Research Article

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Some results in cone metric spaces with applications in homotopy theory

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Abstract: The self-mappings satisfying implicit relations were introduced in a previous study [Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cerc. St. Ser. Mat. Univ. Bacău **7** (1997), 129–133]. In this study, we introduce self-operators satisfying an ordered implicit relation and hence obtain their fixed points in the cone metric space under some additional conditions. We obtain a homotopy result as an application.

Keywords: fixed point, implicit relation, cone metric space, contraction

MSC 2010: 47H09, 47H10, 54H25

1 Introduction

The importance of the Banach contraction principle lies in the fact that it is an indispensable tool to check the existence of solutions of differential equations, integral equations, matrix equations, and functional equations, which are formed by mathematical modeling of real-word problems. There have been fixed point theorists to enhance both the underlying space and contractive condition (explicit type) used by Banach [1] under the effect of one of the structures like order metric structure [2,3], graphic metric structure [4], multivalued mapping structure [5], α-admissible mapping structure [6], comparison functions, and auxiliary functions

Recently, Huang and Zhang [7] introduced the structure of cone metric by replacing real numbers with an ordering Banach space and established a convergence criterion for sequences in a cone metric space to generalize the Banach fixed point theorem. Huang and Zhang [7] considered the concept of a normal cone for their findings; however, Rezapour and Hamlbarani [8] omitted this concept in some results by Huang. Many authors have investigated fixed point theorems and common fixed point theorems of self-mappings for normal and non-normal cones in cone metric spaces [9–12].

On the other hand, Popa [13] introduced a new class of functions with three properties and obtained fixed points of self-mappings satisfying an implicit relation under the effect of function from this new class. Popa [13–15] obtained some fixed point theorems in metric spaces; however, investigation of fixed points of self-mappings satisfying implicit relations in order metric structure was carried out by Beg and Butt [16,17], and some common fixed point theorems were established by Berinde and Vetro [18,19] and Sedghi et al. [20].

In this study, we investigate fixed points of self-operators satisfying an ordered implicit relation in the framework of the cone metric spaces. These results are supported by an example and an application in homotopy theory.

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2 Preliminaries

Definition 2.1. A binary relation $\mathcal{R}$ over a set $X \neq \phi$ defines a partial order if it has following axioms:
(1) $\mathcal{R}$ is reflexive;
(2) $\mathcal{R}$ is antisymmetric;
(3) $\mathcal{R}$ is transitive.

A set having partial order $\mathcal{R}$ is known as a partially ordered set denoted by $(X, \mathcal{R})$.

Let $(\mathcal{E}, \|\cdot\|)$ be a real Banach space. A subset C of $\mathcal{E}$ is called a cone if and only if
(1) $C \neq \phi$, closed and $C \neq \{\mathbf{0}\}$;
(2) for all $a, b \in \mathbb{R}$ with $a, b \geq 0$ and $\sigma, \xi \in C$, we have $a\sigma + b\xi \in C$;
(3) $C \cap (-C) = \{\mathbf{0}\}$.

Given $C \subset \mathcal{E}$, the partial order $\preccurlyeq$ with respect to C is defined as follows:

$$\sigma \preccurlyeq \xi \text{ if and only if } \xi - \sigma \in C \text{ for all } \sigma, \xi \in \mathcal{E}.$$

We shall write $\sigma \prec \xi$ to indicate that $\sigma \preccurlyeq \xi$ but $\sigma \neq \xi$, while $\sigma \ll \xi$ stands for $\xi - \sigma \in C^{\circ}$ (interior of C).

The cone C is said to be a normal cone if there exists a positive constant K, such that

$$\sigma \preccurlyeq \xi \text{ implies } \|\sigma\| \leq K\|\xi\|, \quad \text{for all } \sigma, \xi \in \mathcal{E}.$$

Throughout this article, we let $X = (X, \mathcal{R})$ and $\preccurlyeq$ being a partial order with respect to cone C defined in $\mathcal{E}$. If $X \subseteq \mathcal{E}$, then $\mathcal{R}$ and $\preccurlyeq$ are identical, otherwise are different.

Definition 2.2. [7] A mapping $d\colon X \times X \mapsto \mathcal{E}$ is said to be a cone metric if for all $\sigma, \xi, \nu, \in X$ the following conditions are satisfied:
(d1) $\mathbf{0} \preccurlyeq d(\sigma, \xi)$ and $d(\sigma, \xi) = \mathbf{0}$ if and only if $\sigma = \xi$;
(d2) $d(\sigma, \xi) = d(\xi, \sigma)$;
(d3) $d(\sigma, \xi) \preccurlyeq d(\sigma, \nu) + d(\nu, \sigma)$.

The cone metric space is denoted by (X,d).

Definition 2.3. [7] Let $\mathcal{E}$ be a real Banach space, (X, d) be a cone metric space, and $c \in \mathcal{E}$ with $0 \ll c$. A sequence $\{\sigma_n\}$ is called a Cauchy sequence if there exists a natural number $N \in \mathbb{N}$, such that $d(\sigma_n, \sigma_m) \ll c$ for all $n, m > N$. The sequence $\{\sigma_n\}$ is said to be convergent if there exists an $N \in \mathbb{N}$, such that $d(\sigma_n, \sigma) \ll c$ for all $n \geq N$ and $\sigma \in X$.

3 Ordered implicit relations

Fixed point theorem involving implicit relations provides a formula to show the existence of a solution of the nonlinear functional equation. In this regard, many authors have presented different fixed point results, which then were applied to solve nonlinear functional equations [16–19,21,22].

Let $(\mathcal{E}, \|\cdot\|)$ be a real Banach space and $B(\mathcal{E}, \mathcal{E})$ be the space of all bounded linear operators $T\colon \mathcal{E} \to \mathcal{E}$ with $\|T\|_1 < 1$, where $\|\cdot\|_1$ is the usual norm defined in $B(\mathcal{E}, \mathcal{E})$.

