

**Some results in the theory of a third-order
linear differential equation**

by MICHAL GREGUŠ and MILAN GERA (Bratislava)

Dedicated to the memory of Jacek Szarski

Abstract. This paper investigates the existence of oscillatory and non-oscillatory solutions of a third-order linear differential equation. Besides, conditions are derived for the asymptotic behavior of its non-trivial non-oscillatory solutions.

1. Consider the differential equation

$$(a) \quad y''' + p(x)y'' + q(x)y' + r(x)y = 0,$$

where p, r are continuous functions and q is continuously differentiable function on the interval $I = (a, \infty)$, $-\infty \leq a < \infty$.

Suppose that the coefficients p, q, r satisfy one of the following conditions on the interval I :

$$(P) \quad p(x) \geq 0, \quad 2r(x) - q'(x) - p(x)q(x) \geq 0,$$

moreover, the sign of the equality in the second inequality does not hold on any subinterval of I ;

$$(\bar{P}) \quad p(x) \leq 0, \quad 2r(x) - q'(x) - p(x)q(x) \leq 0,$$

moreover, the sign of the equality in the second inequality does not hold on any subinterval of I .

The adjoint differential equation of equation (a) is

$$(b) \quad [(z' - p(x)z)' + q(x)z]' - r(x)z = 0.$$

Write

$$lu = u'' + p(x)u' + q(x)u, \quad u \in C^2(I); \quad E(x, x_0) = \exp \int_{x_0}^x p(t) dt$$

for $x, x_0 \in I$ and transform equation (b) by the substitution $z = uE(x, x_0)$.

Then we get the equation

$$(\bar{b}) \quad [luE(x, x_0)]' - r(x)E(x, x_0)u = 0.$$

A non-trivial solution of a linear differential equation will be called *oscillatory on I* if its set of zeros is not bounded above. Other solutions will be called *non-oscillatory on I*.

A linear differential equation is termed *oscillatory on I* if it has at least one oscillatory solution, and *non-oscillatory on I* in the opposite case.

We say that a linear differential equation of order n is *disconjugated* on an interval $J \subset I$ if none of its non-trivial solutions has more than $n-1$ zeros (including multiplicity) on J .

We shall derive some conditions for the existence of oscillatory and non-oscillatory solutions of the differential equations (a) and (\bar{b}).

2. First we introduce the notion of bands of solutions of the differential equation (a) [6]. It will enable us to look into the structure of solutions of the differential equation (a).

Let y_1, y_2 be solutions of the differential equation (a) with the properties $y_1(x_0) = y_1'(x_0) = 0, y_1''(x_0) \neq 0; y_2(x_0) = y_2''(x_0) = 0, y_2'(x_0) \neq 0, x_0 \in I$.

We call the set of solutions $y = c_1y_1 + c_2y_2$ of the differential equation (a) (c_1, c_2 are arbitrary constants) the band of solutions of equation (a) at the point x_0 .

The band of solutions at x_0 fulfils the second order differential equation

$$(c) \quad uy'' - u'y' + luy = 0,$$

where $u = y_1y_2' - y_1'y_2$ is a solution of the differential equation (\bar{b}) with the properties $u(x_0) = u'(x_0) = 0, u''(x_0) \neq 0$.

If $u(x) \neq 0$ for $x > x_0$, then the band of solutions at x_0 of equation (a) is called *regular* on the interval (x_0, ∞) and equation (a) is oscillatory on I iff equation (c) is oscillatory on (α, ∞) . Moreover, two linearly independent solutions of the band at x_0 separate the null-points of each other to the right from the point x_0 .

