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SOME RESULTS FROM THE COMBINATORIAL APPROACH TO QUANTUM LOGIC*

The combinatorial approach to quantum logic focuses on certain interconnections between graphs, combinatorial designs, and convex sets as applied to a quantum logic $(\mathscr{L}, \mathscr{S})$, that is, to a σ-orthocomplete orthomodular poset $\mathscr{L}$ and a full set of σ-additive states $\mathscr{S}$ on $\mathscr{L}$. Combinatorial results of interest in quantum logic appear in Gerelle *et al.* (1974), Greechie (1968, 1969, 1971a, b), Greechie and Gudder (1973), and Greechie and Miller (1970, 1972). In this article I shall be concerned only with orthomodular *lattices* $\mathscr{L}$ and associated structures.

I derive a class of complete atomic irreducible semimodular orthomodular lattices which may not be represented as linear subspaces of a vector space over a division ring. Each of these lattices is a proposition system of dimension three. Moreover each of them admits a state σ which violates the following condition:

$$\sigma(a)=\sigma(b)=1 \quad \text{implies} \quad \sigma(a\wedge b)=1.$$

This answers a question posed by Jauch (this volume). These proposition systems form orthocomplemented non-Desarguesian projective geometries. Knowledge of the existence of such structures is apparently new.

Roughly, the construction proceeds as follows: given *any* cubic (all maximal Boolean subalgebras have cardinality eight) orthomodular lattice L, we consider the associated *structure space* $(X, \mathscr{E})$ where X is the set of atoms of L and $\mathscr{E}$ is the set of all blocks or bases (maximal orthogonal sets of atoms) of L. We call a subset $\{x, y\} \subset X$ *bad* in case no element $z \in X$ is orthogonal to both x and y. We then augment $(X, \mathscr{E})$ by adding to X enough elements to eliminate all bad subsets of X and by adding enough new blocks to maintain orthomodularity in the extended lattice L_1; an infinite iteration of this process yields the desired proposition system L_∞. L_∞ may or may not be a quantum logic – depending on L.

We show that, under a mild assumption, the automorphism group of L_∞ is isomorphic to that of L. Finally, we associate to each semi-

Synthese **29** (1974) 113–127.

modular atomistic lattice a cubic orthomodular lattice L and thus a non-Desarguesian proposition system L_∞. In particular every lattice of (closed) subspaces of a Hilbert space $\mathscr{H}$ – of dimension $\geqslant 3$ – may be associated with a non-Desarguesian proposition system L_∞. Although the initial structure is needed (at present) for computations, any group representable on $\mathscr{H}$ is representable on L_∞.

I. THE STRUCTURE SPACE GESTALT OF ORTHOCOMPLEMENTED PROJECTIVE PLANES

In this part I review some basic definitions such as graph, orthomodular lattice, semimodular lattice, projective geometry and projective plane, as well as introduce the notions of cubic, wide and tight structure spaces. I associate to each graph (X, R), or equivalently to each structure space $(X, \mathscr{E})$, an orthocomplemented lattice $\mathscr{C}(X, \mathscr{E})$ of all R-closed sets. For wide cubic structure spaces $(X, \mathscr{E})$, $\mathscr{C}(X, \mathscr{E})$ is an orthomodular lattice. If $(X, \mathscr{E})$ is tight and cubic, then $\mathscr{C}(X, \mathscr{E})$ is an orthocomplemented projective plane with atoms $\{\{x\} \mid x \in X\}$ and conversely.

A *space* is a pair $(X, \mathscr{A})$ where X is a nonempty set and $\mathscr{A}$ is a set of nonempty subsets of X. A *graph* is a pair (X, R) where X is a nonempty set and R is a symmetric irreflexive relation on X (that is, $R \subset X \times X$, $(x, y) \in R$ implies $(y, x) \in R$, and $(x, x) \notin R$ for all $x \in X$). An R-*set* of a graph (X, R) is a subset $Y \subset X$ such that if $x, y \in Y$ and $x \neq y$ then $(x, y) \in R$. R-sets which are maximal under set theoretic inclusion are called R-blocks or simply *blocks*. Let $\mathscr{E}_R$, or simply $\mathscr{E}$, denote the set of all R-blocks of the graph (X, R). $(X, \mathscr{E}_R)$ is called the *structure space of the graph* (X, R). Clearly R may be recaptured from $\mathscr{E}_R$ by noting that $(x, y) \in R$ if and only if $x \neq y$ and there exists $E \in \mathscr{E}_R$ with $\{x, y\} \subset E$.

A space $(X, \mathscr{E})$ is a *structure space* if there exists a (unique) graph (X, R) such that $\mathscr{E} = \mathscr{E}_R$. Such spaces were characterized in Gerelle *et al.* (1974). For convenience we define $R' = \{(x, y) \in X \times X \mid x \neq y \text{ and } (x, y) \notin R\}$. If M is any set, the cardinality of M is denoted by $|M|$.

A *minimal cycle* in a graph (X, R), or in a structure space $(X, \mathscr{E}_R)$ is an ordered n-tuple $(a_0, a_1, \ldots, a_{n-1})$ such that $a_i \in X$, $(a_i, a_j) \in R$ if and only if $|i-j| = 1 (\text{mod}\, n)$, and $n \geqslant 4$; n is called the *order* of the minimal cycle $(a_0, a_1, \ldots, a_{n-1})$.

Let (X, R) be a graph. If $M \subset X$, define

$M^R=\{x\in X \mid (x, m)\in R$ for all $m\in M\}$ and
$M^{RR}=(M^R)^R$. $\{x\}^R$ is usually written x^R.

Let

$$\mathscr{C}(X, \mathscr{E}_R)=\mathscr{C}(X, R)=\{M\subset X \mid M=M^{RR}\}$$

be the set of all R-closed subsets of X. Then, partially ordered under set-theoretic inclusion, $\mathscr{C}(X, R)$ is a complete lattice. The join and meet, respectively, of a family $\{M_\alpha\}\subset\mathscr{C}(X, R)$ are given by the formulas

$$\bigvee M_\alpha=(\bigcup M_\alpha)^{RR}$$

and

$$\bigwedge M_\alpha=\bigcap M_\alpha.$$

Recall that a lattice L is *orthocomplemented* in case there exists a function $':L\to L$ such that (i) $x''=(x')'=x$, (ii) $x\leqslant y$ implies $y'\leqslant x'$, and (iii) $x\vee x'=1$ and $x\wedge x'=0$.