In this section, consistent with [13], we introduce the following:

Let $\mathcal{A}\colon \mathcal{E}^6 \to \mathcal{E}$ be an operator, which satisfies the conditions given below.

($\mathcal{A}_1$) $\sigma_5 \preccurlyeq \upsilon_5$ and $\sigma_6 \preccurlyeq \upsilon_6 \Rightarrow \mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6) \preccurlyeq \mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \sigma_5, \sigma_6)$.

($\mathcal{A}_2$) if either

$$\mathcal{A}(\sigma, \upsilon, \upsilon, \sigma, \sigma + \upsilon, \mathbf{0}) \preccurlyeq \mathbf{0}$$

or

$$\mathcal{A}(\sigma, \upsilon, \sigma, \upsilon, \mathbf{0}, \sigma + \upsilon) \preccurlyeq \mathbf{0},$$

then there exists $T \in B(\mathcal{E}, \mathcal{E})$, such that $\sigma \preccurlyeq T(\upsilon)$ (for all $\sigma, \upsilon \in \mathcal{E}$).

($\mathcal{A}_3$) $\mathcal{A}(\sigma, \mathbf{0}, \mathbf{0}, \sigma, \sigma, 0) \succ \mathbf{0}$ whenever $\|\sigma\| > 0$.

Let $\mathcal{G} = \{\mathcal{A}: \mathcal{E}^6 \to \mathcal{E} | \mathcal{A}$ satisfies conditions $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$.

Example. Let $\preccurlyeq$ be the partial order with respect to cone C, as defined in Section 2. Let $(\mathcal{E}, \|\cdot\|)$ be a real Banach space and $\mathcal{A}: \mathcal{E}^6 \to \mathcal{E}$ is defined as

$$\mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6) = \upsilon_1 - \alpha \max\{\upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6\} \text{ for all } \upsilon_i \in \mathcal{E}(i = 1 \text{ to } 6) \text{ and } \alpha \in \left[0, \frac{1}{2}\right).$$

Then, the operator $\mathcal{A} \in \mathcal{G}$:

($\mathcal{A}_1$). Let $\upsilon_5 \preccurlyeq \gamma_5$ and $\upsilon_6 \preccurlyeq \gamma_6$, then $\gamma_5 - \upsilon_5 \in C$ and $\gamma_6 - \upsilon_6 \in C$. Now, we show that $\mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6) - \mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \gamma_5, \gamma_6) \in C$. Consider,

$$\begin{aligned}\mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6) - \mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \gamma_5, \gamma_6) &= \upsilon_1 - \alpha \max\{\upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6\} - (\upsilon_1 - \alpha \max\{\upsilon_2, \upsilon_3, \upsilon_4, \gamma_5, \gamma_6\}) \\ &= \alpha \max\{\mathbf{0}, \mathbf{0}, \mathbf{0}, \gamma_5 - \upsilon_5, \gamma_6 - \upsilon_6\} \in C.\end{aligned}$$

Thus, $\mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \gamma_5, \gamma_6) \preccurlyeq \mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6)$.

($\mathcal{A}_2$). Let $\upsilon, \gamma \in \mathcal{E}$ be such that $\mathbf{0} \preccurlyeq \upsilon, \mathbf{0} \preccurlyeq \gamma$. If $\mathcal{A}(\gamma, \upsilon, \upsilon, \gamma, \gamma + \upsilon, \mathbf{0}) \preccurlyeq \mathbf{0}$, then we have

$$-\gamma + \alpha \max\{\upsilon, \upsilon, \gamma, \gamma + \upsilon, \mathbf{0}\} \in C.$$

For if $\gamma = \mathbf{0}$, then $\alpha\upsilon \in C$. Thus, there exists $T: \mathcal{E} \to \mathcal{E}$ defined by $T(\upsilon) = \eta\upsilon$ ($0 \le \eta < \frac{1}{2}$ and $\eta = \alpha$ is a scalar), such that $\gamma \preccurlyeq T(\upsilon)$. Now, if $\gamma \neq \mathbf{0}$, then $-\gamma + \alpha \max\{\upsilon, \upsilon, \gamma, \gamma + \upsilon, \mathbf{0}\} \in C$ implies $\alpha(\gamma + \upsilon) - \gamma \in C$, which then gives $\alpha\upsilon - (1 - \alpha)\gamma \in C$. Thus, $(1 - \alpha)\gamma \preccurlyeq \alpha\upsilon$, so there exists $T: \mathcal{E} \to \mathcal{E}$ defined by $T(\upsilon) = \eta\upsilon$ ($\eta = \frac{\alpha}{1-\alpha}$ is a scalar), such that $\gamma \preccurlyeq T(\upsilon)$.

($\mathcal{A}_3$). Let $\upsilon \in \mathcal{E}$ be such that $\|\upsilon\| > 0$ and consider $\mathbf{0} \preccurlyeq \mathcal{A}(\upsilon, \mathbf{0}, \mathbf{0}, \upsilon, \upsilon, \mathbf{0})$, then $\upsilon - \alpha \max\{\mathbf{0}, \mathbf{0}, \upsilon, \upsilon, \mathbf{0}\} \in C$. This implies $\alpha\upsilon \preccurlyeq \upsilon$, which holds whenever $\|\upsilon\| > 0$.

Similarly, the operators $\mathcal{A}: \mathcal{E}^6 \to \mathcal{E}$ defined by
(1) $\mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6) = \upsilon_1 - \alpha\upsilon_2$; $\alpha \in [0,1)$;
(2) $\mathcal{A}(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6) = \upsilon_1 - a\upsilon_2 - b\upsilon_3 - c\upsilon_4$; $a, b, c \ge 0$ with $a + b + c < 1$
are members of $\mathcal{G}$.

Following remark will be essential in the sequel.

