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Some results on (a, d)-distance antimagic labeling

S. K. Patel[1] orcid.org/0000-0003-0295-4177
Jayesh Vasava[2] orcid.org/0000-0002-0361-3078

[1]Government Engineering College-Bhuj, Dept. of Mathematics, Bhuj, GJ, India.
skpatel27@yahoo.com
[2]Gujarat University, Dept. of Mathematics, Ahmedabad, GJ, India.
jayeshvasava1910@gmail.com

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Abstract:

Let $G = (V,E)$ be a graph of order N and $f : V \rightarrow \{1, 2,...,N\}$ be a bijection. For every vertex v of graph G, we define its weight $w(v)$ as the sum $\sum_{u \in N(v)} f(u)$, where $N(v)$ denotes the open neighborhood of v. If the set of all vertex weights forms an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (N - 1)d\}$, then f is called an (a, d)-distance antimagic labeling and the graph G is called (a, d)-distance antimagic graph. In this paper we prove the existence or non-existence of (a, d)- distance antimagic labeling of some well-known graphs.

Keywords: Distance magic graphs; (a, d)-distance antimagic graphs; Circulant graphs; Cartesian and corona product of graphs.

MSC (2010): 05C78.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. Throughout the paper $N(v)$ and $deg(v)$ denote respectively the open neighborhood and the degree of the vertex v of a given graph. Further, the Greek letters Δ and δ denote respectively the maximum and the minimum degree of a vertex in a given graph. We refer to Gross and Yellen [5] for the standard graph theoretic terminology and notations. We begin with the definition of distance magic labeling which naturally gives rise to the concept of (a, d)-distance antimagic labeling.

Definition 1.1 (6). *Let $G = (V, E)$ be a graph of order N and $f : V \to \{1, 2, \ldots, N\}$ be a bijection. If there exists a positive integer k such that $\sum_{u \in N(v)} f(u) = k$ for every $v \in V$, then we say that f is a distance magic labeling and the graph G is distance magic. Further, the number k is called the magic constant of graph G.*

The sum $\sum_{u \in N(v)} f(u)$ is called the weight of the vertex v and is denoted by $w(v)$. The distance magic labeling was first introduced by Vilfred [11] under the name sigma labeling and it is believed to be motivated by the construction of magic squares in which all the elements of any row, column or diagonal add to a same number. This concept was also introduced by Miller et al. [6] under the name 1-vertex-magic vertex labeling but the term distance magic labeling was used first time by Sugeng et al. [10]. A natural variant of distance magic labeling is distance antimagic labelings where in, all the vertex weights are required to be distinct integers. Distance magic and distance antimagic labeling have very interesting applications in scheduling fair, equalized and handicap incomplete tournaments and they are widely studied for this purpose. See for instance [2] and [3]. A complete survey on the distance magic labeling and its variants is available in [4].

2. A quick review on (a, d)-distance antimagic labeling

(a, d)-distance antimagic labeling is one form of distance antimagic labeling and it emerges in a logical way from distance magic labeling. Let G^c denote the complement of a graph G. If G is a distance magic graph of order N with magic constant k, then it is easy to verify that the set of all vertex weights in G^c is $\{\frac{N(N+1)}{2} - k - i : 1 \leq i \leq N\}$, which is in an arithmetic progression with common difference $d = 1$. This observation motivated Arumugam and Kamatchi [1] to introduce the concept of (a, d)-distance antimagic labeling.

Definition 2.1 (1). *Let $G = (V, E)$ be a graph of order N and $f : V \to \{1, 2, \dots, N\}$ be a bijection. If the set of all vertex weights is $\{a, a + d, a + 2d, \dots, a + (N - 1)d\}$, where a and d are fixed positive integers, then f is called an (a, d)-distance antimagic labeling and the graph G is called (a, d)-distance antimagic graph.*

Although G^c is $(a, 1)$-distance antimagic whenever G is distance magic, the converse is not true in general. The counter example is the cycle graph C_6 which is not distance magic but its complement graph is $(a, 1)$-distance antimagic [1]. The study of (a, d)-distance antimagic labeling for the graphs C_n, $C_n \times K_2$ and paths is available in [1]. R. Simanjuntak and K. Wijaya [9] proved: Wheel graph W_n is (a, d)-distance antimagic if and only if $3 \leq n \leq 5$; the fan graph $F_n = P_n \times K_1$ is (a, d)-distance antimagic if and only $n = 2$ or $n = 4$; the friendship graph f_n, which is obtained by identifying a vertex from n copies of complete graphs of order 3 is (a, d)-distance antimagic if and only $n = 1$ or $n = 2$. M. Nalliah [7] proved that graph mC_n is (a, d)-distance antimagic if and only if mn is odd and $d = 1$. He also proved that the path P_n of order upto 15 except $n = 3, 4$ and 5 is (a, d)-distance antimagic.

