Internat. J. Math. & Math. Sci.
VOL. 11 NO. 2 (1988) 343-350

SOME RESULTS CONCERNING EXPONENTIAL DIVISORS

PETER HAGIS, JR.

Mathematics Department
Temple University
Philadelphia, PA 19122

(Received November 18, 1986 and in revised form February 9, 1987)

ABSTRACT. If the natural number n has the canonical form $p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ then $d = p_1^{b_1}p_2^{b_2}\cdots p_r^{b_r}$ is said to be an exponential divisor of n if $b_i|a_i$ for $i = 1,2,\ldots,r$. The sum of the exponential divisors of n is denoted by $\sigma^{(e)}(n)$. n is said to be an e-perfect number if $\sigma^{(e)}(n) = 2n$; $(m;n)$ is said to be an e-amicable pair if $\sigma^{(e)}(m) = m+n = \sigma^{(e)}(n)$; $n_0,n_1,n_2,\ldots$ is said to be an e-aliquot sequence if $n_{i+1} = \sigma^{(e)}(n_i)-n_i$. Among the results established in this paper are: the density of the e-perfect numbers is .0087; each of the first 10,000,000 e-aliquot sequences is bounded.

KEYS WORDS AND PHRASES. Exponential divisors, e-perfect numbers, e-amicable numbers, e-aliquot sequences.
1980 AMS SUBJECT CLASSIFICATION CODE. 10A20.

1. INTRODUCTION.

If n is a positive integer greater than one whose prime-power decomposition is given by

$$n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r} \tag{1.1}$$

then d is said to be an "exponential divisor" of n if $d = p_1^{b_1}p_2^{b_2}\cdots p_r^{b_r}$ where $b_i|a_i$ for $i = 1,2,\ldots,r$. The sum of all of the exponential divisors of n is denoted by $\sigma^{(e)}(n)$. This function was first studied by Subbarao [1] who also initiated the study of exponentially perfect (or e-perfect) numbers.

The positive integer n is said to be an e-perfect number if $\sigma^{(e)}(n) = 2n$. If $\sigma^{(e)}(n) = kn$, where k is an integer which exceeds 2, n is said to be an e-multiperfect number. The properties of e-perfect and e-multiperfect numbers have been investigated by Straus and Subbarao [2] and Fabrykowski and Subbarao [3]. It has been proved, for example, that all e-perfect and e-multiperfect numbers are even. Also, if n is an e-perfect number and $3\nmid n$ then $2^{110}|\ n$ and $n > 10^{618}$.

While it is easy to show that there are an infinite number of e-perfect numbers, whether or not any e-multiperfect numbers exist is still an open question. Subbarao, Hardy and Aiello [4] have conjectured that there are no e-multiperfect numbers. They have proved that any which exist are very large.

In Section 2 of the present paper the density of the set of e-perfect numbers is investigated. Section 3 is devoted to a study of e-amicable pairs, integers m and n such that $\sigma^{(e)}(m) = m+n = \sigma^{(e)}(n)$. Finally, e-aliquot sequences $n_0, n_1, n_2, \ldots$ where $n_{i+1} = \sigma^{(e)}(n_i) - n_i$ for $i = 0,1,2,\ldots$ are studied in Section 4.

2. THE DENSITY OF THE e-PERFECT NUMBERS.

By definition, $\sigma^{(e)}(1) = 1$ and it is easy to see that $\sigma^{(e)}(n)$ is multiplicative. Therefore, since $\sigma^{(e)}(p) = p$ if p is a prime, we see that $\sigma^{(e)}(m) = m$ if m is squarefree.

Now suppose that n, as given by (1.1), is a powerful e-perfect number (so that $a_i \geq 2$ for $i = 1,2,\ldots,r$ and $\sigma^{(e)}(n) = 2n$). Then if $(m,n) = 1$ and m is squarefree then $\sigma^{(e)}(mn) = 2mn$ so that mn is an e-perfect number. Therefore, if x is a (fixed) positive number and $n_1 < n_2 < \ldots < n_s$ are the powerful e-perfect numbers which do not exceed x then E(x), the set of (all) e-perfect numbers less than or equal to x, is given by $E(x) = \bigcup_{i=1}^{s} A_i$ where

$$A_i = \{mn_i : (m,n_i) = 1,\ m \leq x/n_i \text{ and m is squarefree}\} \tag{2.1}$$

