Al-Omari et al. *Advances in Difference Equations* (2018) 2018:124
https://doi.org/10.1186/s13662-018-1567-1

Advances in Difference Equations
a SpringerOpen Journal

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Some results for Laplace-type integral operator in quantum calculus

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Abstract

In the present article, we wish to discuss q-analogues of Laplace-type integrals on diverse types of q-special functions involving Fox's H_q-functions. Some of the discussed functions are the q-Bessel functions of the first kind, the q-Bessel functions of the second kind, the q-Bessel functions of the third kind, and the q-Struve functions as well. Also, we obtain some associated results related to q-analogues of the Laplace-type integral on hyperbolic sine (cosine) functions and some others of exponential order type as an application to the given theory.

Keywords: $J_v(x;q)$ function; $Y_v(x;q)$ function; $K_v(x;q)$ function; $H_v(x;q)$ function; Laplace-type integral

1 Introduction and preliminaries

Quantum calculus is a version of calculus where derivatives are differences and antiderivatives are sums, and no further limits are required. The quantum calculus or q-calculus, compared to the differential and integral calculus, has been very recently named. Hence some rules and definitions need to be recalled. For $0 < q < 1$, the q-calculus starts with the definition of the q-analogue of the differential and the q-analogue of derivatives as well. The q-analogue of the integer n, the factorial of n, and the binomial coefficient are respectively given as

$$[n]_q = \frac{1-q^n}{1-q}, \qquad ([n]_q)! = \begin{Bmatrix} \prod_1^n [k]_q, & n \in \mathbb{N} \\ 1, & n = 0 \end{Bmatrix}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_1^n \frac{1-q^{n-k+1}}{1-q^k}. \tag{1}$$

The q-analogue of $(x+a)^n$ $(n \in \mathbb{N})$ and its q-derivative are respectively given as

$$(x+a)_q^n = \prod_{j=0}^{n-1} (x + q^j a), \qquad D_q(x+a)_q^n = [n]_q (x+a)_q^{n-1}, \qquad (x+a)_q^0 = 1. \tag{2}$$

The q-Jackson integrals from 0 to a and from a to b are given as follows (see [1], see also [2]):

$$\int_0^a f(x)\, d_q x = (1-q)a \sum_0^\infty f(aq^k) q^k \tag{3}$$

and

$$\int_b^a f(x)\,d_qx = \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx. \tag{4}$$

The improper q-Jackson integral is given as follows (see [1]):

$$\int_0^{\frac{\infty}{A}} f(x)\,d_qx = (1-q)\sum_{n\in\mathbb{Z}} \frac{q^k}{A} f\left(\frac{q^k}{A}\right), \quad A \in \mathbb{C}.$$

The q-analogues of the gamma function are defined by

$$\Gamma_q(\alpha) = \int_0^{\frac{1}{1-q}} x^{\alpha-1} E_q\bigl(q(1-q)x\bigr)\,d_qx$$

and

$${}_q\Gamma(\alpha) = K(A;\alpha)\int_0^{\frac{\infty}{A(1-q)}} x^{\alpha-1} e_q\bigl(-(1-q)x\bigr)\,d_qx,$$

where $\alpha > 0$ and, for every $t \in \mathbb{R}$,

$$K(A;t) = A^{t-1}\frac{(-q/A;q)_\infty}{(-q^t/A;q)_\infty}\frac{(-A;q)_\infty}{(-Aq^{1-t};q)_\infty}.$$

Here

$$(a;q)_n = \prod_0^{n-1}\bigl(1-aq^k\bigr), \qquad (a;q)_\infty = n \overset{\lim}{\rightarrow} \infty\ (a;q)_n.$$

The very useful identities used in this article are (cf. [2])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad \text{and} \quad (a;q)_t = \frac{(a;q)_\infty}{(aq^t;q)_\infty}, \quad t \in \mathbb{R}.$$