The solutions of the differential equation (a) fulfil the following integral identity

$$(1) \quad \begin{aligned} Fx_0[y(x)] &\equiv [y'^2(x) - q(x)y^2(x) - 2y(x)y''(x)]E(x, x_0) \\ &= Fx_0[y(x_0)] + \int_{x_0}^x p(t)E(t, x_0)y'^2(t)dt + \\ &\quad + \int_{x_0}^x [2r(t) - q'(t) - p(t)q(t)]E(t, x_0)y^2(t)dt. \end{aligned}$$

The solutions of the differential equation (\bar{b}) fulfil the integral identity

$$\begin{aligned}
 (\bar{1}) \quad \bar{F}x_0[u(x)] &\equiv [2u(x)u''(x) + 2p(x)u(x)u'(x) + \\
 &\quad + q(x)u^2(x) - u'^2(x)]E(x, x_0) \\
 &= \bar{F}x_0[u(x_0)] + \int_{x_0}^x p(t)E(t, x_0)u'^2(t)dt + \\
 &\quad + \int_{x_0}^x [2r(t) - q'(t) - p(t)q(t)]E(t, x_0)u^2(t)dt.
 \end{aligned}$$

The following lemma is an immediate consequence of the integral identities.

LEMMA 1. *Let $x_0 \in I$ and $y(u)$ be a solution of the differential equation (a) (\bar{b}) with the properties $y(x_0) = y'(x_0) = 0$, $y''(x_0) \neq 0$ ($u(x_0) = u'(x_0) = 0$, $u''(x_0) \neq 0$).*

(a) *Let further assumption (P) hold. Then the solution $y(u)$ does not have any null-point on the interval I to the left (to the right) of x_0 . Moreover, if $q(x) \leq 0$ on I , then it holds for $y(u)$, that $y(x)y'(x) < 0$, $y(x)y''(x) > 0$ for $x \in (\alpha, x_0)$ ($u(x)u'(x) > 0$, $u(x)[u'(x)E(x, x_0)]' > 0$ for $x > x_0$).*

(b) *Let further assumption (\bar{P}) hold. Then the solution $y(u)$ does not have any null-point on I to the right (to the left) of x_0 . If, besides, $q(x) \leq 0$ on I , then it holds for $y(u)$, that $y(x)y'(x) > 0$, $y(x)y''(x) > 0$ for $x > x_0$ ($u(x)u'(x) < 0$, $u(x)[u'(x)E(x, x_0)]' > 0$ for $x \in (\alpha, x_0)$).*

Using Lemma 1 we can prove the following theorem in the similar way as in paper [7], Theorem 14.

THEOREM 1. *Let assumption (P) hold. Then the differential equation (a) (\bar{b}) has at least one solution $y(u)$ without null-points on I for which $Fx_0[y(x)] < 0$ ($\bar{F}x_0[u(x)] > 0$) holds on the interval I . If $q(x) \leq 0$ on I , then $y(x)y'(x) < 0$, $y(x)y''(x) > 0$ ($u(x)u'(x) > 0$, $u(x)[u'(x)E(x, x_0)]' > 0$) for $x \in I$.*

Remark 1. The assertion of Theorem 1 holds also under assumption (\bar{P}), but the inequalities $y(x)y'(x) > 0$, $y(x)y''(x) > 0$ ($u(x)u'(x) < 0$, $u(x)[u'(x)E(x, x_0)]' > 0$) hold on the interval I for the solution $y(u)$ in the case $q(x) \leq 0$.

LEMMA 2. *Let the differential equation $lu = 0$ be disconjugated on I and let $r(x) \geq 0$ (≤ 0) for $x \in I$. Then the solution y of the differential equation (a) with the properties $y(x_0) = y'(x_0) = 0$, $y''(x_0) \neq 0$, $x_0 \in I$ does not have further null-point and further null-point of the derivative to the left (right) of x_0 on I .*

This lemma follows from Corollary 1 of Lemma 1.1 in paper [2].

It follows from Lemma 1 resp. Lemma 2 and from the properties of bands [6] (or from Corollary 1 of Theorem 1 in paper [3], too).

COROLLARY 1. *Let the differential equation $lu = 0$ be disconjugated on I and let $r(x) \leq 0$ (≥ 0) for $x \in I$. If (P) ((\bar{P})) holds, then the differential equation (a) is disconjugated on I .*

COROLLARY 1'. *Let $q(x) \leq 0$, $r(x) \leq 0$ ($r(x) \geq 0$) for $x \in I$ and, moreover, let (P) ((\bar{P})) hold. Then the differential equation (a) is disconjugated on I .*

The following theorem follows from Theorem 1 and from Theorem 1 in paper [1].

