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SOME RESULTS OF η-RICCI SOLITONS ON $(LCS)_n$-MANIFOLDS

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Abstract. In this paper, we consider an η-Ricci soliton on the $(LCS)_n$-manifolds (M, ϕ, ξ, η, g) satisfying certain curvature conditions likes: $R(\xi, X) \cdot S = 0$ and $W_2(\xi, X) \cdot S = 0$. We show that on the $(LCS)_n$-manifolds (M, ϕ, ξ, η, g), the existence of η-Ricci soliton implies that (M, g) is a quasi-Einstein. Further, we discuss the existence of Ricci solitons with the potential vector field ξ. In the end, we construct the non-trivial examples of η-Ricci solitons on the $(LCS)_n$-manifolds.

1 Introduction

In 2003, Shaikh [33] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifold) with an example, which generalize the notion of LP-Sasakian manifolds introduced by Matsumoto [27] and also by Mihai and Rosca [28]. The properties of $(LCS)_n$-manifolds have been studied by many geometer, for instance we refer ([7], [8], [22]-[25], [29], [34], [36], [39]-[42]).

The Ricci solitons are natural generalization of Einstein metrics on a Riemannian manifold, being generalized fixed points of Hamilton's Ricci flow $\frac{\partial}{\partial t} g = -2S$ [20]. The evolution equation defining the Ricci flow is a kind of nonlinear diffusion equation, an analogue of heat equation for metrics. Under Ricci flow, a metric can be improved to evolve into more canonical one by smoothing out its irregularities, depending on the Ricci curvature of the manifold: it will expand in the directions of negative Ricci curvature and shrink in the positive case. The geometrical properties of the Ricci solitons have been studied in ([1]-[5], [7]-[13], [17]-[21], [26], [31], [37], [38], [43]) and by others. In paracontact geometry, the Ricci soliton first appeared in the paper of G. Calvaruso and D. Perrone [6]. C. L. Bejan and M. Crasmareanu studied the properties of Ricci solitons on the 3-dimensional normal paracontact manifolds [3]. A more general notion of a Ricci soliton is that of η-Ricci soliton introduced by J. T. Cho and M. Kimura [18], which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex-space-forms [4]. Metrics satisfying

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Ricci flow equations are interesting and useful in physics and are often referred as quasi-Einstein ([12]-[16]).

2 $(LCS)_n$-manifolds (M, ϕ, ξ, η, g)

Let M be an n-dimensional smooth connected paracontact Hausdroff manifold equipped with a Lorentzian metric g. Then (M, g) is a Lorentzian manifold, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \Re$ is a non degenerate inner product of signature $(-, +, ..., +)$, where T_pM denotes the tangent space of M at p and $\Re$ is the real number. A non-zero vector field $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, and spacelike) if it satisfies $g_p(v, v) < 0$ (resp., $\leq 0, =, > 0$) [30].

Definition 1. *A non-vanishing vector field ρ on a Lorentzian manifold (M, g) defined by $g(X, \rho) = A(X)$, $\forall\ X \in \chi(M)$ is said to be a concircular vector field [41] if*

$$(\nabla_X A)(Y) = \alpha \{g(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form.

If the Lorentzian manifold M admits a unit timelike concircular vector field ξ, called the *generator* of the manifold, then we have

$$g(\xi, \xi) = -1, \ \ g(X, \xi) = \eta(X), \ \ (\nabla_X \eta)(Y) = \alpha \{g(X, Y) + \eta(X)\eta(Y)\}, \quad (2.1)$$

where $\alpha \neq 0$ and η is a non-zero 1-form. It is obvious from (2.1) that

$$\nabla_X \xi = \alpha\{X + \eta(X)\xi\} \quad (2.2)$$

for all vector field X on M. Here ∇ denotes the operator of the covariant differentiation with respect to the Lorentzian metric g and α satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \quad (2.3)$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\alpha\, \phi X = \nabla_X \xi, \quad (2.4)$$

then (2.2) and (2.4) give

$$\phi X = X + \eta(X)\xi, \quad (2.5)$$

where ϕ is a $(1, 1)$-tensor, called the structure tensor of M. Thus the Lorentzian manifold M together with a unit timelike concircular vector field ξ, its associated 1-form η and $(1, 1)$-tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [33]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [27]. For details, we refer [11] and the references therein. In an $(LCS)_n$-manifold, $n > 2$, the following relations