The q-hypergeometric functions are represented by

$${}_r\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_r \\ \alpha_1,\alpha_2,\ldots,\alpha_s \end{matrix}\middle| q,z\right) = \sum_0^\infty \frac{(a_1,a_2,\ldots,a_r;q)_n}{(\alpha_1,\alpha_2,\ldots,\alpha_s;q)_n}\frac{z^n}{(q;q)_n}$$

and

$${}_{m-k}\Phi_{m-1}\left(\begin{matrix} a_1,a_2,\ldots,a_{m-k} \\ \alpha_1,\alpha_2,\ldots,\alpha_{m-1} \end{matrix}\middle| q,z\right) = \sum_0^\infty \frac{(a_1,\ldots,a_{m-k};q)_n}{(\alpha_1,\ldots,\alpha_{m-1};q)_n}\bigl[(-1)^n q^{\binom{n}{2}}\bigr]^k \times \frac{z^n}{(q;q)_n},$$

where $(a_1,a_2,\ldots,a_p;q)_n = \prod_{k=0}^p (a_k;q)_n$.

2 *H*-Function and related functions

The H-function, which is an extension of the hypergeometric functions ${}_pF_q$, introduced by Fox [3] (see also [4, 5]), has found various applications in a huge range of problems associated with reaction, reaction diffusion, communication, engineering, fractional differential equations, integral equations, theoretical physics, and statistical distribution theory as well. The H-functions have also been recognized to play a fundamental role in fractional calculus with its applications. Fox's H-function, admitting to a standard notation, is presented as

$$H_{p,q}^{m,n}(\eta) = \frac{1}{2\pi i}\int_P J_{p,q}^{m,n}(w)\eta^w\,dw, \tag{5}$$

where P is a suitable complex path, $\eta^w = \exp\{w(\log|\eta| + i\arg\eta)\}$, $J_{p,q}^{m,n}(w) = \frac{A(w)B(w)}{C(s)D(w)}$, and

$$A(w) = \prod_1^m \Gamma(b_j - \beta_j w), \qquad B(w) = \prod_1^n \Gamma(1 - a_j + \alpha_j w),$$

$$C(w) = \prod_{m+1}^q \Gamma(1 - b_j - \beta_j w), \qquad D(w) = \prod_{n+1}^p \Gamma(a_j + \alpha_j w),$$

$0 \le n \le p$, $1 \le m \le q$, $\{a_j, b_j\} \in \mathbb{C}$, $\{\alpha_j, \beta_j\} \in \mathbb{R}^+$. Let α_j and β_j be positive integers and $0 \le m \le N$; $0 \le n \le M$. Then the q-analogue of Fox's H-function is given as (see [6])

$$H_{M,N}^{m,n}\left(x;q\,\middle|\,\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\dots,(a_\mu,\alpha_M)\\(b_1,\beta_1),(b_2,\beta_2),\dots,(b_N,\beta_N)\end{matrix}\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^m G(q^{b_j-\beta_j s})\prod_{j=1}^n G(q^{1-a_j+\alpha_j s})\pi x^s}{\prod_{j=m+1}^N G(q^{1-b_j+\beta_j s})\prod_{j=n+1}^M G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\,d_q s,$$

where G is defined in terms of the product

$$G(q^\alpha) = \prod_{k=0}^\infty \left(1 - q^{\alpha-k}\right)^{-1} = \frac{1}{(q^\alpha;q)_\infty}. \tag{6}$$

The contour C is parallel to $\operatorname{Re}(ws) = 0$, such that all poles of $G(q^{b_j-\beta_j s})$, $1 \le j \le m$, are its right and those of $G(q^{1-a_j+\alpha_j s})$, $1 \le j \le n$, are the left of C. The above integral converges if $\operatorname{Re}(s\log x - \log\sin\pi s) < 0$, for huge values of $|s|$ on C. Hence,

$$\left|\arg(x) - w_2 w_1^{-1}\log|x|\right| < \pi, \qquad |q| < 1, \qquad \log q = -w = -w_1 - iw_2,$$

where w_1 and w_2 are real numbers.