Let N be a positive integer and let X be a positive real number. If Q(N,X) is the number of positive, squarefree integers which do not exceed X and which are relatively prime to N, then E. Cohen (Lemma 5.2 in [5]) has shown that

$$Q(N,X) = \beta(N)\cdot X + O(\theta(N)\cdot X^{1/2}) \tag{2.2}$$

where $\beta(N) = (\zeta(2) \prod_{p|N} (1+1/p))^{-1}$ and $\theta(N)$ is the number of squarefree divisors of N. It is easy to see that $\theta(N) = \prod_{p|N} 2$. $\zeta(k)$ is the Riemann Zeta function, so that $\zeta(2) = \pi^2/6$, and the constant implied by the O-term is independent of N and X.

If Q(e,x) is the number of e-perfect numbers which do not exceed x (so that Q(e,x) is the cardinality of E(x)) it follows from (2.1) and (2.2) that

$$Q(e,x) = x \sum_{i=1}^{s} \beta(n_i)/n_i + O(x^{1/2} \sum_{i=1}^{s} \theta(n_i/n_i^{1/2}).$$

Therefore,

$$Q(e,x)/x = \sum_{i=1}^{s} \beta(n_i)/n_i + O(x^{-1/2} \sum_{i=1}^{s} \theta(n_i)/n_i^{1/2}). \tag{2.3}$$

The following results concerning powerful numbers will be needed in what follows. Proofs may be found in Golomb [6].

LEMMA 1. If $r_1 < r_2 < \ldots$ is the sequence of powerful numbers then $\sum_{i=1}^{\infty} 1/r_i$ is convergent.

LEMMA 2. If P(X) is the number of powerful numbers not exceeding x then $P(x) < 2.2x^{1/2}$ for large x.

Now let ε be a given positive number and let P_i denote the ith prime. There exists a positive integer k such that

$$2/P_k < \varepsilon \cdot (2.2K)^{-1}/3 \tag{2.4}$$

where K is the constant implied by the O-term in (2.3).

Since there are only a finite number of powerful e-perfect numbers which are divisible by fewer that k distinct primes (see Theorem 2.3 in [2]) there exists a positive integer J such that if $n_1 < n_2 < \dots$ is the sequence of powerful e-perfect numbers then for all $i > J$ n_i has at least k distinct prime factors and n_i has a prime factor, say Q_i, such that $Q_i \geq P_k$. Since n_i is powerful, $n_i^{1/2} \geq \Pi p$ where the product is taken over the distinct prime factors of n_i, and it follows from (2.4) that if $i > J$ then

$$\theta(n_i)/n_i^{\frac{1}{2}} \leq \prod_{p|n_i} 2/p < 2/Q_i \leq 2/P_k < \varepsilon\cdot(2.2K)^{-1}/3. \tag{2.5}$$

Splitting the sum in the O-term in (2.3) at $i = J$ (with J held fixed) we can take x large enough so that $x^{-1/2}\cdot K\cdot \sum_{i=1}^{J} \theta(n_i)/n_i^{1/2} < \varepsilon/3$. At the same time, since every n_i is powerful, we see from (2.5) and Lemma 2 that we can also take x large enough so that

$$x^{-1/2}\cdot K\cdot \sum_{i=J+1}^{s} \theta(n_i)/n_i^{1/2} < x^{-1/2}\cdot K\cdot \sum_{i=J+1}^{s} \varepsilon\cdot(2.2K)^{-1}/3$$

$$< x^{-1/2}\cdot P(x)\cdot\varepsilon\cdot(2.2)^{-1}/3 < \varepsilon/3.$$

Finally, since $\beta(n_i) < 1$ and every n_i is powerful we see from Lemma 1 that $\sum_{i=1}^{\infty} \beta(n_i)/n_i$ is convergent. (This series may be finite since whether or not the set of powerful e-perfect numbers is finite or infinite is an open question). It follows that we can take x (and consequently s) large enough so that the tail of this series is less that $\varepsilon/3$. Therefore, from (2.3) we have for all large values of x,

$$\left|Q(e,x)/x - \sum_{i=1}^{\infty} \beta(n_i)/n_i\right| < \varepsilon \quad . \tag{2.6}$$

We have proved

THEOREM 1. Let $Q(e,x)$ denote the number of e-perfect numbers which do not exceed x and let $n_1 < n_2 < n_3 < \dots$ be the sequence of powerful numbers. Then

$$\lim_{x\to\infty} Q(e,x)/x = \sum_{i=1}^{\infty} \beta(n_i)/n_i = C$$

where $\beta(n) = 6\pi^{-2} \prod_{p|n} (1+1/p)^{-1}$. Correct to ten decimal places, $C = .0086941940$.

