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SOME RESULTS OF FIXED POINT THEOREMS IN CONVEX METRIC SPACES

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Abstract. In this paper we study some fixed point theorems for self-mappings satisfying certain contraction principles on a convex complete metric space. In addition, we also improve and extend some very recently results in [9].

1. Introduction and Preliminary

In 1970, Takahashi [11] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. A convex metric space is a generalized space. For example, every normed space and cone Banach space is a convex metric space and convex complete metric space, respectively. Subsequently, many mathematicians in [2]-[7], [10, 12] and recently, Moosaei [9] studied fixed point theorems in convex metric spaces.

Our results improve and extend some of Moosaei's results in [9] and Karapinar's results in [8] from a cone Banach space to a convex complete metric space. For instance, Karapinar proved that

Theorem 1.1. ([8, Theorem 2.4]) *Let C be a closed and convex subset of a cone Banach space X with the norm $\|x\|_p = d(x,0)$, and $T : C \to C$ be a mapping which satisfies the condition*

$$\exists\, q \in [2,4), \quad \forall\, x,y \in C, \quad d(x,Tx) + d(y,Ty) \le q d(x,y).$$

Then T has at least one fixed point.

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Letting $x = y$ in the above inequality, it is easy to see that T is an identity mapping. In this paper, results in [8, 9] is improved and extended to a convex complete metric space.

Theorem 1.2. ([8, Theorem 2.6]) *Let C be a closed and convex subset of a cone Banach space X with the norm $\|x\|_p = d(x, 0)$, and $T : C \to C$ be a mapping which satisfies the condition*

$$\exists\, r \in [2, 5), \quad \forall\, x, y \in C, \quad d(Tx, Ty) + d(x, Tx) + d(y, Ty) \le rd(x, y).$$

Then T has at least one fixed point.

Definition 1.3. ([1]) Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure W is called a convex metric space, which is denoted by (X, d, W).

Example 1.4. Let $(X, d, \|.\|)$ be a normed space. The mapping $W : X \times X \times I \to X$ defined by $W(x, y, \lambda) = \lambda x + (1 - \lambda)\mathrm{y}$ for each $x, y \in X$, $\lambda \in I$ is a convex structure on X.

Definition 1.5. ([1]) Let (X, d, W) be a convex metric space. A nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times I$.

Lemma 1.6. ([9]) *Let (X, d, W) be a convex metric space, then the following statements hold:*

(i) $d(x, y) = d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))$ *for all* $(x, y, \lambda) \in X \times X \times I$.
(ii) $d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y)$ *for all* $x, y \in X$.
(iii) $d(y, W(x, y, \lambda)) = \lambda d(x, y)$ *for all* $x, y \in X$.

Proof. To prove (i) see [9, Lemma 3.1].
By definition, we have

$$d(x, W(x, y, \lambda)) \le (1 - \lambda)d(x, y)$$

and on the other hand

$$\begin{aligned}(1 - \lambda)d(x, y) &= d(x, y) - \lambda d(x, y)\\ &= [d(x, W(x, y, \lambda)) + d(y, W(x, y, \lambda))] - \lambda d(x, y)\end{aligned}$$

but

$$d(y, W(x, y, \lambda)) \le \lambda d(x, y).$$

Therefore

$$(1-\lambda)d(x,y) \leq d(x,W(x,y,\lambda)).$$

Thus $d(x,W(x,y,\lambda)) = (1-\lambda)d(x,y)$ for all $x,y \in X$. This completes proof of (ii).

For (iii), by (i) and (ii), we have

$$\begin{aligned} d(x,y) &= d(x,W(x,y,\lambda)) + d(y,W(x,y,\lambda)) \\ &= (1-\lambda)d(x,y) + d(y,W(x,y,\lambda)). \end{aligned}$$

So $d(y,W(x,y,\lambda)) = \lambda d(x,y)$ for all $x,y \in X$. □

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a convex complete metric space (X,d,W) and T be a self-mapping of C. If there exist a,b,c,e,f, and k such that*

$$\frac{b+e-|f|(1-\lambda)-|c|\lambda}{1-\lambda} \leq k < \frac{a+b+c+e+f-|c|\lambda-|f|(1-\lambda)}{1-\lambda} \tag{2.1}$$

$$ad(x,Tx) + bd(y,Ty) + cd(Tx,Ty) + ed(x,Ty) + fd(y,Tx) \leq kd(x,y) \tag{2.2}$$

for all $x,y \in C$, then T has at least one fixed point.

