# SOME RESULTS CONCERNING EXPONENTIAL DIVISORS

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ABSTRACT. If the natural number n has the canonical form  $p_1^{a_1}p_2^{a_2}\dots p_r^{a_r}$  then  $d=p_1^{b_1}p_2^{b_2}\dots p_r^{e_r} \text{ is said to be an exponential divisor of n if } b_i|_{a_i} \text{ for } i=1,2,\dots,r.$  The sum of the exponential divisors of n is denoted by  $\sigma^{(e)}(n)$ . n is said to be an e-perfect number if  $\sigma^{(e)}(n)=2n$ ; (m;n) is said to be an e-amicable pair if  $\sigma^{(e)}(m)=m+n=\sigma^{(e)}(n)$ ;  $n_0,n_1,n_2,\dots$  is said to be an e-aliquot sequence if  $n_{i+1}=\sigma^{(e)}(n_i)-n_i$ . Among the results established in this paper are: the density of the e-perfect numbers is .0087; each of the first 10,000,000 e-aliquot sequences is bounded.

KEYS WORDS AND PHRASES. Exponential divisors, e-perfect numbers, e-amicable numbers, e-aliquot sequences.

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## 1. INTRODUCTION.

If n is a positive integer greater than one whose prime-power decomposition is given by

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$$
 (1.1)

then d is said to be an "exponential divisor" of n if  $d = p_1^{b_1} p_2^{b_2} \dots p_r^{b_r}$  where  $b_i|a_i$  for  $i = 1, 2, \dots, r$ . The sum of all of the exponential divisors of n is denoted by  $\sigma^{(e)}(n)$ . This function was first studied by Subbarao [1] who also initiated the study of exponentially perfect (or e-perfect) numbers.

The positive integer n is said to be an e-perfect number if  $\sigma^{(e)}(n) = 2n$ . If  $\sigma^{(e)}(n) = kn$ , where k is an integer which exceeds 2, n is said to be an e-multi-perfect number. The properties of e-perfect and e-multiperfect numbers have been investigated by Straus and Subbarao [2] and Fabrykowski and Subbarao [3]. It has been proved, for example, that all e-perfect and e-multiperfect numbers are even. Also, if n is an e-perfect number and 3/n then  $2^{110}/n$  and  $n > 10^{618}$ .

While it is easy to show that there are an infinite number of e-perfect numbers, whether or not any e-multiperfect numbers exist is still an open question. Subbarao, Hardy and Aiello [4] have <u>conjectured</u> that there are no e-multiperfect numbers. They have <u>proved</u> that any which exist are very large.

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In Section 2 of the present paper the density of the set of e-perfect numbers is investigated. Section 3 is devoted to a study of e-amicable pairs, integers m and n such that  $\sigma^{(e)}(m) = m+n = \sigma^{(e)}(n)$ . Finally, e-aliquot sequences  $n_0, n_1, n_2, \ldots$  where  $n_{i+1} = \sigma^{(e)}(n_i) - n_i$  for  $i = 0, 1, 2, \ldots$  are studied in Section 4.

2. THE DENSITY OF THE e-PERFECT NUMBERS.

By definition,  $\sigma^{(e)}(1) = 1$  and it is easy to see that  $\sigma^{(e)}(n)$  is multiplicative. Therefore, since  $\sigma^{(e)}(p) = p$  if p is a prime, we see that  $\sigma^{(e)}(m) = m$  if m is square-free.

Now suppose that n, as given by (1.1), is a <u>powerful</u> e-perfect number (so that  $a_i \ge 2$  for  $i=1,2,\ldots,r$  and  $\sigma^{(e)}(n)=2n)$ . Then if (m,n)=1 and m is squarefree then  $\sigma^{(e)}(mn)=2mn$  so that mn is an e-perfect number. Therefore, if x is a (fixed) positive number and  $n_1 < n_2 < \ldots < n_s$  are the powerful e-perfect numbers which do not exceed x then E(x), the set of (all) e-perfect numbers less than or equal to x, is given by  $E(x)=\bigcup_{i=1}^{s}A_i$  where

$$A_{i} = \{mn_{i}: (m,n_{i}) = 1, m \le x/n_{i} \text{ and m is squarefree}\}$$
 (2.1)