Indeed, for $\alpha_i = \beta_j = 1$, for all i, j, we write the q-analogue of Meijer's G-function as

$$G_{M,N}^{m,n}\left(x;q\,\middle|\,\begin{matrix}a_1,a_1,\dots,a_M\\b_1,b_2,\dots,b_N\end{matrix}\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^m G(q^{b_j-s})\prod_{j=1}^n G(q^{1-a_j+s})\pi x^2}{\prod_{j=m+1}^N G(q^{1-b_j+s})\prod_{j=n+1}^M G(q^{a_j-s})G(q^{1-s})\sin\pi s}\,d_q s, \tag{7}$$

where $0 \le m \le N$; $0 \le n \le M$ and $\operatorname{Re}(s\log x - \log\sin\pi s) < 0$.

Additionally, the q-analogues of the Bessel function $J_\nu(x)$ of the first kind, the Bessel function of $Y_\nu(x)$, the Bessel function of the third kind $K_\nu(x)$, and Struve's function $H_\nu(x)$ are, respectively, defined in terms of Fox's H_q-function by [7] as follows:

$$J_\nu(x;q) = \{G(a)\}^2 H_{0,3}^{1,0}\left(\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \\ (\frac{\nu}{2},1),(-\frac{\nu}{2},1)(1,1)\end{matrix}\right), \tag{8}$$

$$\begin{aligned} Y_\nu(x;q) &= \{G(a)\}^2 \\ &\quad\times H_{1,4}^{2,0}\left(\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} (-\frac{\nu-1}{2},1) \\ (\frac{\nu}{2},1),(-\frac{\nu}{2},1)(-\frac{\nu-1}{2},1)(1,1)\end{matrix}\right), \end{aligned} \tag{9}$$

$$K_\nu(x;q) = (1-q)H_{0,3}^{2,0}\left(\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \\ (\frac{\nu}{2},1),(-\frac{\nu}{2},1)(1,1)\end{matrix}\right), \tag{10}$$

$$\begin{aligned} H_\nu(x;q) &= \left(\frac{1-q}{2}\right)^{1-\alpha} \\ &\quad\times H_{1,4}^{3,1}\left(\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} (\frac{1+\alpha}{2},1) \\ (\frac{\nu}{2},1),(-\frac{\nu}{2},1)(\frac{\nu+\alpha}{2},1)(1,1)\end{matrix}\right). \end{aligned} \tag{11}$$

In [8] (see also [9]), some q-analogues of the natural exponential functions, sine functions, cosine functions, hyperbolic sine functions, and hyperbolic cosine functions are, respectively, given in terms of Fox's H-function as follows:

$$e_q(-x) = G(q)H_{0,2}^{1,0}\left(x(1-q);q \,\middle|\, \begin{matrix} \\ (0,1)(1,1)\end{matrix}\right), \tag{12}$$

$$\begin{aligned} \sin_q(x) &= \sqrt{\pi}(1-q)^{-\frac{1}{2}}\{G(q)\}^2 \\ &\quad\times H_{0,3}^{1,0}\left(\frac{x^2(1-q)^2}{4};q \,\middle|\, (\tfrac{1}{2},1)(0,1)(1,1)\right), \end{aligned} \tag{13}$$

$$\begin{aligned} \cos_q(x) &= \sqrt{\pi}(1-q)^{-\frac{1}{2}}\{G(q)\}^2 \\ &\quad\times H_{0,3}^{1,0}\left(\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \\ (0,1)(\frac{1}{2},1)(1,1)\end{matrix}\right), \end{aligned} \tag{14}$$

$$\begin{aligned} \sinh_q(x) &= \frac{\sqrt{\pi}}{i}(1-q)^{-\frac{1}{2}}\{G(q)\}^2 \\ &\quad\times H_{0,3}^{1,0}\left(-\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \\ (\frac{1}{2},1)(0,1)(1,1)\end{matrix}\right), \end{aligned} \tag{15}$$

$$\begin{aligned} \cosh_q(x) &= \sqrt{\pi}(1-q)^{-\frac{1}{2}}\{G(q)\}^2 \\ &\quad\times H_{0,3}^{1,0}\left(-\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \\ (0,1)(\frac{1}{2},1)(1,1)\end{matrix}\right). \end{aligned} \tag{16}$$

On the other hand, some impressive integral transforms also have the corresponding q-analogues in the concept of q-calculus; they include the q-Laplace transforms [10], the q-Sumudu transforms [9, 11–13], the q-Wavelet transform [14], the q-Mellin transform [15], q-$E_{2,1}$-transform [16], q-Mangontarum transforms [17, 18], q-natural transforms [19], and so on. Recently, a number of authors have studied various image formulas for these q-integral transforms, associated with a variety of special functions. In this sequel, we aim to investigate the q-analogues of Laplace-type integrals on diverse types of q-special functions involving Fox's H_q-function.