(There are eight powerful e-perfect numbers less than 10^{10}: 36; 1800; 2700; 17,424; 1,306,800; 4,769,856; 238,492,800; 357,739,200. The approximate value of C given above was calculated using these eight numbers).

The "theoretical" density of the e-perfect numbers as given in Theorem 1 agrees very nicely with the following exact computational results: $Q(e,10^5)/10^5 = .00871$; $Q(e,10^6)/10^6 = .008690$; $Q(e,10^7)/10^7 = .0086940$; $Q(e,10^8)10^8 = .00869417$.

3. EXPONENTIALLY AMICABLE NUMBERS.

We shall say that m and n are exponentially amicable (or e-amicable) numbers if

$$\sigma^{(e)}(m) = m + n = \sigma^{(e)}(n). \tag{3.1}$$

LEMMA 3. If $(m;n)$ is an e-amicable pair and p is a prime, then $p|m$ if and only if $p|n$.

PROOF. Suppose that $p^a||m$ where $a \geq 1$. Then $p|\sigma^{(e)}(m)$ since $p|\sigma^{(e)}(p^a)$ and $\sigma^{(e)}$ is a multiplicative function. It is now obvious from (3.1) that $p|n$. By the same argument, if $p|n$ then $p|m$.

COROLLARY 3.1. If $(m;n)$ is an e-amicable pair then $m \equiv n \pmod 2$.

If $(m;n)$ is an e-amicable pair and there is no prime p such that $p||m$ and $p||n$ we shall say that m and n are primitive e-amicable numbers. It is easy to see that if $(m;n)$ is a primitive e-amicable pair and r is a squarefree positive integer such that $(m,r) = 1$, then $(rm;rn)$ is an amicable pair.

A search was made for all primitive e-amicable pairs $(m;n)$ such that $m < n$ and $m < 10^7$. The search required about 1.5 hours on the CDC CYBER 750 and three pairs were found. They are as follows: $(2^2 3^2 7 \cdot 19^2;\ 2^2 3^3 7^2 19)$; $(2^2 3^2 7 \cdot 61^2;\ 2^2 3^4 7^2 61)$; $(2^3 3^2 5^2 7 \cdot 19^2;\ 2^3 3^3 5^2 7^2 19)$.

This list suggests the following questions. Are there any odd e-amicable numbers? Are there any powerful e-amicable numbers? Is every e-amicable number divisible by at least four distinct primes? (It is easy to show that every e-amicable number has at least three different prime factors).

The following result can sometimes be used to generate new e-amicable pairs from known pairs.

THEOREM 2. Suppose that $(aM;aN)$ is an e-amicable pair such that $(a,M) = (a,N) = 1$. If $(b,M) = (b,N) = 1$ and $\sigma^{(e)}(a)/a = \sigma^{(e)}(b)/b$ then (bM,bN) is an e-amicable pair.

PROOF. $\sigma^{(e)}(bM) = \sigma^{(e)}(b) \cdot \sigma^{(e)}(M) = a^{-1}b\sigma^{(e)}(a) \cdot \sigma^{(e)}(M) = a^{-1}b\sigma^{(e)}(aM) = \sigma$ $a^{-1}b(aM + aN) = bM + bN$. Similarly, $\sigma^{(e)}(bN) = bM + bN$.

The results of a computer search for powerful numbers a and b such that $4 \leq a < b \leq 10000$ and $\sigma^{(e)}(a)/a = \sigma^{(e)}(b)/b$ are given in Table I.