THEOREM 2. *Let (P) or (\bar{P}) hold. Then the differential equation (a) is oscillatory on I iff the differential equation (\bar{b}) is oscillatory.*

THEOREM 3. *Let (P) ((\bar{P})) hold and let the differential equation (a) be oscillatory on I . Then a non-trivial solution y (u) of (a) ((\bar{b})) is non-oscillatory on I iff $Fx_0[y(x)] < 0$ ($\bar{F}x_0[u(x)] > 0$) for $x \in I$.*

The proof of Theorem 3 is similar to the proof of Theorem 3.2 in paper [10].

COROLLARY 2. *Let (P) ((\bar{P})) hold. A necessary and sufficient condition for equation (a) ((\bar{b})) to be oscillatory on I is that every its non-trivial solution y (u) satisfying the condition $Fx_0[y(x_0)] \geq 0$ ($\bar{F}x_0[u(x_0)] \leq 0$) at some point $x_0 \in I$, be oscillatory on I .*

COROLLARY 2'. *Let (P) ((\bar{P})) hold. Then the following three conditions are equivalent:*

- (a) *The differential equation (a) ((\bar{b})) is oscillatory on I .*
- (b) *Every non-trivial solution of (a) ((\bar{b})) with at least one zero in I is oscillatory on I .*
- (c) *Any non-trivial non-oscillatory solution of (a) ((\bar{b})) has not any null-points on I .*

COROLLARY 3. *Let (P) ((\bar{P})) hold. Then equation (a) ((\bar{b})) is non-oscillatory on I if there exists such a non-trivial, non-oscillatory solution y (u) of (a) ((\bar{b})) and such $x_0 \in I$ that $Fx_0[y(x_0)] \geq 0$ ($\bar{F}x_0[u(x_0)] \leq 0$).*

THEOREM 4. *Let $q(x) \leq 0$ on I and let (P) hold. If the differential equation (a) is oscillatory on I , then for any non-trivial non-oscillatory solution y of (a) on I we have $y(x)y'(x) < 0$, $y(x)y''(x) > 0$ for all $x \in I$.*

Proof. Let y be non-trivial non-oscillatory solution of the oscillatory equation (a). It follows from Theorem 3 that y fulfils the inequality $Fx_0[y(x)] < 0$ for $x \in I$. The integral identity (1) implies $y(x)y''(x) > 0$ on I , because $q(x) \leq 0$ for $x \in I$. We can suppose, without loss of generality, that $y(x) > 0$ for $x \in I$. Then $y''(x) > 0$ on I . It is necessary to prove that $y'(x) < 0$ for $x \in I$. Let, on the contrary, there exists such a number $\tau \in I$ that $y'(\tau) \geq 0$. Since $y''(x) > 0$ on I , therefore $y'(x) > 0$ for $x > \tau$. It follows from Theorem 1 that there exists a solution v of (a) with the properties $v(x) > 0$, $v'(x) < 0$ for $x \in I$.

Therefore

$$\left| \begin{matrix} v(x), & y(x) \\ v'(x), & y'(x) \end{matrix} \right| > 0$$

for $x \in I$. Thus every non-trivial linear combination of solutions y and v is non-oscillatory on I . This is in a contradiction with the assumption that (a) is oscillatory on I . The solution $\bar{y}(x) = y(x_0)v(x) - v(x_0)y(x)$ of (a) with a null-point at $x_0 \in (\tau, \infty)$ fulfils at x_0 the relation $Fx_0[\bar{y}(x_0)] = \bar{y}'^2(x_0) > 0$, that is, it oscillates on I by Corollary 2 and the theorem is proved.