3 q-Laplace-type transforms for H_q-function

A Laplace-type integral was introduced in [20, 21]. The q-analogues of the Laplace-type integral of the first kind were defined later by [22] as follows:

$$\begin{aligned} {}_qL_2\big(f(\xi);y\big) &= \frac{1}{1-q^2}\int_0^{y^{-1}} \xi E_{q^2}\big(q^2y^2\xi^2\big)f(\xi)\,d\xi \\ &= \frac{(q^2;q^2)_\infty}{[2]_q y^2}\sum_{i=0}^{\infty}\frac{q^{2i}}{(q^2;q^2)_i}f\big(q^i y^{-1}\big), \end{aligned} \tag{17}$$

whereas the q-analogues of the Laplace-type integral of the second kind were defined by

$$\begin{aligned} {}_q\ell_2\big(f(\xi);y\big) &= \frac{1}{1-q^2}\int_0^{\infty} \xi e_{q^2}\big(y^2\xi^2\big)\,d_q\xi \\ &= \frac{1}{[2]_q(-y^2;q^2)_\infty}\sum_{i\in\mathbb{Z}} q^{2i}f\big(q^i\big)\big(-y^2;q^2\big)_i. \end{aligned} \tag{18}$$

For the sake of convenience, we establish some formulas for the ${}_qL_2$ operator. A similar argument can give certain corresponding results for the operator ${}_q\ell_2$.

Theorem 1 *Let β be a positive real number. Then*

$${}_qL_2\big(\xi^{2\beta-2}\big)(y) = \frac{(q^2;q^2)_\infty}{[2]_q y^2 (q^\beta;q^2)_\infty}.$$

Proof By using (17), we have

$$\begin{aligned} {}_qL_2\big(\xi^{2\beta-2};y\big) &= \frac{(q^2;q^2)_\infty}{[2]_q y^{2\beta}}\sum_{i=0}^{\infty}\frac{q^{2i}}{(q^2;q^2)_i}\big(q^i y^{-1}\big)^{2\beta-2} \\ &= \frac{(q^2;q^2)_\infty}{[2]_q y^{2\beta}}\sum_{i=0}^{\infty}\frac{q^{2\beta i}y^{2\beta-2}}{(q^2;q^2)_i}. \end{aligned}$$

That is,

$${}_qL_2\big(\xi^{2\beta-2};y\big) = \frac{(q^2;q^2)_\infty}{[2]_q y^2}\sum_{i=0}^{\infty}\frac{q^{2\beta i}}{(q^2;q^2)_i}. \tag{19}$$

By the fact that

$$e_q(z) = \sum_{i=0}^{\infty}\frac{z^i}{(q;q)_i},$$

we have

$$
\begin{aligned}
{}_qL_2\left(\xi^{2\beta-2};y\right) &= \frac{(q^2;q^2)_\infty}{[2]_q y^2} e_{q^2}\left(q^\beta\right) \\
&= \frac{(q^2;q^2)_\infty}{[2]_q y^2} \frac{1}{(q^{2\beta};q^2)_\infty}.
\end{aligned}
$$