Remark 3.1. If $T \in B(\mathcal{E}, \mathcal{E})$, the Neumann series $I + T + T^2 + \ldots + T^n + \ldots$ converges if $\|T\|_1 < 1$ and diverges otherwise. Also if $\|T\|_1 < 1$, then there exists $\lambda > 0$, such that $\|T\|_1 < \lambda < 1$ and $\|T^n\|_1 \le \lambda^n < 1$.

4 Results

Recently, Popa [13] has employed an implicit type contractive condition on self-mapping to obtain some fixed point theorems. Ran and Reurings [2] have presented an analogue of the Banach fixed point theorem for monotone self-mappings in an ordered metric space. Huang and Zhan [7] introduced the idea of the cone metric space and obtained analogues of the Banach fixed point theorem, Kannan fixed point theorem, and Chatterjea fixed point theorem in a cone metric space. In this section, we prove some fixed point results for ordered implicit relations in a cone metric space, which improve the results in [2,7,13]. We derive these results under two different partial orders one defined in the underlying set and other in a real Banach space.

Theorem 4.1. *Let (X, d) be a complete cone metric space and $C \subset \mathcal{E}$ be a cone. Let $f: X \to X$. If there exist $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T\|_1 < 1$, identity operators $I: \mathcal{E} \to \mathcal{E}$, and $\mathcal{A} \in \mathcal{G}$, such that for all comparable elements σ, $\kappa \in X$*

$$(I - T)(d(\sigma, f(\sigma))) \preccurlyeq d(\sigma, \kappa) \text{ implies}$$

$$\mathcal{A}(d(f(\sigma), f(\kappa)), d(\sigma, \kappa), d(\sigma, f(\sigma)), d(\kappa, f(\kappa)), d(\sigma, f(\kappa)), d(\kappa, f(\sigma))) \preccurlyeq \mathbf{0} \tag{4.1}$$

and

(1) *there exists $\sigma \in X$, such that $\sigma_0 \mathcal{R} f(\sigma_0)$;*
(2) *for all σ, $\kappa \in X$, $\sigma \mathcal{R} \kappa$ implies $f(\sigma) \mathcal{R} f(\kappa)$;*
(3) *for a sequence $\{\sigma_n\}$ with $\sigma_n \to x^\star$ whose all sequential terms are comparable, we have $\sigma_n \mathcal{R} x^\star$ for all $n \in \mathbb{N}$.*

Then, $x^\star = f(x^\star)$.

Proof. Let $\sigma_0 \in X$ be such that $\sigma_0 \mathcal{R} f(\sigma_0)$. We construct a sequence $\{\sigma_n\}$ by $f(\sigma_{n-1}) = \sigma_n$. Then, $\sigma_0 \mathcal{R} \sigma_1$. By (4.1), for $\sigma = \sigma_0$, we have

$$(I - T)(d(\sigma_0, f(\sigma_0))) = (I - T)(d(\sigma_0, \sigma_1)) \preccurlyeq d(\sigma_0, \sigma_1) \text{ implies}$$

$$\mathcal{A}(d(f(\sigma_0), f(\sigma_1)), d(\sigma_0, \sigma_1), d(\sigma_0, f(\sigma_0)), d(\sigma_1, f(\sigma_1)), d(\sigma_0, f(\sigma_1)), d(\sigma_1, f(\sigma_0))) \preccurlyeq \mathbf{0},$$

that is,

$$\mathcal{A}(d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1), d(\sigma_0, \sigma_1), d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_2), d(\sigma_1, \sigma_1)) \preccurlyeq \mathbf{0}. \tag{4.2}$$

By $(d3)$, we have

$$d(\sigma_0, \sigma_2) \preccurlyeq d(\sigma_0, \sigma_1) + d(\sigma_1, \sigma_2)$$

and so we rewrite (4.2) employing condition $(\mathcal{A}_1)$ as follows:

$$\mathcal{A}(d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1), d(\sigma_0, \sigma_1), d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1) + d(\sigma_1, \sigma_2), \mathbf{0}) \preccurlyeq \mathbf{0}$$

and thus by $(\mathcal{A}_2)$, there exists $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T\|_1 < 1$, such that

$$d(\sigma_1, \sigma_2) \preccurlyeq T(d(\sigma_0, \sigma_1)).$$

Since $\sigma_0 \mathcal{R} \sigma_1$, assumption (2) implies $\sigma_1 = f(\sigma_0) \mathcal{R} f(\sigma_1) = \sigma_2$ and by (4.1), for $\sigma = \sigma_1$, we have

$$(I - T)(d(\sigma_1, f(\sigma_1))) = (I - T)(d(\sigma_1, \sigma_2)) \preccurlyeq d(\sigma_1, \sigma_2) \text{ implies}$$

$$\mathcal{A}(d(f(\sigma_1), f(\sigma_2)), d(\sigma_1, \sigma_2), d(\sigma_1, f(\sigma_1)), d(\sigma_2, f(\sigma_2)), d(\sigma_1, f(\sigma_2)), d(\sigma_2, f(\sigma_1))) \preccurlyeq \mathbf{0},$$

that is,

$$\mathcal{A}(d(\sigma_2, \sigma_3), d(\sigma_1, \sigma_2), d(\sigma_1, \sigma_2), d(\sigma_2, \sigma_3), d(\sigma_1, \sigma_3), d(\sigma_2, \sigma_2)) \preccurlyeq \mathbf{0}.$$

By (d3), we have

$$d(\sigma_1, \sigma_3) \preccurlyeq d(\sigma_1, \sigma_2) + d(\sigma_2, \sigma_3)$$

and $(\mathcal{A}_1)$ implies

$$\mathcal{A}(d(\sigma_2, \sigma_3), d(\sigma_1, \sigma_2), d(\sigma_1, \sigma_2), d(\sigma_2, \sigma_3), d(\sigma_1, \sigma_2) + d(\sigma_2, \sigma_3), \mathbf{0}) \preccurlyeq \mathbf{0}.$$