THEOREM 5. *Let (P) hold and let the integral $\int_{x_0}^{\infty} p(t)dt$, $x_0 \in I$, be convergent. Let further $q(x) \leq 0$ and let equation (a) be oscillatory on I . Then there exists one and only one non-oscillatory solution $v(x) \neq 0$ (up to the linear dependence) of the differential equation (a), for which, on the interval I , $v(x)v'(x) < 0$, $v(x)v''(x) > 0$ hold.*

Proof. Theorem 1 implies the existence of at least one solution v with the desired property. Assume that there exists one more non-oscillatory solution \bar{v} of (a), independent of v . It follows from Theorem 4, that \bar{v} has on I the same properties as v . Clearly $y = \bar{v}(x_0)v(x) - v(x_0)\bar{v}(x)$ is a solution of (a), and $y(x_0) = 0$. By Corollary 2', y is an oscillatory solution on I . Since $\lim_{x \rightarrow \infty} v'(x) = \lim_{x \rightarrow \infty} \bar{v}'(x) = 0$, we have

$$(2) \quad \lim_{x \rightarrow \infty} y'(x) = 0.$$

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence of null-points of y on (x_0, ∞) such that $\lim_{n \rightarrow \infty} x_n = \infty$. The supposition (P) and the integral identity (1) for y imply $Fx_1[y(x)] \geq Fx_1[y(x_1)]$ for all $x \geq x_1$ and therefore

$$y'^2(x_n) \geq y'^2(x_1)E(x_1, x_n), \quad n = 1, 2, \dots$$

It follows from Lemma 1, that $y'(x_1) \neq 0$.

From these facts and from the convergency of $\int_{x_0}^{\infty} p(t)dt$ we get that $\lim_{n \rightarrow \infty} y'^2(x_n)$ is not equal to zero. This is a contradiction with (2) and the theorem is proved.

The proofs of the following theorems are similar to the proofs of Theorems 4 and 5.

THEOREM 4'. *Let $q(x) \leq 0$ on I and let (\bar{P}) hold. If the differential equation (a) is oscillatory, then $u(x)u'(x) < 0$, $u(x)[u'(x)E(x, x_0)]' > 0$ hold for every non-trivial non-oscillatory solution u of the differential equation (\bar{b}) on I .*

THEOREM 5'. *Let (\bar{P}) hold and let the integral $\int_{x_0}^{\infty} p(t)dt$, $x_0 \in I$, be convergent. Let further $q(x) \leq 0$ and (a) be oscillatory on I . Then there exists*

one and only one non-trivial, non-oscillatory solution u of (\bar{b}) (up to the linear dependence) for which $u(x)u'(x) < 0$, $u(x)[u'(x)E(x, x_0)]' > 0$, $x \in I$, hold.

From Theorems 4, 4' it follows

COROLLARY 4. Let $q(x) \leq 0$ and let (P) (\bar{P}) hold. If the differential equation (a) is oscillatory on I , then every non-trivial solution y (u) of (a) (\bar{b}) is oscillatory on I , if it fulfils one of the following conditions at some point $x_0 \in I$:

$$y(x_0)y'(x_0) \geq 0; \quad y(x_0)y''(x_0) \leq 0 \quad (u(x_0)u'(x_0) \geq 0; \\ u(x_0)(u'(x)E(x, x_0))'_{x=x_0} \leq 0).$$

3. We shall derive some sufficient conditions for the differential equation (a) in order to be oscillatory on I .

LEMMA 3. Let w be such a positive, twice differentiable function on $J \subset I$ that $lw \leq 0$ for $x \in J$. Then the differential equation $ly = 0$ is disconjugated on J .

This lemma is proved in paper [9] (see [4], too).

THEOREM 6. Let (P) hold and let the second order differential equation

$$(3) \quad v'' + \frac{1}{4}q(x)v = 0$$

be oscillatory on I . Then the differential equation (a) is oscillatory on I , too.

Proof. Let y be a non-trivial solution of (a) with the property $Fx_0[y(x_0)] \geq 0$ for some $x_0 \in I$ and be non-oscillatory on I . Then there exists such $x_1 \geq x_0$ that $y(x) \neq 0$ for $x > x_1$. Assume e.g. $y(x) > 0$ for $x > x_1$. We can suppose it since $Fx_0[y(x)] = Fx_0[-y(x)]$. It follows from (P) that $Fx_0[y(x)] > 0$ for $x > x_0$ and therefore $Fx_1[y(x)] > 0$ for $x > x_1$. Denote $Y(x) = \sqrt{y(x)}$ for $x > x_1$. Then