An orthocomplemented lattice $(L, \leqslant, ')$ is *orthomodular* in case $x\leqslant y$ implies $y=x\vee(y\wedge x')$, or equivalently $x\leqslant y$ and $y\wedge x'=0$ imply $x=y$. The lattice $\mathscr{C}(X, R)$ is orthocomplemented by the function $M\to M^R$. It is orthomodular if and only if for each $M\in\mathscr{C}(X, R)$ and each maximal R-subset D of M, $D^{RR}=M$.

An orthomodular lattice is *cubic* when every maximal Boolean sub-orthomodular lattice has eight elements.

If $R=\{(x, y)\in X\times X \mid x\neq y\}$ then $\mathscr{C}(X, R)$ is the set of all subsets of X and $M^R=X\backslash M$. If $(\mathscr{H}, \langle,\rangle)$ is a Hilbert space, $X=\mathscr{H}\backslash\{\circ\}$ and $R=\{(x, y)\in X\times X \mid \langle x, y\rangle=0\}$ then $\mathscr{C}(X, R)$ is the lattice of all closed linear subspaces of $\mathscr{H}$ and M^R is the usual 'orthogonal complement' of the subspace M. Because Hilbert space 'orthogonality' is, in some sense, a prototypical example of the relations R that we have in mind we shall frequently use $\perp$ for R and $\perp'$ for R'.

A structure space $(X, \mathscr{E})$ is called *cubic* if $|E|=3$ for all $E\in\mathscr{E}$; it is called *wide* if $|x^R\cap y^R|\leqslant 1$ for all distinct $x, y\in X$; it is called *tight* if $|x^R\cap y^R|=1$ for all distinct $x, y\in X$. A cubic structure space $(X, \mathscr{E})$ is called *nontrivial* if $\bigcap\mathscr{E}=\phi$, that is, no $x\in X$ is in every $E\in\mathscr{E}$; otherwise it is called *trivial.*

LEMMA 1. *Let $(X, \mathscr{E})$ be a cubic structure space. These are equivalent:*

(i) *$(X, \mathscr{E})$ is wide,*

(ii) *all minimal cycles of $(X, \mathscr{E})$ have order at least* 5,

(iii) $\mathscr{C}(X, \mathscr{E})$ *is an orthomodular lattice with atoms* $\{\{x\} \mid x \in X\}$.

Proof. See Greechie (1971a), Theorem 3.

Let L be a lattice with 0, i.e., $0 \leqslant x$ for all $x \in L$. For $x, y \in L$ we say that x covers y, written $x \succ y$ or $y \prec x$, in case $x > y$ and if $x \geqslant z \geqslant y$ then $z = x$ or $z = y$. Elements which cover 0 are called *atoms*. L is *atomic* if for all $x \in L$ with $x \neq 0$, there exists an atom $a \in L$ with $a \leqslant x$. L is *atomic* in case every element is the supremum of some set (possibly empty) of atoms. An atomistic lattice L is *upper semimodular* in case $x \in L$ and a an atom of L with $a \nleqslant x$ imply $x \vee a \succ x$. If L is of finite height (i.e., all maximal chains in L are finite), then L is upper semimodular if and only if $x \vee y \succ x, y$ whenever $x, y \succ x \wedge y$. Recall that a lattice L is modular if $(x \vee y) \wedge z = x \vee (y \wedge z)$ for all $x, y, z \in L$ with $x \leqslant z$. For an orthocomplemented lattice of finite height the concepts 'upper semimodular', 'semimodular', and 'modular' coincide.

A lattice L is *reducible* if it may be written as a nontrivial Cartesian product of lattices $L_1 \times L_2$; otherwise, it is called *irreducible*. A lattice with 0 and 1 is *complemented* if for each $x \in L$ there exists $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$. (Clearly an orthocomplemented lattice is complemented.) A *projective geometry* is an irreducible complemented modular lattice of finite height. A *projective plane* is a projective geometry of height 3. An essentially equivalent and more classical definition of a projective plane is a pair $(P, \mathscr{L})$ together with a notion of incidence such that (for 'points' in P and 'lines' in $\mathscr{L}$)

(a) two distinct points lie on exactly one line,
(b) two distinct lines meet in exactly one point,
(c) each line contains at least three points, and
(d) there exist at least three noncollinear points.

Let $(L, \leqslant)$ be a lattice which is a projective plane. Assume that L is orthocomplemented by $' : L \to L$. Let A be the set of atoms of L and, for $x, y \in A$, define $x \perp y$ in case $x \leqslant y'$ so that $(A, \perp)$ is a graph. Let $\mathscr{E}$ be the set of all maximal $\perp$-sets of atoms so that $(A, \mathscr{E})$ is the structure space of the graph $(A, \perp)$. Note that $(L, \leqslant, ')$ is isomorphic to $(\mathscr{C}(A, \perp), \subset, ^{\perp})$ under the mapping $x \to \{a \in A \mid a \leqslant x\}$ (for each $x \in L$).

Note also that $(A, \mathscr{E})$ is a nontrivial, tight cubic structure space. It is nontrivial because L is irreducible, tight because L is semimodular and cubic because L is cubic. The following theorem shows that every such space gives rise to an orthocomplemented projective plane.

THEOREM 1. *Let* $(X, \mathscr{E})$ *be a structure space.* $(X, \mathscr{E})$ *is nontrivial cubic and tight if and only if* $\mathscr{C}(X, \mathscr{E})$ *is an orthocomplemented projective plane with atoms* $X_1 = \{\{x\} \mid x \in X\}$.