TABLE I

$\sigma^{(e)}(a)/a$	a	b
3/2	2^2	$2^3 5^2$ or $2^4 11^2$
4/3	3^2	$3^3 5^2$
2	$2^2 3^2$	$2^3 3^5 5^2$ or $2^2 3^3 5^2$
39/32	2^6	$2^7 5^2$
5/3	$2^3 3^2$	$2^2 3^3$ or $2^3 3^3 5^2$
12/7	$2^2 7^2$	$2^3 5^2 7^2$
65/48	$2^7 3^2$	$2^6 3^3$
40/21	$2^3 3^2 7^2$	$2^2 3^3 7^2$

EXAMPLE. Since $(2^2 \cdot 3^2 \cdot 7 \cdot 19^2;\ 2^2 \cdot 3^3 \cdot 7^2 \cdot 19)$ is an e-amicable pair and since $\sigma^{(e)}(2^2)/2^2 = \sigma^{(e)}(2^4 \cdot 11^2)/2^4 \cdot 11^2$ it follows from Theorem 2 that $(2^4 \cdot 11^2 \cdot 3^2 \cdot 7 \cdot 19^2;\ 2^4 \cdot 11^2 \cdot 3^3 \cdot 7^2 \cdot 19)$ is an e-amicable pair.

4. EXPONENTIAL ALIQUOT SEQUENCES.

The function $s^{(e)}$ is defined by $s^{(e)}(n) = \sigma^{(e)}(n) - n$, the sum of the exponential aliquot divisors of n. $s^{(e)}(1) = s^{(e)}(r) = 0$ for every squarefree number r and we define $s^{(e)}(0) = 0$. A t-tuple of distinct natural numbers $(n_0;n_1;\ldots;\ n_{t-1})$ with $n_i = s^{(e)}(n_{i-1})$ for $i = 1,2,\ldots,\ t-1$ and $s^{(e)}(n_{t-1}) = n_0$ is called an exponential t-cycle. An exponential 1-cycle is an e-perfect number and an exponential 2-cycle is an e-amicable pair. A search was made for all exponential t-cycles with smallest member not exceeding 10^7. None with $t > 2$ was found.

The exponential aliquot sequence (or e-aliquot sequence) $\{ n_i \}$ with leader n is defined by $n_0 = n, n_1 = s^{(e)}(n_0),\ n_i = s^{(e)}(n_{i-1}),\ldots$. Such a sequence is said to be terminating if n_k is squarefree for some index k (so that $n_i = 0$ for $i > k$). An exponential aliquot sequence is said to be periodic if there is an index k such that $(n_k;n_{k+1};\ldots;\ n_{k+t-1})$ is an exponential t-cycle. An e-aliquot sequence which is neither terminating nor periodic is unbounded.

An investigation was made of all aliquot sequences with leader $n \leq 10^7$. About 2.3 hours of computer time was required. 9,896,235 were found to be terminating and 103,765 were periodic (103,694 ended in 1-cycles and 71 ended in 2-cycles).

The fact that the first ten million exponential aliquot sequences are bounded might tempt one to conjecture that the set of unbounded e-aliquot sequences is empty. However, the following theorem shows that e-aliquot sequences exist which contain arbitrarily long strings of monotonically increasing terms. Therefore, whether or not unbounded e-aliquot sequences exist would seem to be a very open and difficult question.

THEOREM 3. Let N be a positive integer which exceeds 2. Then there exist infinitely many exponential aliquot sequences such that $n_0 < n_1 < n_2 < \ldots < n_{N-2}$.

PROOF. Let $q_1,q_2,\ldots,\ q_N$ be a sequence of N primes such that $q_1 = 2,\ q_2 = 3$ and $q_i^2 | (q_{i+1} + 1)$ for $i = 2,3,\ldots,\ N-1$. (Infinitely many such sequences exist since, by Dirichlet's theorem, the arithmetic progression $aq_i^2 - 1$ contains an infinite number of primes.) We shall write $q_{i+1} + 1 = K_i\ q_i^2$.

Now let $n_0,n_1,n_2,\ldots$ be the exponential aliquot sequence with leader n_0 given by $n_0 = q_1^2\ q_2^2 \ldots q_N^2$. Then

$$\sigma^{(e)}(n_0) = \prod_{i=1}^{N} (q_i + q_i^2) = 3 \cdot q_1q_2 \ldots q_N \cdot \prod_{i=2}^{N} (1 + q_i)$$

$$= 3 \cdot q_1q_2 \ldots q_N \cdot \prod_{i=1}^{N-1} K_iq_i^2 ,$$

and

$$n_1 = \sigma^{(e)}(n_0) - n_0 = (3 \cdot q_1q_2 \ldots q_N \cdot K_1 \ldots K_{N-1} - q_N^2) \cdot \prod_{i=1}^{N-1} q_i^2 .$$

Therefore, $n_1 = M_1 \prod_{i=1}^{N-1} q_i^2$ where $(M_1,q_i) = 1$ for $i = 1,2,\ldots,\ N-1$.