This completes the establishment of the belief. □

Theorem 2 *Let λ be a complex number. Then*

$$
\begin{aligned}
&{}_qL_2\left(x^{2\lambda} H^{m,n}_{M,N}\left(\gamma x^{2k};q^2 \,\middle|\, \begin{matrix}(a_1,\alpha_1),\dots,(a_M,\alpha_M)\\(b_1,\beta_1),\dots,(b_N,\beta_N)\end{matrix}\right)\right)(y) \\
&\quad = \frac{(q^2;q^2)_\infty}{y^{2\lambda+2}[2]_q} H^{m,n+1}_{M+1,N}\left(\frac{\gamma}{y^{2k}},q^2 \,\middle|\, \begin{matrix}(-\lambda,k),(a_1,\alpha_1),\dots,(a_M,\alpha_M)\\(b_1,\beta_1),\dots,(b_N,\beta_N)\end{matrix}\right),
\end{aligned}
$$

where $0 \le n \le m$ *and* $0 \le m \le N$ *and λ is an arbitrary complex number.*

Proof Let λ be a complex number. Then by (17) we obtain

$$
\begin{aligned}
&{}_qL_2\left(x^{2\lambda} H^{m,n}_{M,N}\left(\gamma x^{2k};q^2 \,\middle|\, \begin{matrix}(a_1,\alpha_1),\dots,(a_M,\alpha_M)\\(b_1,\beta_1),\dots,(b_N,\beta_N)\end{matrix}\right)\right)(y) \\
&\quad = \frac{1}{2\pi i}\int_c \frac{\prod_{j=1}^m G(q^{2b_j-2\beta_j z})\prod_{j=1}^n G(q^{2-2a_j+2\alpha_j z})\pi\gamma^z}{\prod_{j=m+1}^N G(q^{2-2b_j+2\beta_j z})\prod_{j=n+1}^M G(q^{2a_j-2\alpha_j z})G(q^{2-2z})\sin\pi z} \\
&\qquad \times_q L_2\left(x^{2\lambda+2kz}\right)(y)\,d_qz.
\end{aligned}
\tag{20}
$$

Let $\beta = \lambda + kz + 1$, then by Theorem 1 we have

$$
{}_qL_2\left(x^{2(\lambda+kz)}\right)(y) =_q L_2\left(x^{2B-2}\right)(y) = \frac{(q^2;q^2)_\infty}{[2]_q y^2(q^{2(\lambda+zk+1)};q^2)_\infty}.
\tag{21}
$$

By invoking (21) in (20), we get

$$
\begin{aligned}
&{}_qL_2\left(x^{2\lambda} H^{m,n}_{M,N}\left(\gamma x^{2k};q^2 \,\middle|\, \begin{matrix}(a_1,\alpha_1),\dots,(a_M,\alpha_M)\\(b_1,\beta_1),\dots,(b_N,\beta_N)\end{matrix}\right)\right)(y) \\
&\quad = \frac{1}{2\pi i}\int_c \frac{\prod_{j=1}^m G(q^{2b_j-2\beta_j z})\prod_{j=1}^n G(q^{2-2a_j+2\alpha_j z})\pi\gamma^z}{\prod_{j=m+1}^N G(q^{2-2b_j+2\beta_j z})\prod_{j=n+1}^M G(q^{2a_j-2\alpha_j z})G(q^{2-2z})\sin\pi z} \\
&\qquad \times \frac{(q^2;q^2)_\infty}{[2]_q y^2(q^{2(\lambda+zk+1)};q^2)_\infty}\,d_qz.
\end{aligned}
\tag{22}
$$

By inserting the identity

$$
G\left(q^{2\lambda+2kz+2}\right) = \frac{1}{(q^{2\lambda+2kz+2};q^2)_\infty}
$$

in (22) yields

$$
\begin{aligned}
&{}_qL_2\left(x^{2\lambda}H_{M,N}^{m,n}\left(\gamma x^{2k};q^2\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_M,\alpha_M)\\(b_1,\beta_1),\dots,(b_N,\beta_N)\end{matrix}\right.\right)\right)(y)\\
&\quad=\frac{(q^2;q^2)_\infty}{2\pi iy^{2\lambda+2}[2]_q}\int_c\frac{\prod_{j=1}^{m}G(q^{2b_j-2\beta_jz})\prod_{j=1}^{n}G(q^{2-2a_j+2\alpha_jz})}{\prod_{j=m+1}^{N}G(q^{2-2b_j+2\beta_jz})\prod_{j=n+1}^{M}G(q^{2a_j-2\alpha_jz})}\\
&\qquad\times\frac{G(q^{1+\lambda+kz})}{G(q^{2(1-z)})\sin\pi z}\pi\left(\frac{\gamma}{y^2k}\right)^z\,d_qz.
\end{aligned}
$$