Let N be a positive integer and let X be a positive real number. If Q(N,X) is the number of positive, squarefree integers which do not exceed X and which are relatively prime to N, then E. Cohen (Lemma 5.2 in [5]) has shown that

$$Q(N,X) = \beta(N) \cdot X + O(\theta(N) \cdot X^{1/2})$$
 (2.2)

where  $\beta(N) = (\zeta(2) \prod_{p \mid N} (1+1/p))^{-1}$  and  $\theta(N)$  is the number of squarefree divisors of N. It is easy to see that  $\theta(N) = \prod_{p \mid N} 2$ .  $\zeta(k)$  is the Riemann Zeta function, so that  $\zeta(2) = \pi^2/6$ , and the constant implied by the 0-term is independent of N and X.

If Q(e,x) is the number of e-perfect numbers which do not exceed x (so that Q(e,x) is the cardinality of E(x)) it follows from (2.1) and (2.2) that

$$Q(e,x) = x \sum_{i=1}^{s} \beta(n_i)/n_i + 0(x^{1/2} \sum_{i=1}^{s} \theta(n_i/n_i^{1/2}).$$

Therefore,

$$Q(e,x)/x = \sum_{i=1}^{s} \beta(n_i)/n_i + O(x^{-1/2} \sum_{i=1}^{s} \theta(n_i)/n_i^{1/2}).$$
 (2.3)

The following results concerning powerful numbers will be needed in what follows. Proofs may be found in Golomb [6].  $_{\infty}$ 

LEMMA 1. If  $r_1 < r_2 < \dots$  is the sequence of powerful numbers then  $\sum_{i=1}^{\infty} 1/r_i$  is convergent.

LEMMA 2. If P(X) is the number of powerful numbers not exceeding x then  $P(x) < 2.2x^{1/2}$  for large x.

Now let  $\epsilon$  be a given positive number and let  $P_{\hat{\bf 1}}$  denote the ith prime. There exists a positive integer k such that

$$2/P_{k} < \varepsilon \cdot (2.2K)^{-1}/3$$
 (2.4)

where K is the constant implied by the 0-term in (2.3).

Since there are only a finite number of powerful e-perfect numbers which are divisible by fewer that k distinct primes (see Theorem 2.3 in [2]) there exists a positive integer J such that if  $n_1 < n_2 < \ldots$  is the sequence of powerful e-perfect numbers then for all i > J  $n_i$  has at least k distinct prime factors and  $n_i$  has a prime factor, say  $Q_i$ , such that  $Q_i \ge P_k$ . Since  $n_i$  is powerful,  $n_i^{1/2} \ge \pi_p$  where the product is taken over the distinct prime factors of  $n_i$ , and it follows from (2.4) that if i > J then

$$\theta(n_{\underline{i}})/n_{\underline{i}}^{\frac{1}{2}} \le \prod_{p|n_{\underline{i}}} 2/p < 2/Q_{\underline{i}} \le 2/P_{\underline{k}} < \varepsilon \cdot (2.2K)^{-1}/3.$$
 (2.5)

Splitting the sum in the 0-term in (2.3) at i = J (with J held fixed) we can take x large enough so that  $x^{-1/2} \cdot K \cdot \sum_{i=1}^{J} \theta(n_i)/n_i^{1/2} < \varepsilon/3$ . At the same time, since every  $n_i$  is powerful, we see from (2.5) and Lemma 2 that we can also take x large enough so that

$$x^{-1/2} \cdot K \cdot \sum_{i=J+1}^{S} \theta(n_i)/n_i^{1/2} < x^{-1/2} \cdot K \cdot \sum_{i=J+1}^{S} \epsilon \cdot (2.2K)^{-1}/3$$

$$< x^{-1/2} \cdot P(x) \cdot \epsilon \cdot (2.2)^{-1}/3 < \epsilon/3.$$

Finally, since  $\beta(n_i) < 1$  and every  $n_i$  is powerful we see from Lemma 1 that  $\sum_{i=1}^{\infty} \beta(n_i)/n_i$  is convergent. (This series <u>may</u> be finite since whether or not the set of powerful e-perfect numbers is finite or infinite is an open question). It follows that we can take x (and consequently s) large enough so that the tail of this series is less that  $\varepsilon/3$ . Therefore, from (2.3) we have for all large values of x,

$$\left|Q(e,x)/x - \sum_{i=1}^{\infty} \beta(n_i)/n_i\right| < \varepsilon . \qquad (2.6)$$