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \phi^2 X = X + \eta(X)\xi,$$
$$\eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.6}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\left\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right\}, \tag{2.7}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)\left\{\eta(Y)X - \eta(X)Y\right\}, \tag{2.8}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)\left\{g(X,Y)\xi - \eta(Y)X\right\}, \tag{2.9}$$

$$(\nabla_X \phi)(Y) = \alpha\left\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\right\}, \tag{2.10}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{2.12}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.13}$$

hold for any vector fields X, Y, Z on M, $\beta = -(\xi\rho)$ is a scalar function [34]. Here R is the curvature tensor corresponding to the Lorentzian metric g and S is the Ricci tensor corresponding to the Ricci operator Q, that is, $S(X,Y) = g(QX,Y)$.

3 η-Ricci solitons on $(LCS)_n$-manifolds (M, ϕ, ξ, η, g)

Let (M, ϕ, ξ, η, g) be an $(LCS)_n$-manifold, then the quartet (g, ξ, λ, μ) on M is said to be an η-Ricci soliton [18] if it satisfies

$$L_\xi g + 2S + 2\lambda\, g + 2\mu\, \eta \otimes \eta = 0, \tag{3.1}$$

where L_ξ is the Lie-derivative operator along the vector field ξ, λ and μ are real constants. We write $L_\xi g$ in term of the Levi-Civita connection ∇ as:

$$(L_\xi g)(X,Y) = g(\nabla_Y \xi, X) + g(Y, \nabla_X \xi) = 2\alpha[g(X,Y) + \eta(X)\eta(Y)], \tag{3.2}$$

where equations (2.1) and (2.2) are used. In view of (3.1) and (3.2), we get

$$QX = -(\alpha + \lambda)X - (\alpha + \mu)\eta(X)\xi, \tag{3.3}$$

$$r = -n\lambda - (n-1)\alpha + \mu, \tag{3.4}$$

$$S(X,Y) = -(\alpha + \lambda)g(X,Y) - (\alpha + \mu)\eta(X)\eta(Y), \tag{3.5}$$

$$S(X,\xi) = S(\xi, X) = (\mu - \lambda)\eta(X), \tag{3.6}$$

$$\mu - \lambda = (n-1)(\alpha^2 - \rho) \tag{3.7}$$

for any $X, Y \in \chi(M)$. Here r is the scalar curvature of (M, g) and is defined by $r = S(e_i, e_i)_{i=1}^n$, where $\{e_1, e_2, ..., e_n\}$ is a set of linearly independent vector fields on M. In particular, if $\mu = 0$ then the triplet (g, ξ, λ) is a Ricci soliton [20] and it is called shrinking, steady or expanding according as λ is negative, zero or positive, respectively [19].

Proposition 2. *The following relations hold on an* $(LCS)_n$*-manifold* (M,ϕ,ξ,η,g)

$$(i)\ \ \eta(\nabla_X\xi)=0,\ \ (ii)\ \ \nabla_\xi\xi=0,\ \ (iii)\ \ \nabla_\xi\eta=0,\ (iv)\ \ L_\xi\phi=0,$$

$$(v)\ \ L_\xi\eta=0,\ \ (vi)\ \ L_\xi(\eta\otimes\eta)=0,\ \ (vii)\ \ L_\xi g=2\alpha(g+\eta\otimes\eta).$$