Now, on account of the definition of H_q-function, we may establish that

$$
\begin{aligned}
&{}_qL_2\left(x^{2\lambda}H_{M,N}^{m,n}\left(\gamma x^{2k};q^2\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_M,\alpha_M)\\(b_1,\beta_1),\dots,(b_N,\beta_N)\end{matrix}\right.\right)\right)(y)\\
&\quad=\frac{(q^2;q^2)_\infty}{y^{2\lambda+2}[2]_q}H_{N,M+1}^{n+1,m}\left(\gamma x^{2k};q^2\left|\begin{matrix}(1-b_1,\beta_1),\dots,(1-b_N,\beta_N)\\(1+\lambda,k),(1-a,\alpha_1),\dots,(1-a_M,\alpha_M)\end{matrix}\right.\right),
\end{aligned}
$$

provided $k<0$.

The proof is completed. □

4 Applications to trigonometric and hyperbolic functions

In this part, we shall give certain natural relevance to the leading results.

Theorem 3 *Let e_q be defined in terms of* (12). *Then*

$$
{}_qL_2\big(e_{q^2}(-x)\big)(y)=\frac{G(q^2)(q^2;q^2)_\infty}{[2]_qy^2}H_{1,2}^{1,1}\left(\frac{1-q^2}{y^2};q^2\left|\begin{matrix}(0,1)\\(0,1),(1,1)\end{matrix}\right.\right).
$$

Proof By setting $\lambda=0$, $\gamma=1-q^2$, and $k=1$, Theorem 3 immediately follows from Theorem 2. □

The demonstration of this theorem is finished.

Theorem 4 *Let* $\sin_q$ *be defined in terms of* (13). *Then we have*

$$
\begin{aligned}
{}_qL_2\big(\sin_{q^2}(x)\big)(y)&=\frac{\sqrt{\pi}(1-q^2)^{\frac{-1}{2}}\{G(q^2)\}^2}{[2]_qy^4}(q^2;q^2)_\infty\\
&\quad\times H_{1,3}^{1,1}\left(\frac{(1-q^2)^2}{4y^2};q^2\left|\begin{matrix}(0,1)\\(\frac{1}{2},1)(0,1),(1,1)\end{matrix}\right.\right).
\end{aligned}
$$

Proof The proof of this theorem indeed follows from substituting the values $\lambda=0$, $k=1$, and $\gamma=\frac{(1-q^2)^2}{4}$ and from multiplying by $\sqrt{\pi}(1-q^2)^{\frac{-1}{2}}\{G(q^2)\}^2$.

Hence, the proof is completed. □

Theorem 5 *Let* $\cos_q$ *be defined in terms of* (14). *Then*

$$ {}_qL_2\bigl(\cos_{q^2}(x)\bigr)(y) = \frac{\sqrt{\pi}(1-q^2)^{\frac{-1}{2}}\{G(q^2)\}^2}{[2]_q y^4}\bigl(q^2;q^2\bigr)_\infty \times H^{1,1}_{1,3}\left(\frac{(1-q^2)^2}{4y^2};q^2 \,\middle|\, \begin{matrix}(0,1)\\ (0,1),(\frac{1}{2},1),(1,1)\end{matrix}\right). $$

Proof Proof follows from Theorem 2 for $\lambda = 0$, $k = 1$, $\gamma = \frac{(1-q^2)^2}{4}$.
The proof is completed. □

Theorem 6 *Let* $\sinh_q$ *be defined in terms of* (15). *Then*

$$ {}_qL_2\bigl(\sinh_{q^2}(x)\bigr)(y) = \frac{\sqrt{\pi}(1-q^2)^{\frac{-1}{2}}\{G(q^2)\}^2}{i[2]_q y^4}\bigl(q^2;q^2\bigr)_\infty \times H^{1,1}_{1,3}\left(\frac{(1-q^2)^2}{4y^2};q^2 \,\middle|\, \begin{matrix}(0,1)\\ (\frac{1}{2},1)(0,1),(1,1)\end{matrix}\right). $$