By $(\mathcal{A}_2)$, there exists $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T\|_1 < 1$, such that

$$d(\sigma_2, \sigma_3) \preccurlyeq T(d(\sigma_1, \sigma_2)) \preccurlyeq T^2(d(\sigma_0, \sigma_1)).$$

By continuing this pattern, we can construct a sequence $\{\sigma_n\}$, such that
$\sigma_n \mathcal{R} \sigma_{n+1}$ with $\sigma_{n+1} = f(\sigma_n)$ and

$$(I - T)(d(\sigma_{n-1}, f(\sigma_{n-1}))) = (I - T)(d(\sigma_{n-1}, \sigma_n)) \preccurlyeq d(\sigma_{n-1}, \sigma_n)$$

implies

$$d(\sigma_n, \sigma_{n+1}) \preccurlyeq T(d(\sigma_{n-1}, \sigma_n)) \preccurlyeq T^2(d(\sigma_{n-2}, \sigma_{n-1})) \preccurlyeq \cdots \preccurlyeq T^n(d(\sigma_0, \sigma_1)).$$

For $m, n \in \mathbb{N}$ with $m > n$, consider

$$\begin{aligned} d(\sigma_n, \sigma_m) &\preccurlyeq d(\sigma_n, \sigma_{n+1}) + d(\sigma_{n+1}, \sigma_{n+2}) + \cdots + d(\sigma_{m-1}, \sigma_m) \\ &\preccurlyeq T^n(d(\sigma_0, \sigma_1)) + T^{n+1}(d(\sigma_0, \sigma_1)) + \cdots + T^{m-1}(d(\sigma_0, \sigma_1)) \\ &= (T^n + T^{n+1} + \cdots + T^{m-1})(d(\sigma_0, \sigma_1)), \\ &\preccurlyeq \{T^n(1 + T + \cdots + T^{m-n-1} + \cdots)\}(d(\sigma_0, \sigma_1)) \\ &= \{T^n(I - T)^{-1})\}(d(\sigma_0, \sigma_1)). \quad \text{(By Remark 3.1)} \end{aligned}$$

Since $\|T\|_1 < 1$, so $T^n \to \mathbf{0}$ as $n \to \infty$. Thus, $\lim_{n\to\infty} d(\sigma_n, \sigma_m) = \mathbf{0}$, which implies that $\{\sigma_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete cone metric space, there exists $x^\star \in X$, such that $\sigma_n \to x^\star$ as $n \to \infty$. Thus, there exists a natural number N_2, such that

$$d(\sigma_n, x^\star) \ll c \quad \text{for all } n \geq N_2.$$

We claim that

$$(I - T)(d(\sigma_n, f(\sigma_n))) \preccurlyeq d(\sigma_n, x^\star).$$

We assume against our claim that

$$(I - T)(d(\sigma_n, f(\sigma_n))) \succ d(\sigma_n, x^\star)$$

and

$$(I - T)(d(\sigma_{n+1}, f(\sigma_{n+1}))) \succ d(\sigma_{n+1}, x^\star) \quad \text{for some } n \in \mathbb{N}.$$

By $(d3)$ and (4.1), we have

$$\begin{aligned} d(\sigma_n, f(\sigma_n)) &\preccurlyeq d(\sigma_n, x^\star) + d(x^\star, f(\sigma_n)) \\ &\prec (I - T)(d(\sigma_n, f(\sigma_n))) + d(\sigma_{n+1}, f(\sigma_{n+1}))) \\ &\prec (I - T)(d(\sigma_n, f(\sigma_n))) + T(d(\sigma_n, f(\sigma_n)))) \\ &= (I - T)(I + T)(d(\sigma_n, f(\sigma_n))) = (I - T^2)(d(\sigma_n, f(\sigma_n))). \end{aligned}$$

Thus,

$$T^2(d(\sigma_n, f(\sigma_n))) \prec \mathbf{0},$$

which is an absurdity. Hence, for each $n \geq 1$, we have

$$(I - T)(d(\sigma_n, f(\sigma_n))) \preccurlyeq d(\sigma_n, x^\star)$$

and

$$(I - T)(d(\sigma_{n+1}, f(\sigma_{n+1}))) \preccurlyeq d(\sigma_{n+1}, x^\star).$$

Assume that $\|d(x^\star, f(x^\star)\| > 0$. By assumption (3), we have $\sigma_n \preccurlyeq x^\star$ for all $n \in \mathbb{N}$, and then by (4.1), we get

$$\mathcal{A}(d(f(\sigma_n), f(x^\star)), d(\sigma_n, x^\star), d(\sigma_n, f(\sigma_n)), d(x^\star, f(x^\star)), d(\sigma_n, f(x^\star)), d(x^\star, f(\sigma_n))) \preccurlyeq \mathbf{0}$$

or

$$\mathcal{A}(d(\sigma_{n+1}, f(x^\star)), d(\sigma_n, x^\star), d(\sigma_n, \sigma_{n+1}), d(x^\star, f(x^\star)), d(\sigma_n, f(x^\star)), d(x^\star, \sigma_{n+1})) \preccurlyeq \mathbf{0}.$$

Letting $n \to \infty$, we have

$$\mathcal{A}(d(x^\star, f(x^\star)), \mathbf{0}, \mathbf{0}, d(x^\star, f(x^\star)), d(x^\star, f(x^\star)), \mathbf{0}) \preccurlyeq \mathbf{0}.$$

This is a contradiction to $(\mathcal{A}_3)$. Thus, $\|d(x^\star, f(x^\star)\| = 0$. Hence, $d(x^\star, f(x^\star)) = \mathbf{0}$. It follows from (d1) that $x^\star = f(x^\star)$. □