In this paper we add a few more results to the existing literature on (a, d)-distance antimagic graphs. We show that the circulant graph $Circ(2n, \{1, n\})$ is $(2n + 2, 1)$-distance antimagic for all even n. We show that mK_{2n} is $(n(2mn - 2m + 1), 1)$-distance antimagic for all m and n whereas $3K_{2n+1}$ is $(6n^2 + n - 1, 1)$-distance antimagic for all n. Later we show that certain graphs will never posses such a labeling. This include $2K_{2n+1}$, the Helm graph H_n, the book graph B_n and the graph $K_n \odot K_1$.

3. Positive results about (a, d)-distance antimagic graphs

We begin with the result about circulant graph whose definition is as follows:

Definition 3.1. *Let $s_1, s_2, \dots, s_m, n$ be positive integers such that $1 \leq s_1 < s_2 < \dots < s_m < n$. Then the circulant graph $Circ(n, \{s_1, s_2, \dots, s_m\})$ is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and whose edges are of the type $v_i v_{i+s_j}$ for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$; where $i + s_j$ is taken modulo n.*

M. F. Semeniuta [8] showed that the circulant graph $Circ(2p+2, \{2, 3, 4, \dots, p-1, p+1\})$ is $(2p^2 - p + 5, 1)$-distance antimagic.

Here we show that the circulant graph $Circ(2n, \{1, n\})$ is $(2n+2, 1)$-distance antimagic for all even n.

Theorem 3.2. *The circulant graph $Circ(2n, \{1, n\})$ is $(2n+2, 1)$-distance antimagic for all even n.*

Proof. Let G denote the graph $\mathrm{Circ}(2n, \{1, n\})$ whose vertex set is say $\{v_1, v_2, \ldots, v_{2n}\}$. Define $f : V(G) \to \{1, 2, \ldots, 2n\}$ as

$$f(v_i) = \begin{cases} \frac{i+1}{2}, & i = 1, 3, \ldots, 2n-1 \\ \frac{3n-i}{2}, & i = 2, 4, \ldots, n-2 \\ \frac{5n-i}{2}, & i = n, n+2, \ldots, 2n \end{cases}$$

It is easy to check that f is a bijection. Also the vertex weights are given by

$$w(v_i) = \begin{cases} \frac{4n+i+2}{2}, & i = 2, 4, \ldots, 2n \\ \frac{7n-i+1}{2}, & i = 1, 3, 5, \ldots, n-3 \\ \frac{9n-i+1}{2}, & i = n-1, n+1, \ldots, 2n-1 \end{cases}$$

We observe that the above weights are in the arithmetic progression with common difference $d = 1$ and first term as $a = 2n + 2$ in the following sequence:
$w(v_2), w(v_4), \ldots, w(v_{2n}), w(v_{n-3}), w(v_{n-5}), \ldots, w(v_3), w(v_1), w(v_{2n-1}),$
$w(v_{2n-3}), \ldots, w(v_{n+1}), w(v_{n-1})$. Hence $\mathrm{Circ}(2n, \{1, n\})$ is $(2n+2, 1)$-distance antimagic for all even n. $\square$

A $(22, 1)$-distance antimagic labeling of $Circ(20, \{1, 10\})$ is illustrated in Figure 1. In general when n is odd, it seems difficult to investigate for (a, d)-distance antimagic labeling of the circulant graph $Circ(2n, \{1, n\})$. However, we have managed to show that $Circ(10, \{1, 5\})$ and $Circ(14, \{1, 7\})$ are $(12, 1)$ and $(16, 1)$-distance antimagic respectively. This is illustrated in Figure 2 and Figure 3 respectively. In all the figures the vertex label is indicated in the usual font and its weight in the bold font.