Proof. Fix $\lambda \in (0,1)$. Suppose $x_0 \in C$ is arbitrary. We define a sequence $\{x_n\}_{n=1}$ in the following way:

$$x_n = W(x_{n-1}, T(x_{n-1},\lambda)), \quad n = 1,2,3,\cdots.$$

As C is convex, $x_n \in C$ for all $n \in \mathbb{N}$. By Lemma 1.6 and above relation, we have

$$d(x_{n+1},x_n) = (1-\lambda)d(x_n,Tx_n), \tag{2.3}$$

$$d(x_n,Tx_{n-1}) = \lambda d(x_{n-1},Tx_{n-1}) = \frac{\lambda}{1-\lambda}d(x_n,x_{n-1}). \tag{2.4}$$

By relation (2.3)

$$\begin{aligned} \frac{1}{1-\lambda}d(x_{n+1},x_n) &= d(x_n,Tx_n) && (2.5)\\ &\leq d(x_n,Tx_{n-1}) + d(Tx_{n-1},Tx_n) && (2.6) \end{aligned}$$

and

$$\frac{c}{1-\lambda}d(x_{n+1},x_n) - \frac{|c|\lambda}{1-\lambda}d(x_n,x_{n-1}) \leq cd(Tx_{n-1},Tx_n). \tag{2.7}$$

And also by relation (2.3) and triangle inequality we have

$$\begin{aligned} \frac{1}{1-\lambda}d(x_{n+1},x_n) &= d(x_n,Tx_n) && (2.8)\\ &\leq d(x_n,x_{n-1}) + d(x_{n-1},Tx_n) && (2.9) \end{aligned}$$

and

$$\frac{f}{1-\lambda}d(x_{n+1},x_n)-|f|d(x_n,x_{n-1})\leq fd(Tx_{n-1},Tx_n) \tag{2.10}$$

for all $n\in\mathbb{N}$. Now, by substituting x with x_n and y with x_{n-1} in (2.2), we get

$$\begin{aligned}&ad(x_n,Tx_n)+bd(x_{n-1},Tx_{n-1})+cd(Tx_n,Tx_{n-1})\\&+ed(x_n,Tx_{n-1})+fd(x_{n-1},Tx_{n-1})\\&\leq kd(x_n,x_{n-1})\end{aligned}$$

so by the relations (2.3),(2.4),(2.7) and (2.10), we obtain

$$\begin{aligned}&\left(\frac{a+c+f}{1-\lambda}\right)d(x_{n+1},x_n)+\left(\frac{b-|c|\lambda+e\lambda}{1-\lambda}-|f|\right)d(x_{n-1},x_n)\\&\leq kd(x_n,x_{n-1}).\end{aligned}$$

Thus

$$d(x_n,x_{n+1})\leq\left(\frac{k(1-\lambda)+|f|(1-\lambda)-b-e+|c|\lambda}{a+c+f}\right)d(x_n,x_{n-1})$$

for all $n\in\mathbb{N}$. By the relation (2.1) $\frac{k(1-\lambda)+|f|(1-\lambda)-b-e+|c|\lambda}{a+c+f}\in[0,1)$ and hence, $\{x_n\}\subseteq C$ is a contraction sequence. Therefore, it is a Cauchy sequence. Since C is a closed subset of a complete space, so $\lim_{n\to\infty}x_n=x^*$ for some $x^*\in C$. Now by relation (2.3)

$$\frac{1}{1-\lambda}d(x_{n+1},x_n)=d(x_n,Tx_n)\leq d(x_n,x^*)+d(x^*,Tx_n)$$

we obtain $\lim_{n\to\infty}d(x_n,Tx_n)=0$ and by

$$d(x^*,Tx_n)\leq d(x^*,x_n)+d(x_n,Tx_n)$$

we get $\lim_{n\to\infty}Tx_n=x^*$.
Now, by substituting x with x^* and y with x_n in relation (2.2), we obtain

$$\begin{aligned}&ad(x^*,Tx^*)+bd(x_n,Tx_n)+cd(Tx^*,Tx_n)+ed(x^*,Tx_n)+fd(x_n,Tx^*)\\&\leq kd(x^*,x_n).\end{aligned}$$

So

$$(a+c+f)d(x^*,Tx^*)\leq 0.$$

But by relation (2.1) $a+c+f\geq 0$ thus $Tx^*=x^*$. □

The following corollary improves and extends [9, Theorem 3.2].

Corollary 2.2. *Let C be a nonempty closed convex subset of a convex complete metric space (X,d,W) and T be a self-mapping of C. If there exist a,b,c and k such that*

$$2b-|c|\leq k<2(a+b+c)-|c|, \tag{2.11}$$

$$ad(x,Tx) + bd(y,Ty) + cd(Tx,Ty) \leq kd(x,y) \tag{2.12}$$

for all $x, y \in C$, then T has at least one fixed point.

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