Also, if η *is closed the distribution is involuntary and the Nijenhuis tensor of* ϕ *vanishes identically, i.e., the structure is normal.*

Proof. Since $(\nabla_X\phi)(Y)=\alpha\{g(X,Y)\xi+2\eta(X)\eta(Y)\xi+\eta(Y)X\}$ and therefore

$$\nabla_X\phi Y-\phi(\nabla_XY)=\alpha\{g(X,Y)\xi+2\eta(X)\eta(Y)\xi+\eta(Y)X\}.$$

Taking $Y=\xi$ in the above equation, we have $\phi(\nabla_X\xi)=\alpha\phi X$. Applying ϕ on either sides, we get

$$\nabla_X\xi+\eta(\nabla_X\xi)\xi=\alpha\{X+\eta(X)\xi\}.$$

Since $X(g(\xi,\xi))=2g(\nabla_X\xi,\xi)$ and $\nabla_X\xi=\alpha\phi X$, therefore $\eta(\nabla_X\xi)=0$, and hence $\nabla_\xi\xi=0$. As we know that $\eta(X)=g(X,\xi)$ and ∇ is metric, then we have $\nabla_\xi\eta=0$. The Lie-derivative of ϕ along ξ gives

$$(L_\xi\phi)(X)=[\xi,\phi X]-\phi([\xi,X])=\nabla_\xi\phi X-\phi(\nabla_\xi X)=(\nabla_\xi\phi)(X)=0, i.e., L_\xi\phi=0.$$

Again, $(L_\xi\eta)(X)=\xi(\eta(X)-\eta([\xi,X]))=g(X,\nabla_\xi\xi)+g(\nabla_X\xi,\xi)=0$, *i.e.*, $L_\xi\eta=0$. Also, if $L_\xi\eta=0$, then $L_\xi\eta\otimes\eta=0$, as $L_\xi\eta\otimes\eta=(L_\xi\eta)\otimes\eta+\eta\otimes(L_\xi\eta)$. Again $(L_\xi g)(X,Y)=\xi\, g(X,Y)-g([\xi,X],Y)-g(X,[\xi,Y])$, implies that

$$(L_\xi g)(X,Y)=\alpha[g(\phi X,Y)+g(X,\phi Y)].$$

Using (2.5), we get

$$L_\xi g=2\alpha(g+\eta\otimes\eta).$$

It is well known that

$$(d\eta)(X,Y)=X(\eta(Y))-Y(\eta(X))-\eta([X,Y])$$

implies that

$$(d\eta)(X,Y)=g(Y,\nabla_X\xi)-g(X,\nabla_Y\xi)$$
$$=\alpha\{g(Y,X)+\eta(X)\eta(Y)\}-\alpha\{g(X,Y)+\eta(X)\eta(Y)\}=0, i.e., d\eta=0.$$

Finally,

$$N_\phi(X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]$$

yields that

$$N_\phi(X,Y)=\phi^2(\nabla_XY)-\phi^2(\nabla_YX)-\phi(\nabla_X\phi Y)+\phi(\nabla_Y\phi X)$$
$$+\nabla_{\phi X}\phi Y-\phi(\nabla_{\phi X}Y)-\nabla_{\phi Y}\phi X+\phi(\nabla_{\phi Y}X)=0,$$

i.e., the structure is normal. □

In [7] and [8], Shaikh et al. proved that a second order parallel symmetric tensor on a Lorentzian concircular structure manifold with $\alpha^2-\rho\neq 0$ is a constant multiple of the Ricci tensor. Thus we apply this concept for η-Ricci soliton and prove the following results.