Proof. Let $(X, \mathscr{E})$ be the nontrivial tight cubic structure space corresponding to the graph $(X, \perp)$, let $x \in X$ and $E \in \mathscr{E}$ with $x \in E$. If $y \in x^{\perp\perp} \backslash \{x\}$ then, since $(X, \mathscr{E})$ is cubic, $|x^{\perp} \cap y^{\perp}| \geqslant 2$ contradicting the fact that $(X, \mathscr{E})$ is tight; therefore $x^{\perp\perp} = \{x\}$ and X_1 is the set of atoms of $\mathscr{C}(X, \mathscr{E})$. If $\mathscr{C}(X, \mathscr{E}) \cong L_1 \times L_2$ where L_1 and L_2 are nontrivial then there exists $M \in \mathscr{C}(X, \mathscr{E})$ with $L_1 \cong \{N \in \mathscr{C}(X, \mathscr{E}) \mid \phi \subset N \subset M\}$ and $L_2 \cong \{N \in \mathscr{C}(X, \mathscr{E}) \mid \phi \subset N \subset M^{\perp}\}$. Since $(X, \mathscr{E})$ is cubic either M or $M^{\perp}$ is an atom of $\mathscr{C}(X, \mathscr{E})$, say M is an atom; then $M = \{m\}$ and it follows that $m \in E$ for all $E \in \mathscr{E}$, contradicting the nontriviality of $(X, \mathscr{E})$; therefore $\mathscr{C}(X, \mathscr{E})$ is irreducible.

To see that $\mathscr{C}(X, \mathscr{E})$ is semimodular we need only check that, for any $x, y \in X$ with $x \neq y$, $\{x, y\}^{\perp\perp} \succ \{x\}, \{y\}$; but there exists $z \in X$ such that $\{z\} = x^{\perp} \cap y^{\perp} = x^{\perp} \wedge y^{\perp}$, $X \neq z^{\perp} = \{x, y\}^{\perp\perp}$, and since $(X, \mathscr{E})$ is cubic $z^{\perp} = \{x\} \vee \{y\} \succ \{x\}, \{y\}$. $\mathscr{C}(X, \mathscr{E})$ is therefore semimodular and of height three (since $(X, \mathscr{E})$ is cubic) so that $\mathscr{C}(X, \mathscr{E})$ is modular. Thus $\mathscr{C}(X, \mathscr{E})$ is an orthocomplemented projective plane. The detailed proof of the converse, sketched above, is left to the reader.

COROLLARY 1. *There exist no nontrivial tight cubic structure spaces* $(X, \mathscr{E})$ *such that* $|X|$ *is finite.*

Proof. There exist no finite orthocomplemented projective planes.

I now show that every nontrivial wide cubic structure space $(X, \mathscr{E})$ may be embedded in a nontrivial tight cubic structure space $\pi(X, \mathscr{E})$, $\mathscr{C}(\pi(X, \mathscr{E}))$ is called the *free orthocomplemented projective plane* over $(X, \mathscr{E})$; it is Desarguesian if and only if $(X, \mathscr{E})$ is itself tight and Desarguesian. To my knowledge these are the first known non-Desarguesian orthocomplemented projective planes.

DEFINITION 1. *Let* $(X, \mathscr{E})$ *be a wide cubic structure space and let* $x, y \in X$. *The set* $\{x, y\}$ *is called* bad *in case* $x^{\perp} \cap y^{\perp} = \phi$, *that is* $|x^{\perp} \cap y^{\perp}| = 0$. *Let* Bad$(X, \mathscr{E})$ *be the set of all bad sets in* $(X, \mathscr{E})$. *If* $\{x, y\} \in$ Bad$(X, \mathscr{E})$

let $A_x^y=\{x, (x, y), \{x, y\}\}$ *and note that* $A_x^y \cap A_y^x=\{\{x, y\}\}$. *Let* $\mathrm{d}(X) = X \cup \bigcup \{A_x^y \mid \{x, y\} \in \mathrm{Bad}(X, \mathscr{E})\}$, $\mathrm{d}(\mathscr{E})=\mathscr{E} \cup \{A_x^y \mid \{x, y\} \in \mathrm{Bad}(X, \mathscr{E})\}$ *and* $\mathrm{d}(X, \mathscr{E})=(\mathrm{d}(X), \mathrm{d}(\mathscr{E}))$.

For obvious reasons we assume throughout that, for all $x, y \in X$, neither the set $\{x, y\}$ nor the ordered pair (x, y) are *elements* of X.

Let $(X_0, \mathscr{E}_0)=(X, \mathscr{E})$, $(X_1, \mathscr{E}_1)=\mathrm{d}(X_0, \mathscr{E}_0)$ and, inductively, $(X_n, \mathscr{E}_n) = \mathrm{d}(X_{n-1}, \mathscr{E}_{n-1})$ for all integers $n \geqslant 1$; let

$$X_\infty = \bigcup_{n=0}^{\infty} X_n, \mathscr{E}_\infty = \bigcup_{n=0}^{\infty} \mathscr{E}_n \quad \text{and} \quad \pi(X, \mathscr{E})=(X_\infty, \mathscr{E}_\infty).$$

THEOREM 2. *If* $(X, \mathscr{E})$ *is a wide cubic structure space then* $\pi(X, \mathscr{E})$ *is a tight cubic structure space. If* $(X, \mathscr{E})$ *is also nontrivial then* $\mathscr{C}(\pi(X, \mathscr{E}))$ *is an orthocomplemented projective plane.*

Proof. Let $(X, \mathscr{E})$ be a wide cubic structure space. Note that $\mathrm{d}(X, \mathscr{E})$ is again such a space so that $(X_n, \mathscr{E}_n)$ is one such for each $n \geqslant 0$. Let $x, y \in X_\infty$, then $x \in X_n$ and $y \in X_m$ for some $n, m \geqslant 0$; we may assume $n \geqslant m$ so that $\{x, y\} \subset X_n$ and there exists $z \in X_{n+1} \subset X_\infty$ with $\{z\}=x^\perp \cap y^\perp$ (in X_i, for $i=n+1, n+2, \ldots, \infty$); it follows that $\pi(X, \mathscr{E})$ is tight.