$$y(x) = Y^2(x), \quad y'(x) = 2Y(x)Y'(x), \quad y''(x) = 2Y(x)Y''(x) + 2Y'^2(x)$$

and

$$0 < Fx_1[Y^2(x)] = -4Y^3(x)[Y''(x) + \frac{1}{4}q(x)Y(x)]$$

for $x > x_1$. Hence we get

$$(4) \quad Y''(x) + \frac{1}{4}q(x)Y(x) < 0, \quad x \in (x_1, \infty).$$

By Lemma 3 the differential equation (3) is therefore disconjugated on (x_1, ∞) . This is applied to obtain a contradiction with the assumption on equation (3) to be oscillatory on I and Theorem 6 is proved.

COROLLARY 5. Let (P) hold and let $q(x) \geq 0$ on I . If the differential equation (a) is non-oscillatory on I and if y is an arbitrary non-trivial solution of (a) with the property $Fx_0[y(x_0)] \geq 0$ at some number $x_0 \in I$, then there exists such a number $\tau \in I$ that $y(x)y'(x) > 0$ holds for $x > \tau$.

Proof. Let (a) be non-oscillatory on I and let y be a positive solution of (a), mentioned in the proof of Theorem 6. Then by (4) regarding the assumption $q(x) \geq 0$ on I for $Y''(x)$ we get that $Y''(x) < 0$ for $x > x_1$. Since $Y(x) > 0$ for $x \in (x_1, \infty)$ there exists such a number $\tau \geq x_1$, that $Y'(x) > 0$ for $x > \tau$ and thus $y'(x) > 0$ on (τ, ∞) , too.

Remark 2. (a) In the case $p(x) \equiv 0$ on I from Theorem 6 and Corollary 2 we get Theorem 4 in paper [11].

(b) If (a) is non-oscillatory on I and (P) holds, then the differential equation (3) is non-oscillatory on I .

LEMMA 4. Let $x_0 \in I$ and let $r(x) \geq 0$ for $x \geq x_0$. Then the differential equation (a) is disconjugated on $[x_0, \infty)$ if there exists a function f with continuous third derivative on (x_0, ∞) such that $f(x) > 0$, $f'(x) > 0$ and $f'''(x) + p(x)f''(x) + q(x)f'(x) + r(x)f(x) \leq 0$ for $x > x_0$.

The proof of Lemma 4 is given in [4] as the proof of Corollary 1' of Theorem 2.

The following lemma follows from Lemma 4 and Corollary 5.

LEMMA 5. Let $q(x) \geq 0$, $r(x) \geq 0$ for $x \in I$ and let (P) hold. Then (a) is non-oscillatory on I iff there exists such a solution y of (a) and such a number $\tau \in I$ that $y(x)y'(x) > 0$ for $x > \tau$.

If, moreover,

$$(E) \quad \int_{x_0}^{\infty} E(x_0, s) ds = \infty, \quad x_0 \in I,$$

then $y(x)y''(x) > 0$ holds on (τ, ∞) , too.

From Lemma 5 and Lemma 4 follows

COROLLARY 6. Let $q(x) \geq 0$, $r(x) \geq 0$ for $x \in I$ and let (P) hold. Then (a) is non-oscillatory on I iff there exists such $\tau \in I$ and such a function $w \in C^3((\tau, \infty))$ with the property $w(x) > 0$, $w'(x) > 0$, $w'''(x) + p(x)w''(x) + q(x)w'(x) + r(x)w(x) \leq 0$ for $x > \tau$.

COROLLARY 7. Let $q(x) \geq 0$, $r(x) \geq 0$ for $x \in I$ and let (P) and (E) hold. Besides let the second order differential equation

$$(5) \quad v'' + q(x)v = 0$$

be oscillatory on I . Then the differential equation (a) is oscillatory on I , too.

Proof. Assume the contrary, that is, let (a) be non-oscillatory on I . Then by Lemma 5 there exists such a number $\tau \in I$ and such a solution y of (a), that $y(x) > 0$, $y'(x) > 0$, $y''(x) > 0$ for $x > \tau$. Denote $z = y'(x)$. Then from (a) we get

$$z'' + q(x)z = -p(x)z' - r(x)y \leq 0 \quad \text{for } x > \tau.$$

By Lemma 3 the differential equation (5) is therefore disconjugated on

(τ, ∞) . This is in a contradiction with the supposition and Corollary 7 is proved.