Theorem 3. *Let (M,ϕ,ξ,η,g) is an $(LCS)_n$-manifold. If the symmetric tensor field $h=L_\xi g+2S+2\mu\,\eta\otimes\eta$ of type $(0,2)$ is parallel with respect to the Levi-Civita connection ∇, then (g,ξ,λ) on M yields an η-Ricci soliton.*

Proof. In consequence of (3.2), we have

$$h(X,Y)=2\alpha\, g(X,Y)+2S(X,Y)+2(\alpha+\mu)\eta(X)\eta(Y).$$

Replacing X and Y with ξ in the above equation, we get

$$h(\xi,\xi)=(L_\xi g)(\xi,\xi)+2S(\xi,\xi)+2\mu\eta(\xi)\eta(\xi)=2\lambda,$$

and therefore

$$\lambda=\frac{1}{2}h(\xi,\xi).$$

From [7] and [8], we have

$$h(X,Y)=-h(\xi,\xi)g(X,Y),\ \forall\, X,\, Y\in\chi(M).$$

Thus, $L_\xi g+2S+2\mu\eta\otimes\eta=-2\lambda g$. Hence the statement of the theorem. □

If $\mu=0$, it follows that $L_\xi g+2S+2(n-1)(\alpha^2-\rho)g=0$. Thus we conclude the following corollary:

Corollary 4. *On an $(LCS)_n$-manifold (M,ϕ,ξ,η,g) with the property that a symmetric tensor field $h=L_\xi g+2S$ of type $(0,2)$ is parallel with respect to the Levi-Civita connection associated to g, then the equation (3.1), for $\mu=0$ and $\lambda=(n-1)(\alpha^2-\rho)$, define a Ricci soliton.*

An $(LCS)_n$ manifold (M,ϕ,ξ,η,g) is said to be quasi-Einstein if its Ricci tensor S is a linear combination (with real scalars λ and $\mu(\neq 0)$) of g and the tensor product of a non-zero 1-form η satisfying (2.1) and for an Einstein if S is collinear with g [6]. From (3.5), we state the results in the form of corollary as:

Corollary 5. *If the equation (3.5) define an η-Ricci soliton on an $(LCS)_n$-manifold, then (M,g) is quasi-Einstein.*

Next, we prove the following theorem as:

Theorem 6. *Let (g,ξ,λ,μ) is an η-Ricci soliton on an $(LCS)_n$-manifold (M,ϕ,ξ,η,g). If the Ricci tensor S of M is*
(i) cyclic parallel, then $\mu=-\alpha-\frac{\rho}{2\alpha}$, and $\lambda=-\frac{\rho}{2\alpha}(1-2\alpha(n-1))-\alpha(1+(n-1)\alpha)$.
(ii) cyclic parallel η-recurrent, then there does not exist an η-Ricci soliton or a Ricci soliton with the potential vector field ξ on M.

Proof. It is well known that

$$(\nabla_X S)(Y,Z) = X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{3.8}$$

In view of (2.2), (2.3) and (3.5), the equation (3.8) reduces to

$$(\nabla_X S)(Y,Z) = -\rho g(\phi Y, \phi Z)\eta(X) - \alpha(\alpha+\mu)\{g(\phi X, \phi Z)\eta(Y) + g(\phi X, \phi Y)\eta(Z)\}. \tag{3.9}$$

If possible, we suppose that the Ricci tensor S of M is cyclic parallel, that is, $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(Z,Y) = 0 \ \forall \ X,\ Y,\ Z \ \in \chi(M)$. The cyclic sum of (3.9) together with the last argument give

$$\begin{aligned} &-\rho\{g(\phi Y, \phi Z)\eta(X) + g(\phi X, \phi Z)\eta(Y) + g(\phi Y, \phi X)\eta(Z)\} \\ &\quad -2\alpha(\alpha+\mu)\{g(\phi X, \phi Z)\eta(Y) + g(\phi X, \phi Y)\eta(Z) + g(\phi Y, \phi Z)\eta(X)\} = 0. \end{aligned} \tag{3.10}$$