Clearly $\pi(X, \mathscr{E})$ is a cubic structure space. The rest follows from Theorem 1.

COROLLARY 2. *Let* $(X, \mathscr{E})$ *be a wide cubic structure space.* (i) $(X, \mathscr{E})$ *is tight if and only if* $(X, \mathscr{E})=\pi(X, \mathscr{E})$. (ii) $\mathscr{C}(\pi(X, \mathscr{E}))$ *is a Desarguesian projective plane if and only if* $\mathscr{C}((X, \mathscr{E}))$ *is a Desarguesian projective plane.*

Proof. (i) is evident. For (ii) one must show that a non-Desarguesian configuration always exists if $(X, \mathscr{E}) \neq \pi(X, \mathscr{E})$. We leave the translation of a non-Desargues configuration (that is, a configuration which violates Desargues' 'Theorem') into a wide cubic structure space $(A, \mathscr{A})$ to the reader. (Hint: if a point P is on a line 1 then $P \leqslant 1$ and $P \perp Q$ where $\{Q\}=1^\perp$; thus there exists a point R with $\{P, Q, R\} \in \mathscr{A}$; neither Q nor R need appear in the non-Desargues configuration.)

Let $(X, \mathscr{E})$ be a structure space. By a *substructure* of $(X, \mathscr{E})$ we mean a structure space $(A, \mathscr{A})$ such that $A \subset X$ and $\mathscr{A} \subset \mathscr{E}$. We say that $(A, \mathscr{A})$ is a *confined configuration* of $(X, \mathscr{E})$ in case $(A, \mathscr{A})$ is a substructure satisfying $|\{B \in \mathscr{A} \mid x \in B\}| \geqslant 2$ for all $x \in A$.

If $x \in X_\infty$ we define the *level* of x, $\lambda(x)$, to be the smallest integer n such that $x \in X_n$. A confined configuration $(A, \mathscr{A})$ of $\pi(X, \mathscr{E})$ is called *bounded* in case there exists $n \in \mathbb{N}$ such that $A \subset X_n$, i.e., $\lambda(x) \leqslant n$ for all $x \in A$.

PROPOSITION 1. *Any bounded confined configuration of $\pi(X, \mathscr{E})$ is contained in $(X_0, \mathscr{E}_0)$.*

Proof. Let $(A, \mathscr{A})$ be a confined configuration in $\pi(X, \mathscr{E})$, $N = \max\{\lambda(x) \mid x \in A\} < \infty$ (since A is bounded), and $x_0 \in A$ with $\lambda(x_0) = N$. Suppose $N > 0$, then there exist $y, z \in X_{N-1} \cap A$ such that $x_0 \in \{(y, z), (z, y), \{y, z\}\}$.

Whichever element equals x_0, it follows that $(y, z) \in A$; but $|\{B \in \mathscr{A} \mid (y, z) \in B\}| = 1$, contradiction. Hence $A \subset X_0$.

II. STATES ON NON-DESARGUESIAN PROPOSITION SYSTEMS

One reason that non-Desarguesian orthocomplemented projective planes may be of interest in the foundations of quantum mechanics is that they are proposition systems (Jauch, 1968). Recall that a *proposition system* is an irreducible complete atomic semimodular orthomodular lattice. (For orthomodular lattices 'atomic' is equivalent to 'atomistic'.) The non-Desarguesian orthocomplemented projective planes are precisely the proposition systems which may not be represented by subspaces of a left vector space over a division ring.

In this part I define quantum logics, weights, and states. I show that a (non-Desarguesian) proposition system $\mathscr{L}$ may or may not be a quantum logic and, even if it is, there exists a state $\mu: \mathscr{L} \to [0, 1]$ and $a, b \in \mathscr{L}$ such that $\bar{\mu}(a) = \bar{\mu}(b) = 1$ but $\bar{\mu}(a \wedge b) = 0$.

A *quantum logic* (Greechie and Gudder, 1973) is a pair $(\mathscr{L}, \mathscr{S})$ where $\mathscr{L}$ is a σ-orthocomplete orthomodular poset and $\mathscr{S}$ is a full set of states on $\mathscr{L}$. In order to facilitate the exposition in this article I consider only orthomodular posets $\mathscr{L}$ which are, in fact, complete lattices. A *state* on $\mathscr{L}$ is a function $\sigma: \mathscr{L} \to [0, 1] \subset \mathbb{R}$ such that $\sigma(0) = 0$, $\sigma(1) = 1$ and if $\{x_i \mid i \in \mathbb{N}\}$ is a countable family of mutually orthogonal ($x_i \leqslant x_j'$ if $i \neq j$) elements of $\mathscr{L}$ then $\sigma(\bigvee x_i) = \sum \sigma(x_i)$. A weight on a structure space $(X, \mathscr{E})$ is a function $\omega: X \to [0, 1] \subset \mathbb{R}$ such that

$$\sum_{x\in E} \omega(x)=1 \quad \text{for each} \quad E\in\mathscr{E}.$$

Let $\mathscr{L}=\mathscr{C}(X,\mathscr{E})$ be an orthomodular lattice. For each state σ on $\mathscr{L}$ $\sigma|_X$ is a weight on $(X,\mathscr{E})$. If each $E\in\mathscr{E}$ is finite then every weight ω on $(X,\mathscr{E})$ induces a unique state $\bar{\omega}$ on $\mathscr{L}$ by defining $\bar{\omega}(M)=\sum_{d\in D}\omega(d)$ for any $D\subset X$ such that $D\subset E$ for some $E\in\mathscr{E}$, $D\subset M=M^{\perp\perp}$ and $D^{\perp\perp}=M$. (See Greechie and Miller, 1970, Theorem 1.6.)