Theorem 4.2. *Let (X,d) be a complete cone metric space and f be a self-mapping on X. If for all comparable elements $\sigma, \kappa \in X$, there exist $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T_1\| < 1$, identity operator $I: \mathcal{E} \to \mathcal{E}$, and $\mathcal{A} \in \mathcal{G}$, such that*

$$(I - T)(d(\sigma, f(\sigma))) \preccurlyeq d(\sigma, \kappa) \text{ implies}$$

$$\mathcal{A}(d(f(\sigma), f(\kappa)), d(\sigma, \kappa), d(\sigma, f(\sigma)), d(\kappa, f(\kappa)), d(\sigma, f(\kappa)), d(\kappa, f(\sigma))) \preccurlyeq \mathbf{0} \tag{4.3}$$

and

(1) *there exists $\sigma_0 \in X$, such that $f(\sigma_0)\mathcal{R}\sigma_0$;*
(2) *for any $\sigma, \kappa \in X$, $\sigma\mathcal{R}\kappa$ implies $f(\kappa)\mathcal{R}f(\sigma)$;*
(3) *for a sequence $\{\sigma_n\}$ with $\sigma_0 \to x^\star$ whose all sequential terms are comparable, we have $\sigma_n\mathcal{R}x^\star$ for all $n \in \mathbb{N}$.*

Then, f has a fixed point in X.

Proof. Let σ_0 be an initial point in X. Define the sequence $\{\sigma_n\}$ by $\sigma_n = f(x_{n-1})$ for all n. By assumption (1), we have $\sigma_1 = f(\sigma_0)\mathcal{R}\sigma_0$, and then, assumption (2) implies $f(\sigma_0)\mathcal{R}f(\sigma_1)$, i.e., $\sigma_1\mathcal{R}\sigma_2$. By (4.3), we have

$$(I - T)(d(f(\sigma_0), \sigma_0)) = (I - T)(d(\sigma_1, \sigma_0)) \preccurlyeq d(\sigma_1, \sigma_0) \text{ implies}$$

$$\mathcal{A}(d(f(\sigma_1), f(\sigma_0)), d(\sigma_1, \sigma_0), d(\sigma_1, f(\sigma_1)), d(\sigma_0, f(\sigma_0)), d(\sigma_1, f(\sigma_0)), d(\sigma_0, f(\sigma_1))) \preccurlyeq \mathbf{0}$$

or

$$\mathcal{A}(d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1), d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1), \mathbf{0}, d(\sigma_0, \sigma_2)) \preccurlyeq \mathbf{0}.$$

By (d3), we have

$$d(\sigma_0, \sigma_2) \preccurlyeq d(\sigma_0, \sigma_1) + d(\sigma_1, \sigma_2)$$

and then using $\mathcal{A}_1$, we obtain

$$\mathcal{A}(d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1), d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1), \mathbf{0}, d(\sigma_0, \sigma_1) + d(\sigma_1, \sigma_2)) \preccurlyeq \mathbf{0}.$$

By $(\mathcal{A}_2)$, there exists $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T\|_1 < 1$, such that

$$d(\sigma_1, \sigma_2) \preccurlyeq T(d(\sigma_0, \sigma_1)).$$

Since $\sigma_1 \mathcal{R} \sigma_2$, by assumption (2), we get $\sigma_3 \mathcal{R} \sigma_2$, and thus by (4.3)

$$(I - T)(d(\sigma_1, f(\sigma_1))) = (I - T)(d(\sigma_1, \sigma_2)) \preccurlyeq d(\sigma_1, \sigma_2) \text{ implies}$$

$$\mathcal{A}(d(f(\sigma_1), f(\sigma_2)), d(\sigma_1, \sigma_2), d(\sigma_1, f(\sigma_1)), d(\sigma_2, f(\sigma_2)), d(\sigma_1, f(\sigma_2)), d(\sigma_2, f(\sigma_1))) \preccurlyeq \mathbf{0}.$$

By $(d3)$, $(\mathcal{A}_1)$, and $(\mathcal{A}_2)$, we get

$$d(\sigma_2, \sigma_3) = d(\sigma_3, \sigma_2) \preccurlyeq T(d(\sigma_2, \sigma_1)) \preccurlyeq T^2(d(\sigma_0, \sigma_1)).$$

By continuing the pattern, we construct a sequence $\{\sigma_n\}$, such that

$$d(\sigma_n, \sigma_{n+1}) \preccurlyeq T(d(\sigma_{n-1}, \sigma_n)) \preccurlyeq T^2(d(\sigma_{n-2}, \sigma_{n-1})) \preccurlyeq \cdots \preccurlyeq T^n(d(\sigma_0, \sigma_1)).$$

Hence, by the same reasoning as in the proof of Theorem 4.1, we have $x^\star = f(x^\star)$. □

Theorem 4.3. *Let (X,d) be a complete cone metric space and f be a monotone self-mapping on X. If for all comparable elements $\sigma, \kappa \in X$, there exist $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T\|_1 < 1$, identity operator $I\colon \mathcal{E} \to \mathcal{E}$, and $\mathcal{A} \in \mathcal{G}$, such that*

$$(I - T)(d(\sigma, f(\sigma))) \preccurlyeq d(\sigma, \kappa) \text{ implies}$$

$$\mathcal{A}(d(f(\sigma), f(\kappa)), d(\sigma, \kappa), d(\sigma, f(\sigma)), d(\kappa, f(\kappa)), d(\sigma, f(\kappa)), d(\kappa, f(\sigma))) \preccurlyeq \mathbf{0} \tag{4.4}$$

and
(1) *there exists $\sigma_0 \in X$, such that $\sigma_0 \mathcal{R} f(\sigma_0)$ or $f(\sigma_0) \mathcal{R} \sigma_0$;*
(2) *for a sequence $\{\sigma_n\}$ with $\sigma_n \to x^\star$ whose all sequential terms are comparable, we have $\sigma_n \mathcal{R} x^\star$ for all $n \in \mathbb{N}$.*

Then, f has a fixed point in X.