Replacing $Z = \xi$ in (3.10), we have

$$(\rho + 2\alpha(\alpha+\mu))g(\phi X, \phi Y) = 0$$

for any $X, Y \in \chi(M)$. It follows that $\rho + 2\alpha(\alpha+\mu) = 0$ and thus (3.7) gives $\mu = -\alpha - \frac{\rho}{2\alpha}$, and $\lambda = -\frac{\rho}{2\alpha}(1-2\alpha(n-1)) - \alpha(1+(n-1)\alpha)$. To prove the result (ii), we suppose that M is η-recurrent, that is, $(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z) \ \forall \ X,\ Y,\ Z \in \chi(M)$. If the Ricci tensor S of the η-recurrent $(LCS)_n$-manifold is cyclic parallel, then

$$\begin{aligned} &\eta(X)S(Y,Z) + \eta(Y)S(Z,X) + \eta(Z)S(X,Y) \\ &\quad = -\rho\{g(\phi Y, \phi Z)\eta(X) + g(\phi X, \phi Z)\eta(Y) + g(\phi Y, \phi X)\eta(Z)\} \\ &\quad -2\alpha(\alpha+\mu)\{g(\phi X, \phi Z)\eta(Y) + g(\phi X, \phi Y)\eta(Z) + g(\phi Y, \phi Z)\eta(X)\} = 0 \end{aligned} \tag{3.11}$$

for any $X, Y, Z \in \chi(M)$. Taking $Y = Z = \xi$ in (3.11) and then using (3.5) and (3.6), we get $3(\mu - \lambda)\eta(X) = 0$ for any $X \in \chi(M)$. It follows that $\lambda = \mu$, which is a contradiction. Thus the statements of the theorem are proved. □

In view of the Theorem 6, we can state the following corollaries.

Corollary 7. *In an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) equipped with a cyclic parallel Ricci tensor, there is no Ricci soliton with the potential vector field ξ.*

Corollary 8. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) possesses a cyclic parallel η-recurrent Ricci tensor, then M does not admit η-Ricci soliton or Ricci soliton with the potential vector field ξ.*

Theorem 9. *Let (g, ξ, λ, μ) be an η-Ricci soliton on an $(LCS)_n$-manifold (M, ϕ, ξ, η, g). If the Ricci tensor S of M satisfies*
(i) $\nabla S = 0$, then $\mu = -\alpha + \frac{\xi\alpha}{\alpha}$, and $\lambda = \frac{\xi\alpha}{\alpha} - \alpha - (n-1)(\alpha^2 - \rho)$.
(ii) $\nabla S = \eta \otimes S$, then there does not exist η-Ricci soliton or Ricci soliton with the potential vector field ξ on M.

Proof. Let us suppose that the Ricci tensor S of M satisfies $\nabla S = 0$, that is, M is Ricci symmetric $(LCS)_n$-manifold. Replacing Z by ξ in (3.10), we obtain

$$\{\alpha(\alpha+\mu)+\rho\}g(\phi X, \phi Y) = 0, \quad \forall \;\; X, Y \in \chi(M).$$

It follows that $\mu = -\alpha + \frac{\xi\alpha}{\alpha}$, and $\lambda = \frac{\xi\alpha}{\alpha} - \alpha - (n-1)(\alpha^2 - \rho)$, the statement (i). Let M is η-recurrent $(LCS)_n$-manifold, that is, $\nabla S = \eta \otimes S$. From (3.5) we obtain $\lambda = \mu$, which is not possible. Thus our theorem is proved. □

In consequence of the Theorem 9, we state the following corollaries.

Corollary 10. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) is Ricci symmetric, then there is no Ricci soliton with the potential vector field ξ on M.*

Corollary 11. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) is admitting an η-recurrent Ricci tensor, then there does not exist η-Ricci soliton or Ricci soliton with the potential vector field ξ on M.*

4 η-Ricci solitons satisfying certain curvature conditions on the $(LCS)_n$-manifolds (M, ϕ, ξ, η, g)

In 1970, Pokhariyal et al. [32], defined and studied the properties of W_2-curvature tenor, and is given by

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}\{g(X, Z)QY - g(Y, Z)QX\} \tag{4.1}$$

for $X, Y, Z \in \chi(M)$.