We have proved

THEOREM 1. Let Q(e,x) denote the number of e-perfect numbers which do not exceed x and let  $n_1 < n_2 < n_3 < \dots$  be the sequence of powerful numbers. Then

$$\lim_{x\to\infty} Q(e,x)/x = \sum_{i=1}^{\infty} \beta(n_i)/n_i = C$$

where  $\beta(n) = 6\pi^{-2} \prod_{p \mid n} (1+1/p)^{-1}$ . Correct to ten decimal places, C = .0086941940.

(There are eight powerful e-perfect numbers less than  $10^{10}$ : 36; 1800; 2700; 17,424; 1,306,800; 4,769,856; 238,492,800; 357,739,200. The approximate value of C given above was calculated using these eight numbers).

The "theoretical" density of the e-perfect numbers as given in Theorem 1 agrees very nicely with the following exact computational results:  $Q(e,10^5)/10^5 = .008691$ ;  $Q(e,10^6)/10^6 = .008690$ ;  $Q(e,10^7)/10^7 = .0086940$ ;  $Q(e,10^8)10^8 = .00869417$ .

# 3. EXPONENTIALLY AMICABLE NUMBERS.

We shall say that m and n are exponentially amicable (or e-amicable) numbers if

$$\sigma^{(e)}(m) = m + n = \sigma^{(e)}(n).$$
 (3.1)

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LEMMA 3. If (m;n) is an e-amicable pair and p is a prime, then  $p \mid m$  if and only if  $p \mid n$ .

PROOF. Suppose that  $p^a \mid |m|$  where  $a \ge 1$ . Then  $p \mid \sigma^{(e)}(m)$  since  $p \mid \sigma^{(e)}(p^a)$  and  $\sigma^{(e)}$  is a multiplicative function. It is now obvious from (3.1) that  $p \mid n$ . By the same argument, if  $p \mid n$  then  $p \mid m$ .

COROLLARY 3.1. If (m;n) is an e-amicable pair then  $m=n \pmod{2}$ .

If (m;n) is an e-amicable pair and there is no prime p such that  $p \mid |m|$  and  $p \mid |n|$  we shall say that m and n are primitive e-amicable numbers. It is easy to see that if (m;n) is a primitive e-amicable pair and r is a squarefree positive integer such that (m,r) = 1, then (rm;rn) is an amicable pair.

A search was made for all primitive e-amicable pairs (m;n) such that m < n and m <  $10^7$ . The search required about 1.5 hours on the CDC CYBER 750 and three pairs were found. They are as follows:  $(2^23^27 \cdot 19^2; 2^23^37^219); (2^23^27 \cdot 61^2; 2^23^47^261); (2^33^25^27 \cdot 19^2; 2^33^35^27^219).$ 

This list suggests the following questions. Are there any odd e-amicable numbers? Are there any powerful e-amicable numbers? Is every e-amicable number divisible by at least four distinct primes? (It is easy to show that every e-amicable number has at least three different prime factors).

The following result can sometimes be used to generate new e-amicable pairs from known pairs.

THEOREM 2. Suppose that (aM;aN) is an e-amicable pair such that (a,M) = (a,N) = 1. If (b,M) = (b,N) = 1 and  $\sigma^{(e)}(a)/a = \sigma^{(e)}(b)/b$  then (bM,bN) is an e-amicable pair. PROOF.  $\sigma^{(e)}(bM) = \sigma^{(e)}(b) \cdot \sigma^{(e)}(M) = a^{-1}b\sigma^{(e)}(a) \cdot \sigma^{(e)}(M) = a^{-1}b\sigma^{(e)}(aM) = \sigma^{(e)}(aM) = \sigma^{(e)}(bM) = bM + bN$ . Similarly,  $\sigma^{(e)}(bN) = bM + bN$ .

The results of a computer search for powerful numbers a and b such that  $4 \le a < b \le 10000$  and  $\sigma^{(e)}(a)/a = \sigma^{(e)}(b)/b$  are given in Table I.

