf{A}_j),\quad i,j=1,\dots,p,\\ \{\mathbf{C}_{U,V}\}_{i,j}&=\mathrm{Tr}(\mathbf{U}\mathbf{B}_i\mathbf{V}\mathbf{B}_j'),\quad i,j=1,\dots,p,\\ \mathbf{a}&=[\mathrm{Tr}(\mathbf{A}_1\mathbf{U}),\dots,\mathrm{Tr}(\mathbf{A}_p\mathbf{U})]'.\end{aligned}$$

The set

$$\mathcal{L}_\Delta=\left\{\delta\boldsymbol{\vartheta}\colon\ -\delta\boldsymbol{\vartheta}\mathbf{a}'+t\sqrt{\delta\boldsymbol{\vartheta}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})\delta\boldsymbol{\vartheta}}<c_\varepsilon\right\}$$

can be used as a nonsensitiveness region for the level α of the test for the hypothesis $\boldsymbol{\Delta}=\mathbf{0}$. Here c_ε is a solution of the equation

$$\alpha+\varepsilon=P\{k(\boldsymbol{\vartheta}^*)\geqslant\chi^2_s(0;1-\alpha)-c_\varepsilon\}$$

and t is a sufficiently large real number.

Proof.

$$\begin{aligned}&P\{h(\boldsymbol{\vartheta}^*)+\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}\geqslant\chi^2_s(0;1-\alpha)\}\\ &\quad=P\big\{h(\boldsymbol{\vartheta}^*)+\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}\geqslant\chi^2_s(0;1-\alpha)\big|\ |\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}|<c\big\}\,P\{|\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}|<c\}\\ &\qquad+P\big\{h(\boldsymbol{\vartheta}^*)+\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}\geqslant\chi^2_s(0;1-\alpha)\big|\ |\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}|\geqslant c\big\}\,P\{|\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}|\geqslant c\}.\end{aligned}$$

If c satisfies the condition $P\{|\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}|\geqslant c\}\approx 0$, then

$$\begin{aligned}\alpha+\varepsilon&=P\{h(\boldsymbol{\vartheta}^*)+\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}\geqslant\chi^2_s(0;1-\alpha)\}\\ &\approx P\{h(\boldsymbol{\vartheta}^*)+\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}\geqslant\chi^2_s(0;1-\alpha)\ \&\ |\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*\delta\boldsymbol{\vartheta}|<c\}\\ &\geqslant P\{h(\boldsymbol{\vartheta}^*)\geqslant\chi^2_s(0;1-\alpha)-c\}.\end{aligned}$$

If $c=c_\varepsilon$, then

$$P\{|\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}|<c_\varepsilon\}\approx 1\Leftrightarrow-\mathbf{a}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}+t\sqrt{\delta\boldsymbol{\vartheta}'(\mathrm{Var}[\boldsymbol{\xi}(\boldsymbol{\vartheta}^*)]\delta\boldsymbol{\vartheta}}<c_\varepsilon$$
$$\text{for sufficiently large } t,$$

which means

$$\forall\,\{\delta\boldsymbol{\vartheta}\in\mathcal{L}_\Delta\}P\{h(\boldsymbol{\vartheta}^*)+\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}\geqslant\chi^2_s(0;1-\alpha)\}\leqslant\alpha+\varepsilon,$$

i.e. the probability of the first kind error of the test is smaller than $\alpha+\varepsilon$. □

Theorem 6.5. *If*

$$(\delta\boldsymbol{\vartheta}-\mathbf{u}_0)'[t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V})-\mathbf{a}\mathbf{a}'](\delta\boldsymbol{\vartheta}-\mathbf{u}_0)\leqslant c_\varepsilon\frac{t^2}{t^2-\mathbf{a}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+\mathbf{a}},$$

where

$$\mathbf{u}_0=\frac{c_\varepsilon}{t^2-\mathbf{a}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+\mathbf{a}}(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+\mathbf{a},$$

then $P\{h(\boldsymbol{\vartheta}^*)+\boldsymbol{\xi}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}\geqslant\chi^2_s(0;1-\alpha)\}\leqslant\alpha+\varepsilon.$