Let $\mathscr{S}_{\mathscr{L}}$ be the set of all states on $\mathscr{L}=\mathscr{C}(X,\mathscr{E})$ and let $\Omega(X,\mathscr{E})$ be the set of all weights on $(X,\mathscr{E})$; the mapping $\omega\to\bar{\omega}$ from $\Omega(X,\mathscr{E})$ onto $\mathscr{S}_{\mathscr{L}}$ is a convex bijection between the two convex sets $\Omega(X,\mathscr{E})$ and $\mathscr{S}_{\mathscr{L}}$ (when each $E\in\mathscr{E}$ is finite). Any set Ω of weights on $(X,\mathscr{E})$ is said to be *full* in case, for all $x, y\in X$ with $x\neq y$, $\{x, y\}\subset E$ for some $E\in\mathscr{E}$ if and only if $\omega(x)+\omega(y)<1$ for all $\omega\in\Omega$. Any set $\mathscr{S}$ of states on $\mathscr{L}$ is said to be *full* in case, for all $x, y\in\mathscr{L}$, $x\leqslant y$ if and only if $\sigma(x)\leqslant\sigma(y)$ for all $\sigma\in\mathscr{S}$. The mapping $\omega\to\bar{\omega}$ defined above corresponds full sets of weights with full sets of states (Greechie and Miller, 1970); Ω is full if and only if $\{\bar{\omega}\mid\omega\in\Omega\}$ is full. $\mathscr{S}$ (respectively, Ω) is said to satisfy the *projection postulate* if for all $x\in\mathscr{L}\backslash\{0\}$ (respectively, X) there exists $\sigma\in\mathscr{S}$ ($\omega\in\Omega$) with $\sigma(x)=1$ ($\omega(x)=1$).

A set of states $\mathscr{S}$ on $\mathscr{L}=\mathscr{C}(X,\mathscr{E})$ is said to be *strong* in case $x\leqslant y$ in $\mathscr{L}$ if and only if $\sigma(x)=1$ implies $\sigma(y)=1$ for all $\sigma\in\mathscr{S}$; a set Ω of weights on $(X,\mathscr{E})$ is strong in case, for all $x, y\in X$ with $x\neq y$, $x\perp' y$ if and only if there exists $\mu\in\Omega$ with $\mu(x)=1$, $\mu(y)>0$. A correspondence, similar to that for full sets, exists between strong sets of weights and strong sets of states.

Any state (or weight) is called *dispersion free* if its image is a subset of the two-element set $\{0, 1\}$, that is, it takes on no value other than 0 or 1.

Remark 1. Let $(X,\mathscr{E})$ be a wide cubic structure space, $x, y\in X$ with $x^{\perp}\cap y^{\perp}=\emptyset$. Let $X_{\{x,y\}}=X\cup\{(x, y), (y, x), \{x, y\}\}$ and $\mathscr{E}_{\{x,y\}}=\mathscr{E}\cup\{\{x, (x, y), \{x, y\}\}, \{y, (y, x), \{x, y\}\}$. If $\mu\in\Omega(X,\mathscr{E})$ define $\mu^i_{\{x,y\}}: X_{\{x,y\}}\to[0, 1]$, for $i=1, 2$ as follows

$$\mu^1_{\{x,y\}}(z)=\begin{cases}\mu(z) & \text{if} \quad z\in X\\ \max(\mu(y)-\mu(x), 0) & \text{if} \quad z=(x, y)\\ \max(\mu(x)-\mu(y), 0) & \text{if} \quad z=(y, x)\\ \min(1-\mu(x), 1-\mu(y)) & \text{if} \quad z=\{x, y\}\end{cases}$$

$$\mu^2_{\{x,y\}}(z) = \begin{cases} \mu(z) & \text{if } z \in X \\ 1-\mu(x) & \text{if } z=(x,y) \\ 1-\mu(y) & \text{if } z=(y,x) \\ 0 & \text{if } z=\{x,y\}. \end{cases}$$

Then

$$\mu^i_{\{x,y\}} \in \Omega(X_{\{x,y\}}, \mathscr{E}_{\{x,y\}}).$$

Note: if $\mu(z)=\frac{1}{3}$ for all $z \in X$ then $\mu^1_{\{x,y\}}(\{x,y\})=\frac{2}{3}$ and $\mu^2_{\{x,y\}}((x,y))=\frac{2}{3}$. Also if $b \in X$ and there exists μ, a weight on $(X_{\{x,y\}}, \mathscr{E}_{\{x,y\}})$, with $\mu(\{x,y\}) + \mu(b) > 1$ then there exists a weight on $X_{\{x,y\}}$ with $\mu((x,y))+\mu(b)>1$.

LEMMA 2. *Let $(X, \mathscr{E})$ be a wide cubic structure space and Ω a strong convex set of weights on $(X, \mathscr{E})$. Then Ω induces a full convex set of weights on* $\mathrm{d}(X, \mathscr{E})$.

Proof. We need only show that if $\{x,y\} \in \mathrm{Bad}(X, \mathscr{E})$ then Ω induces a full set of weights on $(X_{\{x,y\}}, \mathscr{E}_{\{x,y\}})$. The statement of the lemma then follows by transfinite induction and Remark 1.

To show that $(X_{\{x,y\}}, \mathscr{E}_{\{x,y\}})$ is full, let $a, b \in X_{\{x,y\}}$ with $a \perp' b$, we must prove that there exists $\mu \in \Omega(X_{\{x,y\}}, \mathscr{E}_{\{x,y\}})$ with $\mu(a)+\mu(b)>1$. We may assume that $a=\{x,y\}$.

If $b \perp' x, y$ then there exist $x_1, y_1 \in X$ with $b \perp' x_1$, $b \perp' y_1$, $x_1 \perp x$, and $y_1 \perp y$. There exist $\mu_1, \mu_2 \in \Omega$ such that $\mu_1(b)=\mu_2(b)=1$, $\mu_1(x_1)>0$ and $\mu_2(y_1)>0$. Let $\mu=\frac{1}{2}(\mu_1+\mu_2)$ so that $\mu(b)=1$, $\mu(x_1)>0$ and $\mu(y_1)>0$; then $\mu^1_{\{x,y\}}(z)=\min(1-\mu(x), 1-\mu(y))>0$ since $\mu(x)<1$ and $\mu(y)<1$. It remains to consider the case in which $b \perp x$ or $b \perp y$. By symmetry we may assume that $b \perp x$. If $b \perp x$ then $b \perp' y$ since $x^\perp \cap y^\perp \cap X = \emptyset$. There exists $y_1 \perp y$ with $b \perp' y_1$. Let $v \in \Omega$ with $v(b)=1$ and $v(y_1)>0$. Again $v^1_{\{x,y\}}(\{x,y\})>0$.