σ<sup>(e)</sup>(a)/a a  $2^{3}5^{2}$  or  $2^{4}11^{2}$ 22 3/2 3<sub>5</sub>2 32 4/3  $2^{3}3^{5}5^{2}$  or  $2^{2}3^{3}5^{2}$ 2232 2 2752 26 39/32  $2^{2}3^{3}$  or  $2^{3}3^{5}2^{2}$ 2332 5/3 235272 272 12/7  $2^{6}3^{3}$ 2<sup>7</sup>3<sup>2</sup> 65/48 223372 233272 40/21

TABLE I

EXAMPLE. Since  $(2^2 \cdot 3^2 \cdot 7 \cdot 19^2; 2^2 \cdot 3^3 \cdot 7^2 \cdot 19)$  is an e-amicable pair and since  $\sigma^{(e)}(2^2)/2^2 = \sigma^{(e)}(2^4 \cdot 11^2)/2^4 \cdot 11^2$  it follows from Theorem 2 that  $(2^4 \cdot 11^2 \cdot 3^2 \cdot 7 \cdot 19^2; 2^4 \cdot 11^2 \cdot 3^3 \cdot 7^2 \cdot 19)$  is an e-amicable pair.

4. EXPONENTIAL ALIQUOT SEQUENCES.

The function  $s^{(e)}$  is defined by  $s^{(e)}(n) = \sigma^{(e)}(n) - n$ , the sum of the <u>exponential aliquot</u> divisors of n.  $s^{(e)}(1) = s^{(e)}(r) = 0$  for every squarefree number r and we define  $s^{(e)}(0) = 0$ . A t-tuple of <u>distinct</u> natural numbers  $(n_0; n_1; ...; n_{t-1})$  with  $n_i = s^{(e)}(n_{i-1})$  for i = 1, 2, ..., t-1 and  $s^{(e)}(n_{t-1}) = n_0$  is called an exponential t-cycle. An exponential 1-cycle is an e-perfect number and an exponential 2-cycle is an e-amicable pair. A search was made for all exponential t-cycles with smallest member not exceeding  $10^7$ . None with t > 2 was found.

The exponential aliquot sequence (or e-aliquot sequence) {  $n_i$  } with leader n is defined by  $n_0 = n, n_1 = s^{(e)}(n_0), n_i = s^{(e)}(n_{i-1}), \ldots$ . Such a sequence is said to be <u>terminating</u> if  $n_k$  is squarefree for some index k (so that  $n_i = 0$  for i > k). An exponential aliquot sequence is said to be <u>periodic</u> if there is an index k such that  $(n_k; n_{k+1}; \ldots; n_{k+t-1})$  is an exponential t-cycle. An e-aliquot sequence which is neither terminating nor periodic is <u>unbounded</u>.

An investigation was made of all aliquot sequences with leader  $n \le 10^7$ . About 2.3 hours of computer time was required. 9,896,235 were found to be terminating and 103,765 were periodic (103,694 ended in 1-cycles and 71 ended in 2-cycles).

The fact that the first ten million exponential aliquot sequences are bounded might tempt one to conjecture that the set of unbounded e-aliquot sequences is empty. However, the following theorem shows that e-aliquot sequences exist which contain arbitrarily long strings of monotonically increasing terms. Therefore, whether or not unbounded e-aliquot sequences exist would seem to be a very open and difficult question.

THEOREM 3. Let N be a positive integer which exceeds 2. Then there exist infinitely many exponential aliquot sequences such that  $n_0 < n_1 < n_2 < \ldots < n_{N-2}$ .

PROOF. Let  $q_1,q_2,\ldots,q_N$  be a sequence of N primes such that  $q_1=2,q_2=3$  and  $q_1^2 \mid (q_{i+1}+1)$  for  $i=2,3,\ldots,N-1$ . (Infinitely many such sequences exist since, by Dirichlet's theorem, the arithmetic progression  $aq_1^2-1$  contains an infinite number of primes.) We shall write  $q_{i+1}+1=K_1$   $q_i^2$ .