Since $n_0/36$ is not squarefree, $n_1 = \sigma^{(e)}(n_0) - n_0 = \sigma^{(e)}(36) \cdot \sigma^{(e)}(n_0/36) - n_0 = 72 \cdot \sigma^{(e)}(n_0/36) - n_0 > 72 \cdot n_0/36 - n_0 = n_0$.

Similarly, we find that for $k = 2,3,\ldots, N-2$

$$n_k = M_k \prod_{i=1}^{N-k} q_i^2 \text{ where } (M_k,q_i) = 1 \text{ for } i = 1,2,\ldots, N-k$$

and

$$n_k = \sigma^{(e)}(n_{k-1}) - n_{k-1} = \sigma^{(e)}(36) \cdot \sigma^{(e)}(n_{k-1}/36) - n_{k-1}$$

$$> 72 \cdot n_{k-1}/36 - n_{k-1} = n_{k-1}.$$

Therefore, $n_o < n_1 < \ldots < n_{N-2}$.

REMARK 1. $n_{N-2} = 36M_{N-2}$ where $(6,M_{N-2}) = 1$. If M_{N-2} is not squarefree, then $n_{N-1} = 72 \cdot \sigma^{(e)}(M_{N-2}) - 36M_{N-2} > 72M_{N-2} - 36M_{N-2} = 36M_{N-2} = n_{N-2}$.

REMARK 2. The proof of Theorem 3 is modeled on that of Theorem 2.1 in [7].

Our next objective is to determine $M(\sigma^{(e)}(n)/n)$, the mean value of $\sigma^{(e)}(n)/n$. The mean value of an arithmetic function f is defined by $M(f) = \lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} f(n)$.

We shall need the following lemma due to van der Corput (See Theorem A in [8].)

LEMMA 4. If f and h are arithmetic functions such that $f(n) = \sum_{d|n} h(d)$ and $\sum_{n=1}^{\infty} h(n)/n$ is absolutely convergent then $M(f) = \sum_{n=1}^{\infty} h(n)/n$.

We wish to apply this lemma to the function $f(n) = \sigma^{(e)}(n)/n$. By the Moebius inversion formula, $h(n) = \sum_{d|n} \mu(n/d)\sigma^{(e)}(d)/d$. h is multiplicative and $h(1) = 1$. If p is a prime and a is a positive integer then $h(p^a) = \sigma^{(e)}(p^a)/p^a - \sigma^{(e)}(p^{a-1})/p^{a-1}$. If $a < 6$ it is easy to verify that $|h(p^a)| < p^{-a/4}$. (For example, $|h(p^3)| = p^{-1} - p^{-2} < p^{-1} < p^{-3/4}$.) Suppose that $a \geq 6$. Then $|h(p^a)| = \sigma^{(e)}(p^a)/p^a - \sigma^{(e)}(p^{a-1})/p^{a-1}$ or $|h(p^a)| = \sigma^{(e)}(p^{a-1})/p^{a-1} - \sigma^{(e)}(p^a)/p^a$. Since $\sigma^{(e)}(p^m)/p^m < 1 + p/(p-1)p^{m/2}$ (see [2] or [4])and $\sigma^{(e)}(p^b)/p^b \geq 1$, $|h(p^a)| < p/(p-1)p^{(a-1)/2}$. Since $a \geq 6$ it follows easily that $|h(p^a)| < p^{-a/4}$. Since h is multiplicative, $|h(n)| \leq n^{-1/4}$ for every positive integer n. It follows that $\sum_{n=1}^{\infty} h(n)/n$ is absolutely convergent so that Lemma 4 applies if $f(n) = \sigma^{(e)}(n)/n$.