P r o o f. Obviously

$$t^2[\delta\boldsymbol{\vartheta}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})\delta\boldsymbol{\vartheta}]\leqslant[c_\varepsilon+\mathbf{a}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}]^2$$
$$\Rightarrow -\mathbf{a}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}+\sqrt{\delta\boldsymbol{\vartheta}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})\delta\boldsymbol{\vartheta}}<c_\varepsilon,$$
$$t^2[\delta\boldsymbol{\vartheta}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})\delta\boldsymbol{\vartheta}]\leqslant c_\varepsilon^2+2c_\varepsilon\mathbf{a}'(\boldsymbol{\vartheta}^*)\delta\boldsymbol{\vartheta}+\delta\boldsymbol{\vartheta}'\mathbf{a}\mathbf{a}'\delta\boldsymbol{\vartheta}$$
$$\Leftrightarrow \delta\boldsymbol{\vartheta}'[t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V}-\mathbf{a}\mathbf{a}']\delta\boldsymbol{\vartheta}-2\mathbf{a}'\delta\boldsymbol{\vartheta}c_\varepsilon\leqslant c_\varepsilon^2$$
$$\Leftrightarrow \{\delta\boldsymbol{\vartheta}-[t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V}-\mathbf{a}\mathbf{a}']^+c_\varepsilon\mathbf{a}\}'t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V}-\mathbf{a}\mathbf{a}']$$
$$\times\{\delta\boldsymbol{\vartheta}-[t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V}-\mathbf{a}\mathbf{a}']^+c_\varepsilon\mathbf{a}\}\leqslant c_\varepsilon^2-c_\varepsilon^2\mathbf{a}'[t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V}-\mathbf{a}\mathbf{a}']^+\mathbf{a}.$$

Here $\mathbf{a}\in\mathcal{M}(2\mathbf{S}_U+4\mathbf{C}_{U,V})$ is taken into account. This is valid since

$$\begin{pmatrix}\mathrm{Tr}(\mathbf{A}_1\mathbf{U})\\ \vdots\\ \mathrm{Tr}(\mathbf{A}_p\mathbf{U})\end{pmatrix}\in\mathcal{M}\begin{pmatrix}\mathrm{Tr}(\mathbf{A}_1\mathbf{U}\mathbf{A}_1\mathbf{U}), & \dots, & \mathrm{Tr}(\mathbf{A}_1\mathbf{U}\mathbf{A}_p\mathbf{U})\\ \mathrm{Tr}(\mathbf{A}_2\mathbf{U}\mathbf{A}_1\mathbf{U}), & \dots, & \mathrm{Tr}(\mathbf{A}_2\mathbf{U}\mathbf{A}_p\mathbf{U})\\ \dots & \dots & \dots\\ \mathrm{Tr}(\mathbf{A}_p\mathbf{U}\mathbf{A}_1\mathbf{U}), & \dots, & \mathrm{Tr}(\mathbf{A}_p\mathbf{U}\mathbf{A}_p\mathbf{U})\end{pmatrix}$$

and also the matrix $\mathbf{C}_{U,V}$ is positive semidefinite.

Since

$$[t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V}-\mathbf{a}\mathbf{a}']^+$$
$$=\frac{1}{t^2}\left((2\mathbf{S}_U+4\mathbf{C}_{U,V})^++\frac{(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+\mathbf{a}\mathbf{a}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+}{t^2-\mathbf{a}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+\mathbf{a}}\right),$$

we have

$$\mathbf{u}_0=c_\varepsilon[t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V}-\mathbf{a}\mathbf{a}']^+\mathbf{a}$$
$$=\frac{c_\varepsilon}{t^2-\mathbf{a}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+\mathbf{a}}(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+\mathbf{a}$$

and

$$c_\varepsilon^2-c_\varepsilon^2\mathbf{a}'[t^2(2\mathbf{S}_U+4\mathbf{C}_{U,V}-\mathbf{a}\mathbf{a}']^+\mathbf{a}=c_\varepsilon\frac{t^2}{t^2-\mathbf{a}'(2\mathbf{S}_U+4\mathbf{C}_{U,V})^+\mathbf{a}}.$$

By virtue of Lemma 6.4, the statement is proved. □

For more on the nonsensitiveness regions in testing statistical hypotheses cf. [2], [6], [7], [8], [9].

7. Numerical example

Let $\mathbf{X}$ be an $n \times 2$ matrix, $\mathbf{B} = \begin{pmatrix} \boldsymbol{\beta}_{1,1}, & \boldsymbol{\beta}_{1,2} \\ \boldsymbol{\beta}_{2,1}, & \boldsymbol{\beta}_{2,2} \end{pmatrix}$, $\mathbf{E} = \mathbf{I}_{2,2} \otimes \mathbf{e}_i^{(n)}$ (i.e. the ith measurements in both observation vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are suspicious) and $\boldsymbol{\Sigma} = \sigma_1^2 \begin{pmatrix} 1, & 0 \\ 0, & 0 \end{pmatrix} + \sigma_2^2 \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix}$. Then