Proof. Let σ_0 be any point in X. Define the sequence $\{\sigma_n\}$ by $\sigma_n = f(\sigma_{n-1})$ for all $n \in \mathbb{N}$. By assumption (1), we have $\sigma_0 \mathcal{R} f(\sigma_0) = \sigma_1$. Also $\sigma_2 \mathcal{R} \sigma_1$ since f is monotone (either order preserving or order reversing). By (4.4), we have

$$(I - T)(d(f(\sigma_0), \sigma_0)) = (I - T)(d(\sigma_1, \sigma_0)) \preccurlyeq d(\sigma_1, \sigma_0) \text{ implies}$$

$$\mathcal{A}(d(f(\sigma_0), f(\sigma_1)), d(\sigma_0, \sigma_1), d(\sigma_0, f(\sigma_0)), d(\sigma_1, f(\sigma_1)), d(\sigma_0, f(\sigma_1)), d(\sigma_1, f(\sigma_0))) \preccurlyeq \mathbf{0},$$

that is,

$$\mathcal{A}(d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1), d(\sigma_0, \sigma_1), d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_2), d(\sigma_1, \sigma_1)) \preccurlyeq \mathbf{0}.$$

By $(d3)$ and $(\mathcal{A}_1)$,

$$\mathcal{A}(d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_1), d(\sigma_0, \sigma_1), d(\sigma_1, \sigma_2), d(\sigma_0, \sigma_2), \mathbf{0}) \preccurlyeq \mathbf{0}.$$

By $(\mathcal{A}_2)$, there exists $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T\|_1 < 1$, such that

$$d(\sigma_1, \sigma_2) \preccurlyeq T(d(\sigma_0, \sigma_1)).$$

Since $\sigma_2 \mathcal{R} \sigma_1$, by assumption (2), we get $\sigma_3 \mathcal{R} \sigma_2$ (since f is monotone), and thus, by (4.3)

$$(I - T)d(\sigma_1, f(\sigma_1)) = (I - T)d(\sigma_1, \sigma_2) \preccurlyeq d(\sigma_1, \sigma_2) \text{ implies}$$

$$\mathcal{A}((d(f(\sigma_1, f(\sigma_2), d(\sigma_1, \sigma_2), d(\sigma_1, f(\sigma_1)), d(\sigma_2, f(\sigma_2), d(\sigma_1, f(\sigma_2)), d(\sigma_2, f(\sigma_1))) \preccurlyeq \mathbf{0}.$$

By (d3), $(\mathcal{A}_1)$, and $(\mathcal{A}_2)$, we get

$$d(\sigma_3, \sigma_2) \preccurlyeq T(d(\sigma_2, \sigma_1)) \preccurlyeq T^2(d(\sigma_0, \sigma_1)).$$

By following the same pattern, we construct a sequence $\{\sigma_n\}$, such that

$$d(\sigma_n, \sigma_{n+1}) \preccurlyeq T(d(\sigma_{n-1}, \sigma_n)) \preccurlyeq T^2(d(\sigma_{n-2}, \sigma_{n-1})) \preccurlyeq \cdots \preccurlyeq T^n(d(\sigma_0, \sigma_1)).$$

Rest of the proof is similar to the proof of Theorem 4.1. □

Remark 4.1. (1). Fixed point in Theorems 4.1–4.3 can be proved to be unique if additionally we assume that for every pair of elements $\sigma, \kappa \in X$, there exists either an upper bound or lower bound of σ, κ.

(2). If cone is taken as normal in the above theorems, then we can replace $\mathcal{A}_3$ by

$$\mathcal{A}(\sigma, \xi, \nu, \alpha, \alpha + \xi, \alpha + \nu) \preccurlyeq \mathbf{0} \text{ for all, } \sigma \ll c,\ \xi \ll c,\ \nu \ll c \text{ and } \alpha \ll c.$$

The following example illustrates the main theorem.

Example. Let $\mathcal{E} = (\mathbb{R}^3, \|\cdot\|)$ with $\|\sigma\| = \max\{(|\sigma_1|, |\sigma_2|, |\sigma_3|)\}$, then $(\mathcal{E}, \|.\|)$ is a real Banach space. Define $C = \{(\sigma, \xi, \nu) \in \mathbb{R}^3 : \sigma, \xi, \nu \geq 0\}$, then, it is a cone in $\mathcal{E}$. Define the cone metric d by $d(\sigma, \xi) = \{|\sigma_1 - \xi_1|, |\sigma_2 - \xi_2|, |\sigma_3 - \xi_3|\}$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$. Let $X = \left\{(0,0,0), \left(0,0,\frac{1}{4}\right), \left(0,\frac{1}{4},0\right)\right\} \subset \mathcal{E}$ and define f by

$$f(0,0,0) = \left(0, 0, \frac{1}{4}\right), f\left(0, 0, \frac{1}{4}\right) = \left(0, 0, \frac{1}{4}\right), f\left(0, \frac{1}{4}, 0\right) = (0, 0, 0),$$

then the mapping f is monotone with respect to the partial order $\preccurlyeq$. Let

$$T = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Define $T(\sigma) = \frac{\sigma}{3}$ and

$$\|T(\sigma)\| = \max\left\{\frac{|\sigma_1|}{3}, \frac{|\sigma_2|}{3}, \frac{|\sigma_3|}{3}\right\} = \frac{1}{3}\max\{|\sigma_1|, |\sigma_2|, |\sigma_3|\} = \frac{1}{3}\|\sigma\|.$$