Proof By using the special case, $\lambda = 0$, $k = 1$, $\gamma = \frac{(1-q^2)^2}{4}$.
The proof is completed. □

Theorem 7 *Let* $\cosh_q$ *be defined in terms of* (16). *Then*

$$ {}_qL_2\bigl(\cosh_{q^2}(x)\bigr)(y) = \frac{\sqrt{\pi}(1-q^2)^{\frac{-1}{2}}\{G(q^2)\}^2}{[2]_q y^4}\bigl(q^2;q^2\bigr)_\infty \times H^{1,1}_{1,3}\left(\frac{(1-q^2)^2}{4y^2};q^2 \,\middle|\, \begin{matrix}(0,1)\\ (0,1),(\frac{1}{2},1),(1,1)\end{matrix}\right). $$

Proof The validation of this theorem is identical to that of the previous theorem. □

Theorem 8 *Let the Bessel function be defined in terms of* (8). *Then*

$$ {}_qL_2\bigl(J_\nu(x;q^2)\bigr)(y) = \frac{\{G(q^2)\}^2}{[2]_q y^4}\bigl(q^2;q^2\bigr)_\infty \times H^{1,1}_{1,3}\left(\frac{(1-q^2)^2}{4y^2};q^2 \,\middle|\, \begin{matrix}(0,1)\\ (\frac{\nu}{2},1),(\frac{-\nu}{2},1),(1,1)\end{matrix}\right). $$

Proof By setting $\lambda = 0$, $k = 1$, $\gamma = \frac{1-q^2}{4}$ and multiplying by $\{G(q^2)\}^2$, the result follows. □

Theorem 9 *Let the q-Bessel function of the second kind be defined in terms of* (9)–(11). *Then*

$$ {}_qL_2\bigl(Y_\nu(x;q^2)\bigr)(y) = \frac{\{G(q^2)\}^2}{[2]_q y^4}\bigl(q^2;q^2\bigr)_\infty \times H^{2,1}_{2,4}\left(\frac{(1-q^2)^2}{4y^2};q^2 \,\middle|\, \begin{matrix}(0,1),(\frac{-\nu}{2},1)\\ (\frac{\nu}{2},1),(\frac{-\nu}{2},1),(\frac{-\nu-1}{2},1)(1,1)\end{matrix}\right), $$

$$
\begin{aligned}
{}_qL_2\big(K_\nu\big(x;q^2\big)\big)(y) &= \frac{(1-q^2)}{[2]_q y^4}\big(q^2;q^2\big)_\infty \\
&\quad \times H^{2,1}_{1,3}\left(\frac{(1-q^2)^2}{4y^2};q^2 \,\middle|\, \begin{matrix}(0,1)\\ (\frac{\nu}{2},1),(\frac{-\nu}{2},1),(1,1)\end{matrix}\right), \\
{}_qL_2\big(H_\nu\big(x;q^2\big)\big)(y) &= \frac{(1-q^2)^{1-\alpha}}{2^{1-\alpha}[2]_q y^4}\big(q^2;q^2\big)_\infty \\
&\quad \times H^{3,2}_{2,4}\left(\frac{(1-q^2)^2}{4y^2};q^2 \,\middle|\, \begin{matrix}(0,1),(\frac{1-\alpha}{2},1)\\ (\frac{\nu}{2},1),(\frac{-\nu}{2},1),(\frac{1+\alpha}{2},1)(1,1)\end{matrix}\right).
\end{aligned}
$$

Proof Proof of this theorem follows from (9)–(11) and the technique quite similar to that of Theorems 3–8. We omit the details. □

Acknowledgements
The authors are thankful to the referee for his/her valuable remarks and comments for the improvement of the paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 13 February 2018 Accepted: 16 March 2018 Published online: 03 April 2018

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