LEMMA 3. *Let $x, y \in \mathrm{d}(X, \mathscr{E})=(X_1, \mathscr{E}_1)$ with $\{x,y\} \in \mathrm{Bad}(X_1, \mathscr{E}_1)$ and assume that Ω is a strong set of weights on $(X, \mathscr{E})$, then Ω induces a full set of weights on $((X_1)_{\{x,y\}}, (\mathscr{E}_1)_{\{x,y\}})$.*

Proof. Let $a, b \in (X_1)_{\{x,y\}}$ with $a \perp' b$. We may assume $a=\{x,y\}$, $b \in X_1$, $b \neq x, y$ *and* $x \in X_1 \backslash X$. If $y \notin X$ then there exists $\mu \in \Omega$ with $\mu(b)=1$; μ may be extended so that $\mu(x)=\mu(y)=0$ and therefore $\mu(\{x,y\})=1$. Thus we may assume $y \in X$. If $y \perp b$ any μ with $\mu(b)=1$ can be extended to $\mu(\{x,y\}) =1$.

If $y \perp' b$ there exists $y_1 \perp y$, $y_1 \perp' b$. Select μ from Ω so that $\mu(b)=1$ and

$\mu(y_1)>0$, extend so that $\mu(x)=0$, $\mu^1_{\{x,y\}}(\{x, y\})>\mu(y_1)>0$. Therefore Ω induces a full set of weights on $(X_{1_{\{x,y\}}}, \mathscr{E}_{1_{\{x,y\}}})$.

THEOREM 3. *If Ω is a strong set of weights on $(X, \mathscr{E})$, then Ω induces a full set of weights on $\pi(X, \mathscr{E})$.*

Proof. From Lemma 3 it follows that Ω induces a full set of weights on $(X_2, \mathscr{E}_2)$. Induction and a proof similar to that of the lemma provides a full set of weights on $\Omega(X_i, \mathscr{E}_i)$, for each $i>0$, and therefore on $\pi(X, \mathscr{E})$.

COROLLARY 3. *If $\mathscr{C}(X, \mathscr{E})$ has a strong set $\mathscr{S}$ of states, then $\mathscr{C}(\pi(X, \mathscr{E}))$ has a full set $\mathscr{S}'$ of states.*

Thus there exist non-Desarguesian proposition systems which are also quantum logics. For example, let $X=\{a, b, c, d, e, f\}$ and let $\mathscr{E}=\{\{a, b, c\}, \{d, e, f\}\}$ then $(X, \mathscr{E})$ is a wide cubic structure space; it is easy to see that there is a strong set of dispersion-free weights (there are nine of them) on $(X, \mathscr{E})$.

Let us single out of them, say $\mu_0 : X \to [0, 1]$ defined by $\mu_0(a)=\mu_0(d)=1$ and $\mu_0(x)=0$ if $x \neq a, d$. Extend μ_0 to a weight μ on $\pi(X, \mathscr{E})$; then μ defines a state $\bar{\mu}$ on $\mathscr{C}(\pi(X, \mathscr{E}))$. Note that, since $a \wedge b=0$,

$$\bar{\mu}(a)=\bar{\mu}(b)=1 \quad \text{but} \quad \bar{\mu}(a \wedge b)=0.$$

Recently, Jauch (this volume) has asked if the condition

$$(4^0) \qquad \sigma(a)=\sigma(b)=1 \quad \text{implies} \quad \sigma(a \wedge b)=1$$

is necessarily true for any state on a proposition system of dimension at least three. The answer is evidently 'no'. However, for dimension at least *four* the question remains unanswered.

Before leaving this example we should note that by Corollary 3, there exists a full set of states on $\mathscr{C}(\pi(X, \mathscr{E}))$ and this set satisfies the projection postulate. However, there does *not* exist a full set of dispersion-free states on $\mathscr{C}(\pi(X, \mathscr{E}))$ – even though each dispersion-free state on $(X, \mathscr{E})$ extends to infinitely many different dispersion-free states on $\mathscr{C}(\pi(X, \mathscr{E}))$.

Remark 2. $\pi(X, \mathscr{E})$ never admits a full set of dispersion-free weights, provided that $(X, \mathscr{E}) \neq \pi(X, \mathscr{E})$.

Proof. The idea of the proof is given by Figure 1. In this structure it is easy to see that no weight maps both $\{x, y\}$ and $\{x_1, y_1\}$ to 1; moreover, Figure 1 is always a substructure of $\pi(X, \mathscr{E})$ if $(X, \mathscr{E}) \neq \pi(X, \mathscr{E})$.

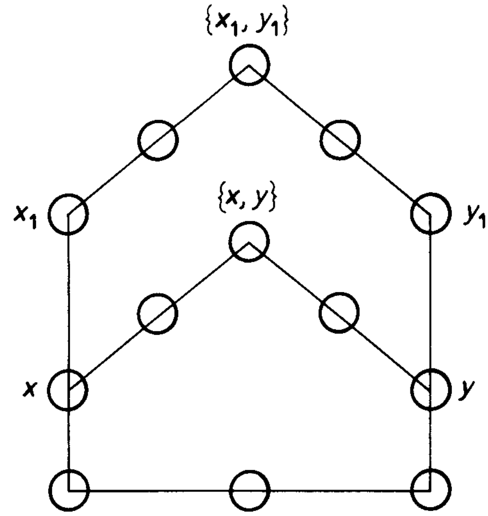

Fig. 1. A cubic structure space which does not admit a full set of dispersion-free weights.