LEMMA 6. Let be $\tau \in I$. Let $g(x)$ be such that $g(x) > 0$, $g'(x) > 0$ for $x \geq \tau$ and let $g''(x)E(x, \tau)$ have a continuous and non-positive derivative on $[\tau, \infty)$ if $p(x) \geq 0$ on $[\tau, \infty)$, then

$$\liminf_{x \rightarrow \infty} \frac{g(x)}{xg'(x)} \geq \frac{1}{2}.$$

The proof of this lemma follows from Corollary 1 of Lemma 1 of paper [5].

THEOREM 7. Let $q(x) \geq 0$, $r(x) \geq 0$ for $x \in I$ and let (P) hold. Let further the second order differential equation

$$(6) \quad v'' + p(x)v' + [q(x) + \frac{1}{2}x\theta r(x)]v = 0$$

be oscillatory on I for some $\theta \in (0, 1)$. Then the differential equation (a) is oscillatory on I .

Proof. Suppose the assertion is false. Let namely equation (a) be non-oscillatory on I . Then by Lemma 5 there exists such a solution y of (a) and such a number $\tau \in I$ that $y(x) > 0$, $y'(x) > 0$ hold for $x > \tau$. From equation (a) we get then

$$(y''(x)E(x, \tau))' = -[q(x)y'(x) + r(x)y(x)]E(x, \tau) \leq 0$$

for $x \geq \tau$. Taking it into consideration and regarding $p(x) \geq 0$ on I from Lemma 6 we get

$$\liminf_{x \rightarrow \infty} \frac{y(x)}{xy'(x)} \geq \frac{1}{2}.$$

Thus there exists for the number $\theta \in (0, 1)$ such a $T > \tau$ ($T > 0$), that

$$\frac{y(x)}{xy'(x)} > \frac{\theta}{2} \quad \text{for } x > T,$$

hence

$$(7) \quad \frac{y(x)}{y'(x)} > \frac{x}{2}\theta \quad \text{for } x > T.$$

Consider now the equation

$$(8) \quad v'' + p(x)v' + \left[q(x) + \frac{y(x)}{y'(x)} r(x) \right] v = 0 \quad (x > T).$$

Since the function $v = y'(x) > 0$ is a solution of this equation on (T, ∞) , equation (8) is disconjugated on (T, ∞) . On account of it regarding inequality (7) by Sturm Comparison Theorem the differential equation (6)

is also disconjugated on (T, ∞) . But this is in a contradiction with the assumption, which completes the proof.

COROLLARY 8. *Let $q(x) \geq 0$, $r(x) \geq 0$ for $x \in I$ and let (P) hold. If the second order differential equation*

$$v'' + p(x)v' + q(x)v = 0$$

is oscillatory on I , then the differential equation (a) is oscillatory on I , too.

Remark 3. By Theorem 7 it follows Theorem 3.1 in [10]. Besides it Theorem 7 completes Theorem 2 in [5].

The proof of the following theorem and corollary is similar to the proof of Theorem 6 and Corollary 5 (using Theorem 2, Corollary 3, respectively).

THEOREM 8. *Let (\bar{P}) hold and let the second order differential equation*

$$v'' + p(x)v' + \frac{1}{4}q(x)v = 0$$

be oscillatory on I . Then equation (a) is oscillatory on I .

COROLLARY 9. *Let $q(x) \geq 0$ on I and let (\bar{P}) hold. If the differential equation (a) is non-oscillatory on I , then for an arbitrary non-trivial solution u of (\bar{b}) with the property $\bar{F}x_0[u(x_0)] \leq 0$ at some number $x_0 \in I$ there exists such a number $\tau \in I$ that $u(x)u'(x) > 0$ for $x > \tau$.*

Remark 4. From Theorem 6 and Theorem 8 follows Theorem 1 of [8].

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DEPARTMENT OF MATHEMATICAL ANALYSIS MFFUK
BRATISLAVA, CZECHOSLOVAKIA

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