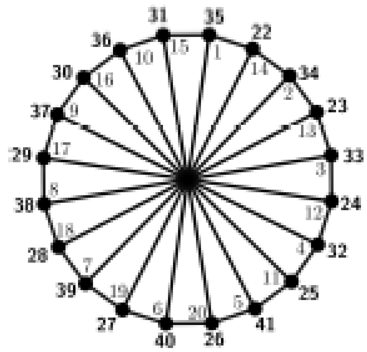

Figure 1: $(22, 1)$-distance antimagic labeling of $Circ(20, \{1, 10\})$

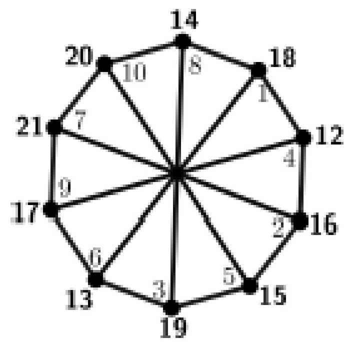

Figure 2: $(12, 1)$-distance antimagic labeling of $Circ(10, \{1, 5\})$

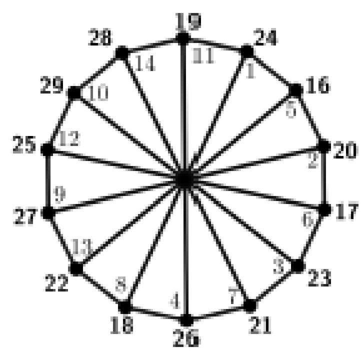

Figure 3: $(16, 1)$-distance antimagic labeling of $Circ(14, \{1, 7\})$

It is quite easy to show that for a complete graph K_n of order n, every bijection f from its vertex set into the set of first n positive integers is (a, d)-distance antimagic labeling with $a = \frac{(n-1)n}{2}$ and $d = 1$. But things are non-trivial and very interesting when we consider union of complete graphs. We get positive as well as negative results as we shall see now. We use the notation mK_n in general to denote the graph which is (disjoint) union of m copies of K_n.

Theorem 3.3. *The graph mK_{2n} is $(n(2mn-2m+1), 1)$-distance antimagic for all m and n.*

Proof. Let $G = mK_{2n}$. Let the vertices of j^{th} copy of K_{2n} be $\{v_1^j, v_2^j, v_3^j, \ldots, v_{2n}^j\}$, where $j = 1, 2, \ldots, m$. Define $f : V(G) \to \{1, 2, \ldots, 2mn\}$ as

$$f(v_i^j) = \begin{cases} n(j-1) + i, & i = 1, 2, \ldots, n \quad \text{and} \quad j = 1, 2, \ldots, m, \\ n(2m-j-1) + i, & i = n+1, n+2, \ldots, 2n \text{ and } j = 1, 2, \ldots, m. \end{cases}$$

It is not difficult to see that f is onto and since the cardinalities of the domain and range sets of f are the same, it follows that f is bijective. Also observe that for each j,

$$\sum_{i=1}^{2n} f(v_i^j) = n^2(2m-2) + \sum_{i=1}^{2n} i = n^2(2m-2) + n(2n+1) = n(2mn+1).$$

Therefore the vertex weights are given by the formula

$$w(v_i^j) = n(2mn+1) - f(v_i^j)$$

and since $f(v_i^j)$ takes each and every value in the set $\{1, 2, \ldots 2mn\}$, it follows that the weights are in the arithmetic progression with first term as $a = n(2mn - 2m + 1)$ and the common difference as $d = 1$. Additionally, it may be verified that the weights are in the arithmetic progression as per the sequence given below:

$$\begin{aligned}
&w(v_{2n}^1), w(v_{2n-1}^1), \ldots, w(v_{n+1}^1), w(v_{2n}^2), w(v_{2n-1}^2), \ldots, w(v_{n+1}^2), \ldots, \\
&w(v_{2n}^m), w(v_{2n-1}^m), \ldots, w(v_{n+1}^m), w(v_n^m), w(v_{n-1}^m), \ldots, \\
&w(v_1^m), w(v_n^{m-1}), w(v_{n-1}^{m-1}), \ldots, w(v_1^{m-1}), \ldots, w(v_n^1), w(v_{n-1}^1), \ldots, w(v_1^1).
\end{aligned}$$

We have shown that the union of any given number of copies of a complete graph of even order is (a, d)-distance antimagic. But this is not always true in case of complete graphs of odd order as we shall prove that the graph

$3K_{2n+1}$ is $(6n^2+n-1,1)$-distance antimagic for all n whereas $2K_{2n+1}$ is never (a,d)-distance antimagic.