PROPOSITION 2. *If $\Omega(X, \mathscr{E})$ is not full then $\Omega(\pi(X, \mathscr{E}))$ is not full. Thus there exist proposition systems $\mathscr{C}(\pi(X, \mathscr{E})$ which are not quantum logics.*

Proof. If $\Omega(\pi(X, \mathscr{E}))$ were full then the restrictions of these weights to $(X, \mathscr{E})$ would be full.

For example, let $(X, \mathscr{E})$ be given by Figure 2. This space does not admit a full set of weights (Bennett, 1970). Thus $\mathscr{C}(\pi(X, \mathscr{E}))$ is not a quantum logic.

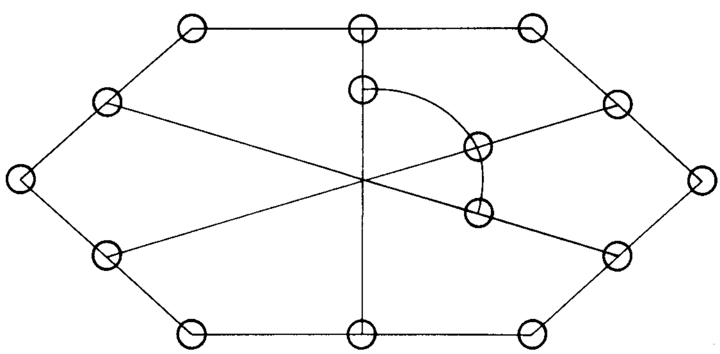

Fig. 2. A structure space which does not admit a full set of weights.

III. AUTOMORPHISM GROUPS

In this part let us consider the automorphism group of $\pi(X, \mathscr{E})$. If each point $x \in X$ is in at least two blocks $E \in \mathscr{E}$ then the automorphism group of $\pi(X, \mathscr{E})$ is isomorphic to the automorphism group of $(X, \mathscr{E})$. This insures that the automorphism group of $\mathscr{C}(\pi(X, \mathscr{E}))$ is isomorphic to the automorphism group of $\mathscr{C}(X, \mathscr{E})$. By taking the structure space $(X, \mathscr{E})$ to be, say, that of Figure 2 one may exhibit a proposition system $\mathscr{C}(\pi(X, \mathscr{E}))$ with an interesting *finite* group of automorphisms. I conclude the part by showing that to each Hilbert space $\mathscr{H}$ one may assign a non-Desarguesian projective plane π such that the unitary group on $\mathscr{H}$ is a subgroup of the automorphisms of π and therefore of the corresponding (non-Desarguesian) proposition system $\mathscr{C}(\pi)$.

Let (X_0, R_0) be a graph. An element $x \in X$ is called an *isolated point* in case $\{x\} \in \mathscr{E}_{R_0}$. (X_0, R_0) is a *triangleless* graph in case $E \in \mathscr{E}_R$ implies $|E| \leqslant 2$. If (X_0, R_0) is a triangleless graph with no isolated points we define $\delta(X_0, R_0)$ to be the graph (X, R) where $X = X_0 \bigcup \{\{\{x, y\}\} \mid (x, y) \in R\}$ and $R = R_0 \cup \bigcup \{\{(x, \{\{x, y\}\}), (\{\{x, y\}\}, x)\} \mid (x, y) \in R_0\}$.

(This construction appears elsewhere in graph theory. I use the symbol δ because it corresponds to a special case of a construction on general structure spaces due to J. C. Dacey, Jr.)

Recall that a graph (X_0, R_0) is a *bigraph relative to the partition* $\{X_1, X_2\}$ in case $X_0 = X_1 \cup X_2$, $X_1 \cap X_2 = \phi$ and $R_0 \subset (X_1 \times X_2) \cup (X_2 \times X_1)$. A *bigraph* is a graph which is a bigraph relative to some partition. Note that every bigraph is triangleless.

Remark 3. Let (X_0, R_0) be a bigraph with no isolated points. The space $(X, \mathscr{E})$ corresponding to $\delta(X_0, R_0)$ is a wide cubic structure space if and only if there are no minimal cycles of order 4 in (X_0, R_0).

EXAMPLE 1. Let L be a semimodular atomistic lattice of height at least 3 and let $A_i = \{x \in L \mid ht(x) = i\}$ for $i = 1, 2$. Let $X_0 = A_1 \cup A_2$, $R_0 = \{(x, y) \in X_0 \times X_0 \mid x < y \text{ or } y < x\}$ be the 'strict comparability' relation restricted to X_0 and let $(X, \mathscr{E})$ be the structure space associated with the graph $\delta(X_0, R_0)$. Since L is a lattice there are no minimal cycles in (X_0, R_0) of order 4. Hence $(X, \mathscr{E})$ is a wide cubic structure space.

EXAMPLE 2. Let $(X_0, \mathscr{E}_0)$ be a wide cubic structure space and let

$L=\mathscr{C}(X_0, \mathscr{E}_0)$. Let $A_1=\{\{x\} \mid x\in X_0\}$ be the atoms of L and let $A_2=\{x^{\perp} \mid x\in X_0\}$ be the 'coatoms' of L. As in Example 1 let R_0 be the 'strict comparability' relation restricted to $Y_0=A_1 \cup A_2$. Then, by Lemma 1 and Remark 3, the structure space $(X, \mathscr{E})$ associated with the graph (Y_0, R_0) is wide and cubic. The process may be iterated.

We now review the definitions of automorphisms of some of the various structures with which we are concerned. An *automorphism of a graph* (X, R) is a bijection $\varphi: X\to X$ such that $(x, y)\in R$ if and only if $(\varphi(x), \varphi(y))\in R$. Every bijection $\varphi: X\to X$ from a set X onto itself induces a bijection $\tilde{\varphi}: 2^X\to 2^X$ defined by $\tilde{\varphi}(E)=\varphi(E)$ for all $E\in\mathscr{E}$ (where $\varphi(E)=\{\varphi(x) \mid x\in E\}$). An *automorphism of a space* $(X, \mathscr{E})$ is a bijection $\varphi: X\to X$ for which $\tilde{\varphi}(\mathscr{E})=\mathscr{E}$.