Then, $\|T\| = \frac{1}{3} < 1$, and hence, $T \in B(\mathcal{E}, \mathcal{E})$. Also, $T(C) \subset C$. Now, if $\sigma = (0, 0, 0)$ and $\kappa = \left(0, 0, \frac{1}{4}\right)$, then $\sigma \preccurlyeq \kappa$. Also

$$d(\sigma, f(\sigma)) = \left(0, 0, \frac{1}{4}\right), \quad d(\kappa, f(\kappa)) = (0, 0, 0), \quad d(\kappa, f(\sigma)) = (0, 0, 0), \quad d(f(\sigma), f(\kappa)) = (0, 0, 0),$$

$$T(d(\sigma, f(\sigma))) = \tfrac{d(\sigma, f(\sigma))}{3} = \left(0, 0, \tfrac{1}{12}\right), \ (I - T)(d(\sigma, f(\sigma))) = \left(0, 0, \tfrac{1}{4}\right) - \left(0, 0, \tfrac{1}{12}\right) = \left(0, 0, \tfrac{1}{6}\right),$$

$$\alpha \max\{d(\sigma,\kappa)), d(\sigma, f(\sigma)), d(\kappa, f(\kappa)), d(\sigma, f(\kappa)), d(\kappa, f(\sigma))\} = \alpha \max\left\{\left(0, 0, \frac{1}{4}\right), \left(0, 0, \frac{1}{4}\right), (0,0,0)\right\} = \left(0, 0, \frac{1}{4}\right).$$

Thus, for every $\alpha \in \left[0, \frac{1}{2}\right)$, $(I - T)(d(\sigma, f(\sigma))) \preccurlyeq d(\sigma, \kappa)$ implies

$$d(f(\sigma), f(\kappa)) \preccurlyeq \alpha \max\{d(\sigma,\kappa)), d(\sigma, f(\sigma)), d(\kappa, f(\kappa)), d(\sigma, f(\kappa)), d(\kappa, f(\sigma))\}.$$

Define

$$\mathcal{A}(d(f(\sigma), f(\kappa)), d(\sigma,\kappa), d(\sigma, f(\sigma)), d(\kappa, f(\kappa)), d(\sigma, f(\kappa)), d(\kappa, f(\sigma))) = d(f(\sigma), f(\kappa)) - \alpha \max\{d(\sigma,\kappa)), d(\sigma, f(\sigma)), d(\kappa, f(\kappa)), d(\sigma, f(\kappa)), d(\kappa, f(\sigma))\}.$$

Then, $(I - T)(d(\sigma, f(\sigma))) \preccurlyeq d(\sigma, \kappa)$ implies

$$\mathcal{A}(d(f(\sigma), f(\kappa)), d(\sigma,\kappa), d(\sigma, f(\sigma)), d(\kappa, f(\kappa)), d(\sigma, f(\kappa)), d(\kappa, f(\sigma))) \preccurlyeq \mathbf{0}.$$

Similarly, all other values of σ, κ satisfy the contractive condition of Theorem 4.3. Note that $\left(0, 0, \frac{1}{4}\right)$ is a fixed point of f.

Corollary 4.1. *Let (X, d) be a complete cone metric space and f be a self-mapping on X. If for all comparable elements $\sigma, \kappa \in X$, there exist $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T\|_1 < 1$, identity operator $I\colon \mathcal{E} \to \mathcal{E}$, and $\mathcal{A} \in \mathcal{G}$, such that*

$$(I - T)(d(\sigma, f(\sigma))) \preccurlyeq d(\sigma, \kappa) \ \text{ implies}$$

$$d(f(\sigma), f(\kappa)) \preccurlyeq T(d(\sigma, \kappa))$$

and

(1) *there exists $\sigma \in X$, such that $\sigma_0 \mathcal{R} f(\sigma_0)$ or $f(\sigma_0)\mathcal{R}\sigma_0$;*
(2) *for all $\sigma, \kappa \in X$, $\sigma\mathcal{R}\kappa$ implies $f(\sigma)\mathcal{R}f(\kappa)$ or $f(\kappa)\mathcal{R}f(\sigma)$;*
(3) *for a sequence $\{\sigma_n\}$ with $\sigma_n \to x^\star$ whose all sequential terms are comparable, we have $\sigma_n \mathcal{R} x^\star$ for all $n \in \mathbb{N}$.*

Then, there exists $x^\star \in X$, such that $x^\star = f(x^\star)$.

5 Homotopy result

In this section, we derive a homotopy result by applying Corollary 4.1 of Theorem 4.1.

Theorem 5.1. *Let $(\mathcal{E}, \|.\|)$ be a real Banach space and $C \subset \mathcal{E}$ be a cone. Let (X,d) be a cone metric space and $U \subset X$ is open. Assume that there exists $T \in B(\mathcal{E}, \mathcal{E})$ with $\|T\|_1 < 1$ and $T(C) \subset C$. Let the operator $h\colon \bar{U} \times [0, 1] \to X$ satisfies the condition (1) of Corollary 4.1 in the first variable and*

(1) *$\sigma \neq h(\sigma,\theta)$ for every $\sigma \in \partial U$ (∂U denotes the boundary of U in X);*
(2) *there exists $M \geq 0$, such that*

$$\|d(h(\sigma,\theta),h(\sigma,\mu))\| \le M|\theta-\mu|$$

for every $\sigma \in \bar{U}$ *and* μ, $\theta \in [0,1]$;

(3) *For some* $\sigma \in U$ *if there exists* κ *with* $\|d(\sigma,\kappa)\| \le r$, *then* $\sigma\mathcal{R}\kappa$, *where r is the radius of open ball in U.*

If $h(\cdot,0)$ *has a fixed point in the open set U, then* $h(\cdot,0)$ *also has a fixed point in the open set U.*

Proof. Let

$$\mathcal{E} = \{\theta \in [0,1] | \sigma = h(\sigma,\theta), \quad \text{for some} \quad \sigma \in U\}.$$