Theorem 12. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) equipped with an η-Ricci soliton (g, ξ, λ, μ) satisfies $R(\xi, X) \cdot S = 0$, then $\mu = -\alpha$ and $\lambda = -\alpha - (n-1)(\alpha^2 - \rho)$.*

Proof. Suppose M satisfies $R(\xi, X) \cdot S = 0$. Then we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$. Using (2.9) and (3.5) in the above equation, we yield

$$(\alpha^2 - \rho)(\mu + \alpha)\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)\} = 0.$$

For $Z = \xi$, we have

$$(\alpha^2 - \rho)(\mu + \alpha)\{g(X, Y) + \eta(X)\eta(Y)\} = 0.$$

It is obvious from the above equation that $\mu = -\alpha$, provided $\alpha^2 - \rho \neq 0$. Equation (3.7) together with the last result give $\lambda = -\alpha - (n-1)(\alpha^2 - \rho)$. Hence the statement of the theorem is proved. □

With the help of the Theorem 12, we state the following corollaries.

Corollary 13. *Let an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) equipped with the η-Ricci soliton satisfies $R(\xi, X) \cdot S = 0$. Then there is no Ricci soliton on M with the potential vector field ξ.*

Corollary 14. *An $(LCS)_n$-manifold (M, ϕ, ξ, η, g) together with the η-Ricci soliton (g, ξ, λ, μ) and $R(\xi, X) \cdot S = 0$ is Einstein.*

Theorem 15. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) with an η-Ricci soliton satisfies $W_2(\xi, X) \cdot S = 0$, then either $\mu = -\alpha$, $\lambda = \alpha - (n-1)(\alpha^2 - \rho)$ or $\lambda = -\alpha$, $\mu = -\alpha + (n-1)(\alpha^2 - \rho)$.*

Proof. If possible, we assume that the $(LCS)_n$-manifolds endowed with the η-Ricci solitons are W_2-Ricci symmetric, that is, $W_2(\xi, X) \cdot S = 0$. Thus we have

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0 \tag{4.2}$$

for any X, Y, $Z \in \chi(M)$. Using (3.5) and (4.1) in (4.2), we get

$$\begin{aligned}
&(\alpha^2 - \rho)\,[g(X, Y)S(\xi, Z) + g(X, Z)S(Y, \xi) - S(X, Z)\eta(Y) - S(X, Y)\eta(Z)] \\
&-\tfrac{1}{n-1}[(\alpha + \lambda)\{S(X, Z)\eta(Y) + \eta(Z)S(Y, X)\} + (\alpha + \mu)\{\eta(X)\eta(Y)S(\xi, Z) \\
&+\eta(X)\eta(Z)S(Y, \xi)\} + (\mu - \lambda)\{g(X, Y)S(\xi, Z) + g(X, Z)S(Y, \xi)\}] = 0.
\end{aligned} \tag{4.3}$$

In consequence of (3.5)-(3.7), equation (4.3) consider the form

$$\frac{(\alpha + \mu)(\alpha + \lambda)}{n-1}\{\eta(Y)g(X, Y) + \eta(Z)g(X, Y) + 2\eta(X)\eta(Y)\eta(Z)\} = 0. \tag{4.4}$$

Taking Z $= \xi$ in (4.4), we yield

$$\frac{(\alpha + \mu)(\alpha + \lambda)}{n-1} g(\phi X, \phi Y) = 0 \tag{4.5}$$

for any X, $Y \in \chi(M)$. In general, $g \neq 0$ on M, therefore (4.5) shows that either $\mu = -\alpha$ or $\lambda = -\alpha$, for $n > 1$. These results together with (3.7) reflect that either $\mu = -\alpha$, $\lambda = \alpha - (n-1)(\alpha^2 - \rho)$ or $\lambda = -\alpha$, $\mu = -\alpha + (n-1)(\alpha^2 - \rho)$ on M. □