Theorem 3.4. *The graph $3K_{2n+1}$ is $(6n^2+n-1,1)$-distance antimagic for all n.*

Proof. Let $G = 3K_{2n+1}$. Let the vertices of j^{th} copy of K_{2n+1} be $\{v_1^j, v_2^j, v_3^j, \ldots, v_{2n+1}^j\}$, where $j = 1,2,3$. The strategy here is to assign the labels in such a way that the sum of all the labels in each of the three copies of K_{2n+1} is a fixed constant. We do this formally by defining $f : V(G) \to \{1,2,\ldots,6n+3\}$ as per the following cases:

Case I: $n \equiv 1 \pmod 3$
Let $A_1 = \{1,4,\ldots,2n-1\}, A_2 = \{2,5,\ldots,2n\}$ and $A_3 = \{3,6,\ldots,2n+1\}$.

$$f(v_i^j) = \begin{cases} 3i-2, & j=1 \text{ and } i \in A_1; j=2 \text{ and } i \in A_3; j=3 \text{ and } i \in A_2 \\ 3i-1, & j=1 \text{ and } i \in A_2; j=2 \text{ and } i \in A_1; j=3 \text{ and } i \in A_3 \\ 3i, & j=1 \text{ and } i \in A_3; j=2 \text{ and } i \in A_2; j=3 \text{ and } i \in A_1 \end{cases}$$

Case II: $n \equiv 2 \pmod 3$
Let $B_1 = \{1,4,\ldots,2n\}, B_2 = \{2,5,\ldots,2n-2\}$ and $B_3 = \{3,6,\ldots,2n-1\}$.

$$f(v_i^j) = \begin{cases} 3i-2, & j=1 \text{ and } i \in B_1; j=2 \text{ and } i \in B_3; j=3 \text{ and } i \in B_2 \bigcup\{2n+1\} \\ 3i-1, & j=1 \text{ and } i \in B_2; j=2 \text{ and } i \in B_1 \bigcup\{2n+1\}; j=3 \text{ and } i \in B_3 \\ 3i, & j=1 \text{ and } i \in B_3 \bigcup\{2n+1\}; j=2 \text{ and } i \in B_2; j=3 \text{ and } i \in B_1 \end{cases}$$

Case III: $n \equiv 0 \pmod 3$
Let $C_1 = \{1,4,\ldots,2n-2\}, C_2 = \{2,5,\ldots,2n-4\}$ and $C_3 = \{3,6,\ldots,2n-3\}$.

$$f(v_i^j) = \begin{cases} 3i-2, & j=1 \text{ and } i \in C_1 \bigcup\{2n\}; j=2 \text{ and } i \in C_3; j=3 \text{ and } i \in C_2 \bigcup\{2n-1,2n+1\} \\ 3i-1, & j=1 \text{ and } i \in C_2; j=2 \text{ and } i \in C_1 \bigcup\{2n-1,2n,2n+1\}; j=3 \text{ and } i \in C_3 \\ 3i, & j=1 \text{ and } i \in C_3 \bigcup\{2n-1,2n+1\}; j=2 \text{ and } i \in C_2; j=3 \text{ and } i \in C_1 \bigcup\{2n\}. \end{cases}$$

Given the above definition of f, it may be verified that depending on the values of n and j, there exist constants $a_{n,j}$ and $b_{n,j}$ so that $2a_{n,j} + b_{n,j} = 2n+1$ and

$$\sum_{i=1}^{2n+1} f(v_i^j) = 3\sum_{i=1}^{2n+1} i - 2a_{n,j} - b_{n,j}.$$

This means that

$$\sum_{i=1}^{2n+1} f(v_i^j) = 3(2n+1)(n+1) - (2n+1) = 6n^2 + 7n + 2.$$

Consequently

$$w(v_i^j) = 6n^2 + 7n + 2 - f(v_i^j), \text{ for all } i \text{ and } j.$$

Since $f(v_i^j)$ takes each and every value in the set $\{1, 2, \ldots 6n+3\}$, it follows that the weights are in the arithmetic progression with first term as $a = 6n^2 + n - 1$ and the common difference $d = 1$.
We include the negative result about the graph $2K_{2n+1}$ in the next section which deals with graphs that are not (a, d)-distance antimagic.

4. Graphs that are not (a, d)-distance antimagic

Theorem 4.1. *The graph $2K_{2n+1}$ is not (a, d)-distance antimagic for all n.*

Proof. If possible, suppose $2K_{2n+1}$ is (a, d)-distance antimagic graph for some a, d and suppose the associated antimagic labeling is mapping f. Let $\{v_1^1, v_2^1, v_3^1, \ldots, v_{2n+1}^1\}$ and $\{v_1^2, v_2^2, v_3^2, \ldots, v_{2n+1}^2\}$ be the sets of vertices of the 1^{st} and the 2^{nd} copy of K_{2n+1} respectively. Then there exists a pair of distinct vertices $v_{i_0}^1$ and $v_{k_0}^1$ such that

$$|w(v_{i_0}^1) - w(v_{k_0}^1)| \geq (2n)d.$$

But $v_{i_0}^1$ and $v_{k_0}^1$ being the distinct vertices of the same copy of a complete graph K_{2n+1}, we have

$$|f(v_{i_0}^1) - f(v_{k_0}^1)| = |w(v_{i_0}^1) - w(v_{k_0}^1)| \geq 2nd.$$

Thus, if $d \geq 3$, then we get $|f(v_{i_0}^1) - f(v_{k_0}^1)| \geq 6n$ which is not possible because the range set of f is $\{1, 2, \ldots, 4n+2\}$. Next we show that the choice $d = 2$ or $d = 1$ also leads to a contradiction.