$$\mathbf{A}_1 = \begin{pmatrix} \{\mathbf{M}_X\}_{i,i}/\sigma_1^4, & 0 \\ 0, & 0 \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} 0, & 0 \\ 0, & \{\mathbf{M}_X\}_{i,i}/\sigma_2^4 \end{pmatrix},$$

$$\mathbf{a} = (\sigma_1^{-2}, \sigma_2^{-2})', \quad \mathbf{U} = \frac{1}{\{\mathbf{M}_X\}_{i,i}} \begin{pmatrix} \sigma_1^2, & 0 \\ 0, & \sigma_2^2 \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} \sigma_1^2, & 0 \\ 0, & \sigma_2^2 \end{pmatrix} \otimes \mathbf{M}_X - \frac{1}{\{\mathbf{M}_X\}_{i,i}} [\mathbf{I}_{2,2} \otimes (\mathbf{M}_X \mathbf{e}_i^{(n)})] \begin{pmatrix} \sigma_1^2, & 0 \\ 0, & \sigma_2^2 \end{pmatrix}$$
$$\times \{\mathbf{I}_{2,2} \otimes [(\mathbf{e}_i^{(n)})' \mathbf{M}_X]\},$$

$$\mathbf{B}_1 = \begin{pmatrix} \sigma_1^{-4}, & 0 \\ 0, & 0 \end{pmatrix} \otimes [(\mathbf{e}_i^{(n)})' \mathbf{M}_X], \quad \mathbf{B}_2 = \begin{pmatrix} 0, & 0 \\ 0, & \sigma_2^{-4} \end{pmatrix} \otimes [(\mathbf{e}_i^{(n)})' \mathbf{M}_X],$$

$$\mathbf{S}_U = \begin{pmatrix} \sigma_1^{-4}, & 0 \\ 0, & \sigma_2^{-4} \end{pmatrix}, \quad \mathbf{C}_{U,V} = \mathbf{0}_{2,2}.$$

If $\alpha = 0.05$, $\varepsilon = 0.04$, $t = 4$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, then $c_\varepsilon = 1.017, \mathbf{u}_0 = (0.000339, 0.001356)'$ and

$$c_\varepsilon \frac{t^2}{t_2 - \mathbf{a}'(2\mathbf{S}_U + 4\mathbf{C}_{U,V})^+ \mathbf{a}} = 1.0848.$$

Thus the nonsitiveness region for the level of the test can be characterized by the ellipse

$$((\delta\boldsymbol{\vartheta} - \mathbf{u}_0)'[t^2(2\mathbf{S}_U + 4\mathbf{C}_{U,V}) - \mathbf{a}\mathbf{a}'](\delta\boldsymbol{\vartheta} - \mathbf{u}_0) =)$$
$$(\delta\boldsymbol{\vartheta} - \mathbf{u}_0)' \begin{pmatrix} 310000, & -2500 \\ -2500, & 19375 \end{pmatrix} (\delta\boldsymbol{\vartheta} - \mathbf{u}_0) = 1.0848$$

with the centre given by the vector $\mathbf{u}_0$, with the first semiaxis equal to $a = 0.007487$ in the direction of the vector $(0.999963, 0.008601)'$ and with the second semiaxis equal to $b = 0.001871$ in the orthogonal direction. Since $\vartheta_1 = \sigma_1^2$, $\vartheta_2 = \sigma_2^2$ and

$\sqrt{a}/\sigma_1 = 0.865$, $\sqrt{b}/\sigma_2 = 0.216$, it can be seen that the value σ_2 must be known much more precisely (21.6 %) than the value σ_1 (86.5 %).

The covariance matrix of the estimator $\hat{\boldsymbol{\vartheta}}$ in this case is

$$\operatorname{Var}(\hat{\boldsymbol{\vartheta}}) = \frac{2}{n-2}\mathbf{S}^{-1}_{\Sigma^{-1}} = \frac{1}{n-2}\begin{pmatrix} 0.00020, & 0 \\ 0, & 0.00320 \end{pmatrix},$$

thus we need at least 6 measurements ($n - 2 \gg 0.00020/0.007487^2 = 3.6$) for the estimation of the parameter ϑ_1, but at least 916 measurements ($n - 2 \gg 0.00320/0.001871^2 = 914$) for the parameter ϑ_2 in order for the resulting level of the test to be smaller than $\alpha + \varepsilon = 0.09$.

This simple example shows how important a good knowledge of variance components is for making order to make a reliable statistical inference.

Analogous consideration can be done with respect to the course of the power function.

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