Let $\mathrm{Aut}(X, R)$ and $\mathrm{Aut}(X, \mathscr{E})$ denote the group of all automorphisms of the graph (X, R) and the space $(X, \mathscr{E})$, respectively. It is easy to see that if $\mathscr{E}=\mathscr{E}_R$ then $\mathrm{Aut}(X, R)=\mathrm{Aut}(X, \mathscr{E})$. If (X, R) is a bigraph relative to the partition $\{X_1, X_2\}$ of X, we define $\mathrm{Aut}_1(X, R)=\{\varphi\in\mathrm{Aut}(X, R) \mid \varphi(X_1)=X_1\}$; thus $\mathrm{Aut}_1(X, R)$ is the subgroup of $\mathrm{Aut}(X, R)$ which maps X_1 to itself; note that the elements of X_1 need not be fixed-points of the automorphism.

If L is an orthomodular lattice $\mathrm{Aut}_{\perp}(L)$ is the group of all 'orthoautomorphisms', that is lattice automorphisms which also preserve the orthocomplementation. If $(X, \mathscr{E})$ is a wide cubic structure space, it is easy to see that

$$\mathrm{Aut}(X, \mathscr{E})\cong\mathrm{Aut}_{\perp}(\mathscr{C}(X, \mathscr{E})).$$

Let $(X, \mathscr{E})$ be a finite wide cubic structure space with weights $\Omega=\Omega(X, \mathscr{E})$. Every automorphism φ on $(X, \mathscr{E})$ induces an affine bijection $\tilde{\varphi}: \Omega\to\Omega$ such that $\bar{\varphi}(e)=e$ where $e(x)=\frac{1}{3}$ for all $x\in X$. Let $\mathrm{Aut}(\Omega, e)$ be the group of all affine bijections on Ω which fix e. If Ω is strong then

$$\mathrm{Aut}(X, \mathscr{E})\cong\mathrm{Aut}(\Omega, e).$$

For a proof of this fact, see Theorem 5.2. of Gerelle *et al* (1974).

PROPOSITION 3. *Let* $(X, \mathscr{E})$ *be a wide cubic structure space. Then* Aut $(X, \mathscr{E})$ *is a subgroup of* $\mathrm{Aut}(\pi(X, \mathscr{E}))$. *If* $|\{E\in\mathscr{E} \mid x\in E\}|\geqslant 2$ *for each* $x\in X$, *then*

$$\mathrm{Aut}(X, \mathscr{E})\cong\mathrm{Aut}(\pi(X, \mathscr{E})).$$

Proof. By a simple induction every automorphism on $(X, \mathscr{E})$ extends uniquely to an automorphism of $\pi(X, \mathscr{E})$. If $(X, \mathscr{E})$ is a confined configuration then, by Proposition 1, $\varphi \in \mathrm{Aut}\,(\pi(X, \mathscr{E}))$ implies $\varphi\,|_X \in \mathrm{Aut}(X, \mathscr{E})$ since $(X, \mathscr{E})$ is bounded in $\pi(X, \mathscr{E})$. The result follows.

Let A_i, L, (X_0, R_0) and $(X, \mathscr{E})$ be defined as in Example 1. Then every lattice automorphism φ of L induces, by restriction, an automorphism $\bar{\varphi} \in \mathrm{Aut}_1(X_0, R_0)$. The map $\varphi \to \bar{\varphi}$ is an injection since L is atomistic. It follows that $\mathrm{Aut}(L)$ is isomorphic to a subgroup of $\mathrm{Aut}(X, \mathscr{E})$.

Now specify L to be the lattice of all closed linear subspaces of a Hilbert space $\mathscr{H}$ of dimension at least 3. $\mathrm{Aut}_\perp(L)$ is a subgroup of $\mathrm{Aut}\,L$ and $\mathrm{Aut}(X, \mathscr{E})$ is a subgroup of $\mathrm{Aut}(\pi(X, \mathscr{E})) \cong \mathrm{Aut}_\perp \mathscr{C}(\pi(X, \mathscr{E}))$. It follows that the general quantum mechanical group on $\mathscr{H}$ is a subgroup of $\mathrm{Aut}_\perp(\mathscr{C}(\pi(X, \mathscr{E})))$. Thus any group representable on $\mathscr{H}$ is representable on the proposition system $\mathscr{C}(\pi(X, \mathscr{E}))$.

IV. CONCLUSION

The construction for free non-Desarguesian projective planes π is a variant of the standard construction of free non-Desarguesian projective planes (Hartshorne, 1967). Just as there are non-Desarguesian projective planes other than the free ones there are likely to be other non-Desarguesian orthocomplemented projective planes. The corresponding proposition systems could prove to be interesting from the point of view of logic and probability in the foundations of quantum mechanics. Perhaps more important – although certainly speculative – is the fact that it is not inconceivable that there is some real world physics modeled by these or related structures.

One interesting class of related structures is that of σ-orthocomplete semimodular atomistic orthomodular *posets P*. An investigation of such posets was begun in Haskins *et al.* (1974) where it was proved that if P is of finite height then perspectivity is transitive. A more detailed investigation of this class of structures is in progress. I wish to point out here only that the construction of the free non-Desarguesian projective planes carries over for cubic posets. Instead of working with the complete lattice $\mathscr{C}(X, \mathscr{E})$ one works with $\mathscr{L}(X, \mathscr{E}) = \{D^{\perp\perp} \subset X \mid D \text{ is a } \perp\text{-set}\}$.

Call a structure space $(X, \mathscr{E})$ a *partial plane* in case $E \in \mathscr{E}$ implies $|E| \geqslant 3$

and $E \neq E_1$ implies $|E \cap E_1| \leqslant 1$. The logic $\mathscr{L}(X, \mathscr{E})$ of a partial plane is an orthocomplete orthomodular poset (Gerelle *et al.*, 1974). If $(X, \mathscr{E})$ is cubic then the process of eliminating 'bad sets' developed in the constructions of the free projective planes yields a semimodular orthomodular poset $\mathscr{L}(\pi(X, \mathscr{E}))$.

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NOTE

* This paper was written while the author was on sabbatical leave at the University of Geneva.

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