Define the relation $\preccurlyeq$ in $\mathcal{E}$ by $e \preccurlyeq w$ if and only if $\|e\| \le \|e\|$ for all $e, w \in \mathcal{E}$. Next, $0 \in B$, since $h(\cdot,0)$ has a fixed point in the open set U. So B is non-empty. Since $d(\sigma,h(\sigma,\theta)) = d(\sigma,\kappa)$, $(I-T)(d(\sigma,h(\sigma,\theta))) \preccurlyeq d(\sigma,\kappa)$ for all $\sigma\mathcal{R}\kappa$, by Corollary 4.1, we have

$$d(h(\sigma,\theta),h(\kappa,\theta)) \preccurlyeq T(d(\sigma,\kappa)).$$

First, we show that B is closed in [0,1]. For this, let $\{\theta_n\}_{n=1}^{\infty} \subseteq B$ with $\theta_n \to \theta \in [0,1]$ as $n \to \infty$. It is necessary to prove that $\theta \in B$. Since $\theta_n \in B$ for $n \in \mathbb{N}$, there exists $\sigma_n \in U$ with $\sigma_n = h(\sigma_n,\theta_n)$. Since $h(\sigma,\cdot)$ is monotone, for $n, m \in \mathbb{N}$, we have $\sigma_m\mathcal{R}\sigma_n$. Since

$$(I-T)(d(\sigma_n,h(\sigma_m,\theta_m))) = (I-T)(d(\sigma_n,\sigma_m)) \preccurlyeq d(\sigma_n,\sigma_m),$$

we have

$$d(h(\sigma_n,\theta_m),h(\sigma_m,\theta_m)) \preccurlyeq T(d(\sigma_n\sigma_m))$$

and

$$\begin{aligned} d(\sigma_n,\sigma_m) &= d(h(\sigma_n,\theta_n),h(\sigma_m,\theta_m)) \preccurlyeq d(h(\sigma_n,\theta_n),h(\sigma_n,\theta_m)) + d(h(\sigma_n,\theta_m),h(\sigma_m,\theta_m)),\\ \|d(\sigma_n,\sigma_m)\| &\le M|\theta_n-\theta_m| + \|T(d(\sigma_n,\sigma_m))\|,\\ \|d(\sigma_n,\sigma_m)\| &\le \frac{M}{1-\|T\|}|\theta_n-\theta_m|. \end{aligned}$$

Since $\{\theta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $[0,1]$, we have

$$\lim_{n,m\to\infty} d(\sigma_n,\sigma_m) = \mathbf{0},$$

that is, $d(\sigma_n,\sigma_m) \ll c$, whenever $n, m \to \infty$. Hence, $\{\sigma_n\}$ is a Cauchy sequence in X. Since X is a complete cone metric space, there exists $\sigma \in \bar{U}$ with $\lim_{n\to\infty} d(\sigma_n,\sigma) \ll c$. Hence, $\sigma_n\mathcal{R}\sigma$ for all $n \in \mathbb{N}$. Now, consider

$$\begin{aligned} d(\sigma_n,h(\sigma,\theta)) &= d(h(\sigma_n,\theta_n),h(\sigma,\theta)) \preccurlyeq d(h(\sigma_n,\theta_n),h(\sigma_n,\theta)) + d(h(\sigma_n,\theta),h(\sigma,\theta)),\\ \|d(\sigma_n,h(\sigma,\theta))\| &\le M|\theta_n-\theta| + \|T(d(\sigma_n,\sigma))\|. \end{aligned}$$

So we have

$$\lim_{n\to\infty} d(\sigma_n,h(\sigma,\theta)) = \mathbf{0}.$$

Thus, $d(\sigma,h(\sigma,\theta)) = \mathbf{0}$. Hence, $\theta \in B$ and so B is closed in $[0,1]$.

Next, we show that B is open in [0,1]. For this, let $\theta_1 \in B$. Then, we have the existence of $\sigma_1 \in U$ with $h(\theta_1,\sigma_1) = \sigma_1$. Since U is open, there exists $r > 0$, such that $B(\sigma_1,r) \subseteq U$. Now, assume

$$l = d(\sigma_1,\partial U) = \inf\{d(\sigma_1,\xi) \colon \xi \in \partial U\}.$$

Then, $r = l > 0$. Fix $\varepsilon > 0$ with $\varepsilon < \frac{(1-\|T\|)l}{M}$. Let $\theta \in (\theta_1 - \varepsilon, \theta_1 + \varepsilon)$. Then,

$$\sigma \in \overline{B(\sigma_1, r)} = \{\sigma \in X\colon \|d(\sigma, \sigma_1)\| \leq r\}, \quad \text{as } \sigma\mathcal{R}\sigma_1.$$

Consider

$$d(h(\sigma, \theta), \sigma_1) = d(h(\sigma, \theta), h(\sigma_1, \theta_1) \preccurlyeq d(h(\sigma, \theta), h(\sigma, \theta_1) + d(h(\sigma, \theta_1), h(\sigma_1, \theta_1)),$$
$$\|d(h(\sigma, \theta), \sigma_1)\| \leq M|\theta_1 - \theta| + \|T(d(\sigma_1, \sigma))\| \leq M\varepsilon + \|T\|l < l.$$

Thus, for every fixed $\theta \in (\theta_1 - \varepsilon, \theta_1 + \varepsilon)$, $h(\cdot,t)\colon \overline{B(\sigma, r)} \to \overline{B(\sigma, r)}$ has a fixed point in $\overline{U}$ and can be deduced by applying Corollary 4.1. But this fixed point should be in U as in the previous case. Hence, $\theta_1 \in B$ for any $\theta_1 \in (\theta - \varepsilon, \theta + \varepsilon)$ and so B is open in [0,1]. Thus, we showed that B is open as well as closed in [0, 1] and by connectedness, $B = [0, 1]$. Hence, $h(\cdot,1)$ has a fixed point in U.

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