Corollary 16. *If an $(LCS)_n$-manifold (M, ϕ, ξ, η, g) satisfies $W_2(\xi, X) \cdot S = 0$, then there is no Ricci soliton with the potential vector field ξ on M.*

5 Examples of η-Ricci soliton on $(LCS)_n$-manifolds

Example 17. *Let a 3-dimensional manifold $M = \left\{(x, y, z) \in \Re^3 : z \neq 0\right\}$, where (x, y, z) are the standard coordinates in $\Re^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame on M given by*

$$E_1 = e^z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = e^z\frac{\partial}{\partial y}, \qquad E_3 = e^{2z}\frac{\partial}{\partial z}.$$

Assume that g be the Lorentzian metric on M, and is defined by

$$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0,\ g(E_1, E_1) = g(E_2, E_2) = 1,\ g(E_3, E_3) = -1.$$

Let η be the 1-form defined by $\eta(V) = g(V, E_3)$ for any $V \in \chi(M)$ and ϕ is a $(1,1)$-tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_3) = -1, \quad \phi^2 V = V + \eta(V)E_3, \ g(\phi V, \phi W) = g(V, W) + \eta(V)\eta(W)$$

for any $V, W \in \chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we obtain

$$[E_1, E_2] = -e^z E_2, \qquad [E_1, E_3] = -e^{2z} E_1, \qquad [E_2, E_3] = -e^{2z} E_2.$$

Taking $E_3 = \xi$ and using the Koszul's formula for the Lorentzian metric g, we have

$$\nabla_{E_1} E_3 = -e^{2z} E_1, \qquad \nabla_{E_1} E_1 = -e^{2z} E_3, \qquad \nabla_{E_1} E_2 = 0,$$

$$\nabla_{E_2} E_3 = -e^{2z} E_2, \qquad \nabla_{E_3} E_2 = 0, \qquad \nabla_{E_2} E_1 = e^z E_2,$$

$$\nabla_{E_3} E_3 = 0, \qquad \nabla_{E_2} E_2 = e^{2z} E_3 - e^z E_1, \qquad \nabla_{E_3} E_1 = 0.$$

From the above equations, it can be easily seen that $E_3 = \xi$ is a unit timelike concircular vector field and hence the structure (ϕ, ξ, η, g) is an $(LCS)_3$-structure on M. Consequently, $M^3(\phi, \xi, \eta, g)$ is an $(LCS)_3$-manifold with $\alpha = -e^{2z} \neq 0$ such that $(X\alpha) = \rho\eta(X)$, where $\rho = 2e^{4z}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R and the Ricci tensor S as follows:

$$R(E_2, E_3)E_3 = -e^{4z} E_2, \quad R(E_1, E_3)E_3 = -e^{4z} E_1, \quad R(E_1, E_2)E_2 = \{e^{4z} - e^{2z}\} E_1,$$

$$R(E_2, E_3)E_2 = e^{4z} E_3 - e^{3z} E_1,\ R(E_1, E_3)E_1 = -e^{4z} E_3,\ R(E_2, E_1)E_1 = \{e^{4z} - e^{2z}\} E_2,$$

$$S(E_1, E_1) = -e^{2z}, \qquad S(E_2, E_2) = -e^{2z}, \qquad S(E_3, E_3) = -2e^{4Z}.$$

Also from the equation (3.5), we can see that

$$S(E_1, E_1) = -(\alpha + \lambda), \qquad S(E_2, E_2) = -(\alpha + \lambda), \qquad S(E_3, E_3) = (\lambda - \mu).$$

Thus we conclude from the last two expressions that for $\alpha = -e^{2z}$, $\lambda = 2e^{2z}$ and $\mu = 2\{e^{2z} + e^{4z}\}$, the structure (g, ξ, λ, μ) is an η-Ricci soliton on $M^3(\phi, \xi, \eta, g)$.