From Theorem 286 in [9] we have

$$\sum_{n=1}^{\infty} h(n)/n = \prod_p \{1 + h(p)/p + h(p^2)/p^2 + \ldots\}$$

$$= \prod_p \{1 + p^{-1}(\sigma^{(e)}(p)/p-1) + p^{-2}(\sigma^{(e)}(p^2)/p^2 - \sigma^{(e)}(p)/p) + \ldots\}$$

$$= \prod_p \{\sum_{j=0}^{\infty} \sigma^{(e)}(p^j)/p^{2j} - p^{-1}\sum_{j=0}^{\infty} \sigma^{(e)}(p^j)/p^{2j}\}$$

$$= \prod_p \{(1 - p^{-1})\sum_{j=0}^{\infty} \sigma^{(e)}(p^j)/p^{2j}\}.$$

Now the last infinite series can be "split up" by first taking all the terms with numerator p^j to form the series $\sum_{j=0}^{\infty} p^j/p^{2j} = \sum_{j=0}^{\infty} 1/p^j$; then taking the remaining terms with numerators p to form the series $\sum_{j=2}^{\infty} p/p^{2j} = p^{-3} \sum_{j=0}^{\infty} (p^{-2})^j$; then taking the terms with numerators p^2 to form the series $\sum_{j=2}^{\infty} p^2/p^{4j} = p^{-6} \sum_{j=0}^{\infty} (p^{-4})^j$; then taking the terms with numerators p^3 to form the series $\sum_{j=2}^{\infty} p^3/p^{6j} = p^{-9} \sum_{j=0}^{\infty} (p^{-6})^j$; etc. It follows that

$$\sum_{n=1}^{\infty} h(n)/n = \prod_p \{(1 - p^{-1})((1 - p^{-1})^{-1} + p^{-3}(1 - p^{-2})^{-1}$$

$$+ p^{-6}(1 - p^{-4})^{-1} + p^{-9}(1 - p^{-6})^{-1} + \ldots)\}$$

$$= \prod_p \{(1 - p^{-1})((1 - p^{-1})^{-1} + (p^3 - p)^{-1} + (p^6 - p^2)^{-1}$$

$$+ (p^9 - p^3)^{-1} + \ldots)\}$$

$$= \prod_p \{1 + (1 - p^{-1}) \sum_{j=1}^{\infty} (p^{3j} - p^j)^{-1}\}.$$

From Lemma 4 we have

THEOREM 4. $M(\sigma^{(e)}(n)/n) = \prod_p \{1 + (1 - p^{-1}) \cdot \sum_{j=1}^{\infty} (p^{3j} - p^j)^{-1}\} = C.$

Correct to 6 decimal places, C = 1.136571.

(This approximate value of C was calculated using all primes less than 10^6 in the infinite product.)

Since $s^{(e)}(n) = \sigma^{(e)}(n) - n$ we have

COROLLARY 4.1. $M(s^{(e)}(n)/n) = .136571.$

Finally, since $n_{i+1}/n_i = s^{(e)}(n_i)/n_i$ we see that, in some sense, the average value of the ratio of two consecutive non-zero terms of an e-aliquot sequence is about .136571.

REFERENCES

1. SUBBARAO, M.V. On some arithmetic convolutions, The Theory of Arithmetic Functions, Lecture Notes in Mathematics 251 (1972), Springer-Verlag, New York, 247-271.
2. STRAUS, E.G. and SUBBARAO, M.V. On exponential divisors, Duke Math. J. 41 (1974), 465-471.
3. FABRYKOWSKI, J. and SUBBARAO, M.V. On e-perfect numbers and a conjecture of Straus and Subbarao, Congressus Numerantium 52 (1986), 79-90.
4. SUBBARAO, M.V., HARDY, G.E. and AIELLO, W. On the existence of e-multiperfect numbers, The Fibonacci Quarterly 25 (1987), 65-71.
5. COHEN, E. Arithmetical functions associated with the unitary divisors of an integer, Math. Zeit. 74 (1960), 66-80.
6. GOLOMB, S.W. Powerful Numbers, Amer. Math. Monthly 77 (1970), 848-855.
7. te RIELE, H.J.J. A theoretical and computational study of generalized aliquot sequences, Mathematical Centre Tracts 74 (1976), Amsterdam.
8. COHEN, E. ARITHMATICAL NOTES, I. On a theorem of van der Corput, Proc. A.M.S. 12 (1961), 214-217.
9. HARDY, G.H. and WRIGHT, E.M. An Introduction to the Theory of Numbers (Fourth Edition) Oxford University Press, New York, 1960.