Now let  $n_0, n_1, n_2, \ldots$  be the exponential aliquot sequence with leader  $n_0$  given by  $n_0 = q_1^2 q_2^2 \ldots q_N^2$ . Then

$$\sigma^{(e)}(n_0) = \prod_{i=1}^{N} (q_i + q_i^2) = 3 \cdot q_1 q_2 \dots q_N \cdot \prod_{i=2}^{N} (1 + q_i)$$

$$= 3 \cdot q_1 q_2 \dots q_N \cdot \prod_{i=1}^{N-1} K_i q_i^2,$$

and

$$n_1 = \sigma^{(e)}(n_0) - n_0 = (3 \cdot q_1 q_2 \dots q_N \cdot K_1 \dots K_{N-1} - q_N^2) \cdot \prod_{i=1}^{N-1} q_i^2$$

Therefore,  $n_1 = M_1 \prod_{i=1}^{N-1} q_1^2$  where  $(M_1, q_i) = 1$  for i = 1, 2, ..., N-1.

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Since  $n_0/36$  is not squarefree,  $n_1 = \sigma^{(e)}(n_0) - n_0 = \sigma^{(e)}(36) \cdot \sigma^{(e)}(n_0/36) - n_0$ =  $72 \cdot \sigma^{(e)}(n_0/36) - n_0 > 72 \cdot n_0/36 - n_0 = n_0$ .

Similarly, we find that for k = 2,3,..., N-2

$$n_k = M_k \prod_{i=1}^{N-k} q_i^2$$
 where  $(M_k, q_i) = 1$  for  $i = 1, 2, ..., N-k$ 

and

$$n_k = \sigma^{(e)}(n_{k-1}) - n_{k-1} = \sigma^{(e)}(36) \cdot \sigma^{(e)}(n_{k-1}/36) - n_{k-1}$$
  
>  $72 \cdot n_{k-1}/36 - n_{k-1} = n_{k-1}$ .

Therefore,  $n_0 < n_1 < \dots < n_{N-2}$ .

REMARK 1.  $n_{N-2} = 36M_{N-2}$  where  $(6,M_{N-2}) = 1$ . If  $M_{N-2}$  is not squarefree, then  $n_{N-1} = 72 \cdot \sigma^{(e)}(M_{N-2}) - 36M_{N-2} > 72M_{N-2} - 36M_{N-2} = 36M_{N-2} = n_{N-2}$ . REMARK 2. The proof of Theorem 3 is modeled on that of Theorem 2.1 in [7].

Our next objective is to determine  $M(\sigma^{(e)}(n)/n)$ , the mean value of  $\sigma^{(e)}(n)/n$ . The mean value of an arithmetic function f is defined by  $M(f) = \lim_{n \to \infty} N^{-1} \int_{n}^{N} f(n)$ .

We shall need the following lemma due to van der Corput (See Theorem A in [8].) LEMMA 4. If f and h are arithmetic functions such that  $f(n) = \sum_{\substack{d \mid n}} h(d)$  and  $\sum_{n=1}^{\infty} h(n)/n \text{ is absolutely convergent then } M(f) = \sum_{n=1}^{\infty} h(n)/n.$ 

We wish to apply this lemma to the function  $f(n) = \sigma^{(e)}(n)/n$ . By the Moebius inversion formula,  $h(n) = \int\limits_{\substack{d \mid n}} \mu(n/d) \sigma^{(e)}(d)/d$ . h is multiplicative and h(1) = 1. If p is a prime and a is a positive integer then  $h(p^a) = \sigma^{(e)}(p^a)/p^a - \sigma^{(e)}(p^{a-1})/p^{a-1}$ . If a < 6 it is easy to verify that  $|h(p^a)| < p^{-a/4}$ . (For example,  $|h(p^3)| = p^{-1} - p^{-2} < p^{-1} < p^{-3/4}$ .) Suppose that  $a \ge 6$ . Then  $\left| h(p^a) \right| = \sigma^{(e)}(p^a)/p^a - \sigma^{(e)}(p^{a-1})/p^{a-1} \underline{\text{ or }} \left| h(p^a) \right| = \sigma^{(e)}(p^{a-1})/p^{a-1} - \sigma^{(e)}(p^a)/p^a.$ Since  $\sigma^{(e)}(p^m)/p^m < 1 + p/(p-1)p^{m/2}$  (see [2] or [4]) and  $\sigma^{(e)}(p^b)/p^b \ge 1$ ,  $|h(p^a)| < p/(p-1)p^{(a-1)/2}$ . Since  $a \ge 6$  it follows easily that  $|h(p^a)| < p^{-a/4}$ . Since h is multiplicative,  $|h(n)| \le n^{-1/4}$  for every positive integer n. It follows that  $\sum_{n=1}^{\infty} h(n)/n$  is absolutely convergent so that Lemma 4 applies if  $f(n) = \sigma^{(e)}(n)/n$ .