Example 18. *Let a 3-dimensional manifold $M = \left\{(x, y, z) \in \Re^3 : z \neq 0\right\}$, where (x, y, z) are the standard coordinates in $\Re^3$. In [35], Shaikh defined the linearly independent vector fields $\{E_1, E_2, E_3\}$ on M as:*

$$E_1 = e^{-z}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = e^{-z}\frac{\partial}{\partial y}, \qquad E_3 = e^{-2z}\frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by

$$g(E_1,E_3)=g(E_2,E_3)=g(E_1,E_2)=0,\ g(E_1,E_1)=g(E_2,E_2)=1,\ g(E_3,E_3)=-1.$$

Let η be the 1-form defined by $\eta(V)=g(V,E_3)$ for any $V\in\chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by $\phi E_1=E_1$, $\phi E_2=E_2$, $\phi E_3=0$. Then using the linearity of ϕ and g we have

$$\eta(E_3)=-1,\quad \phi^2 V=V+\eta(V)E_3,\ g(\phi V,\phi W)=g(V,W)+\eta(V)\eta(W),$$

for any $V,W\in\chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we obtain

$$[E_1,E_2]=-e^{-z}E_2,\qquad [E_1,E_3]=-e^{-2z}E_1,\qquad [E_2,E_3]=-e^{-2z}E_2.$$

Taking $E_3=\xi$ and using the Koszul's formula for the Lorentzian metric g, we have

$$\nabla_{E_1}E_3=e^{-2z}E_1,\qquad \nabla_{E_1}E_1=e^{-2z}E_3,\qquad \nabla_{E_1}E_2=0,$$

$$\nabla_{E_2}E_3=e^{-2z}E_2,\qquad \nabla_{E_3}E_2=0,\qquad \nabla_{E_2}E_1=e^{-2z}E_2,$$

$$\nabla_{E_3}E_3=0,\qquad \nabla_{E_2}E_2=e^{-2z}E_3-e^{-z}E_1,\qquad \nabla_{E_3}E_1=0.$$

From the above equations, it can be easily seen that $E_3=\xi$ is a unit timelike concircular vector field and hence (ϕ,ξ,η,g) is an $(LCS)_3$-structure on M. Thus $M^3(\phi,\xi,\eta,g)$ is an $(LCS)_3$-manifold with $\alpha=e^{-2z}\neq 0$ such that $(X\alpha)=\rho\eta(X)$, where $\rho=2e^{-4z}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R and the Ricci tensor S as follows:

$$R(E_2,E_3)E_3=e^{-4z}E_2,\ \ R(E_1,E_3)E_3=e^{-4z}E_1,\ \ R(E_1,E_2)E_2=\{e^{-4z}-e^{-2z}\}E_1,$$

$$R(E_2,E_3)E_2=e^{-4z}E_3,\ \ R(E_1,E_3)E_1=e^{-4z}E_3,\ \ R(E_1,E_2)E_1=\{-e^{-4z}-e^{-2z}\}E_2,$$

$$S(E_1,E_1)=2e^{-4z}-e^{-2z},\qquad S(E_2,E_2)=2e^{-4z}-e^{-2z},\qquad S(E_3,E_3)=2e^{-4z}.$$

Also from (3.5), we calculate that

$$S(E_1,E_1)=-(\alpha+\lambda),\qquad S(E_2,E_2)=-(\alpha+\lambda),\qquad S(E_3,E_3)=(\lambda-\mu).$$

We conclude from (3.5) that for $\alpha=e^{2z},\lambda=-2e^{-4z}$ and $\mu=-4e^{-4z}$, the data (g,ξ,λ,μ) admits an η-Ricci soliton on $M^3(\phi,\xi,\eta,g)$.

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