From Theorem 286 in [9] we have

$$\sum_{n=1}^{\infty} h(n)/n = \prod_{p} \{1 + h(p)/p + h(p^{2})/p^{2} + ...\}$$

$$= \prod_{p} \{1 + p^{-1}(\sigma^{(e)}(p)/p - 1) + p^{-2}(\sigma^{(e)}(p^{2})/p^{2} - \sigma^{(e)}(p)/p) + ...\}$$

$$= \prod_{p} \{\sum_{j=0}^{\infty} \sigma^{(e)}(p^{j})/p^{2j} - p^{-1} \sum_{j=0}^{\infty} \sigma^{(e)}(p^{j})/p^{2j}\}$$

$$= \prod_{p} \{(1 - p^{-1}) \sum_{j=0}^{\infty} \sigma^{(e)}(p^{j})/p^{2j}\}.$$

Now the last infinite series can be "split up" by first taking all the terms with numerator  $p^j$  to form the series  $\sum_{j=0}^{\infty} p^j/p^{2j} = \sum_{j=0}^{\infty} 1/p^j$ ; then taking the remaining terms with numerators p to form the series  $\sum_{j=2}^{\infty} p/p^{2j} = p^{-3} \sum_{j=0}^{\infty} (p^{-2})^j$ ; then taking the terms with numerators  $p^2$  to form the series  $\sum_{j=2}^{\infty} p^2/p^{4j} = p^{-6} \sum_{j=0}^{\infty} (p^{-4})^j$ ; then taking the terms with numerators  $p^3$  to form the series  $\sum_{j=2}^{\infty} p^3/p^{6j} = p^{-9} \sum_{j=0}^{\infty} (p^{-6})^j$ ; etc. It follows that

$$\sum_{n=1}^{\infty} h(n)/n = \prod_{p} \{ (1 - p^{-1})((1 - p^{-1})^{-1} + p^{-3}(1 - p^{-2})^{-1} + p^{-6}(1 - p^{-4})^{-1} + p^{-9}(1 - p^{-6})^{-1} + \dots \}$$

$$= \prod_{p} \{ (1 - p^{-1})((1 - p^{-1})^{-1} + (p^{3} - p)^{-1} + (p^{6} - p^{2})^{-1} + (p^{9} - p^{3})^{-1} + \dots \} \}$$

$$= \prod_{p} \{ 1 + (1 - p^{-1}) \sum_{j=1}^{\infty} (p^{3j} - p^{j})^{-1} \}.$$

From Lemma 4 we have

THEOREM 4.  $M(\sigma^{(e)}(n)/n) = \prod_{j=1}^{\infty} \{1 + (1 - p^{-1}) \cdot \sum_{j=1}^{\infty} (p^{3j} - p^{j})^{-1}\} = C.$  Correct to 6 decimal places, C = 1.136571.

(This approximate value of C was calculated using all primes less than  $10^6$  in the infinite product.)

Since  $s^{(e)}(n) = \sigma^{(e)}(n) - n$  we have

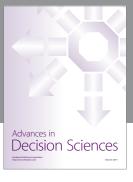
COROLLARY 4.1.  $M(s^{(e)}(n)/n) = .136571$ .

Finally, since  $n_{i+1}/n_i = s^{(e)}(n_i)/n_i$  we see that, in some sense, the average value of the ratio of two consecutive non-zero terms of an e-aliquot sequence is about .136571.

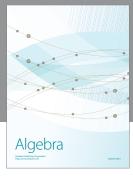
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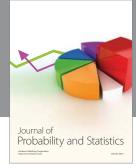
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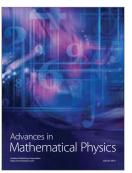


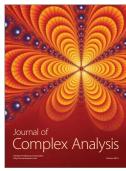


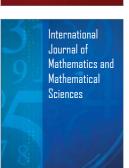


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