Case I: $d = 2$
If $d = 2$, then the weights are either all even or all odd. Therefore $|f(v_i^j) - f(v_k^j)| = |w(v_i^j) - w(v_k^j)|$ implies that $|f(v_i^j) - f(v_k^j)|$ is always an even number for all i, k and $j = 1, 2$. Thus, without loss of generality, if we

assume that $f(v_1^1)$ is odd then it follows that $f(v_i^1)$ is odd for every i and as a result $f(v_i^2)$ is even for every i. Hence

$$\sum_{i=1}^{2n+1} f(v_i^1) = 1+3+5+\cdots+(4n+1) = (2n+1)^2,$$

whereas

$$\begin{aligned}\sum_{k=1}^{2n+1} f(v_k^2) &= 2+4+6+\cdots+(4n+2)\\ &= (2n+1)(2n+2)\\ &= (2n+1)^2+(2n+1).\end{aligned}$$

This implies that for the vertices $v_{i_0}^1$ and $v_{k_0}^2$ for which $f(v_{i_0}^1) = 1$ and $f(v_{k_0}^2) = 2n+2$, we shall have

$$w(v_{i_0}^1) = w(v_{k_0}^2) = (2n+1)^2 - 1.$$

But this is not possible because the weights are in arithmetic progression with a positive common difference.

Case II: $d = 1$

Here

$$\begin{aligned}\sum_{j=1}^{2}\sum_{i=1}^{2n+1} w(v_i^j) &= a+(a+1)+(a+2)+\cdots+(a+(4n+1))\\ &= (4n+2)\left(a+\tfrac{(4n+1)}{2}\right).\end{aligned} \tag{4.1}$$

Moreover every vertex of the graph $2K_{2n+1}$ is adjacent to exactly $2n$ vertices and so

$$\sum_{j=1}^{2}\sum_{i=1}^{2n+1} w(v_i^j) = 2n(1+2+3+\cdot+(4n+2)) = n(4n+2)(4n+3). \tag{4.2}$$

Therefore in view of (4.1) and (4.2)

$$\begin{aligned}a &= n(4n+3) - \tfrac{(4n+1)}{2}\\ &= \tfrac{2n(4n+1)-1}{2}\end{aligned}$$

which is not possible as a is an integer.

Next we derive certain necessary conditions for the existence of (a,d)-distance antimagic labeling of a graph. These conditions are often useful in proving that certain graphs do not admit (a,d)-distance antimagic labeling.

Lemma 4.2. *If G is an (a,d)-distance antimagic graph of order N and if G has $k(>1)$ vertices of degree m, then $d \leq \frac{m(2N-m+1)-\delta(\delta+1)}{2(k-1)}$.*

Proof. Let G be an (a,d)-distance antimagic graph of order N for some a and d. Then there exists a bijection $f : V(G) \to \{1,2,\ldots,N\}$ such that the set of all vertex weights is $\{a, a+d, \ldots, a+(N-1)d\}$. Let $\{u_1, u_2, \ldots, u_k\}$, $1 \leq k \leq N$ be a set of vertices of G of degree m. Then for $1 \leq i \leq k$,

$$\begin{aligned} w(u_i) &\leq N + (N-1) + \cdots + (N-(m-1)). \\ &= \tfrac{m}{2}(2N-m+1). \end{aligned} \tag{4.3}$$

Also in view of pigeonhole principle,

$$max\{w(u_i) : 1 \leq i \leq k\} \geq a + (k-1)d. \tag{4.4}$$

In view of (4.3) and (4.4),

$$a + (k-1)d \leq \frac{m}{2}(2N-m+1).$$

Also, $a \geq 1 + 2 + \cdots + \delta = \frac{\delta(\delta+1)}{2}$. So it follows that

$$\frac{\delta(\delta+1)}{2} + (k-1)d \leq \frac{m}{2}(2N-m+1).$$

Consequently

$$d \leq \frac{m(2N-m+1)-\delta(\delta+1)}{2(k-1)}.$$

In particular, if $m = \delta$ then we get the following corollary which will be used quite often in the subsequent results.

Corollary 4.3. *If G is an (a,d)-distance antimagic graph of order N and the number of vertices of degree δ is $k(>1)$, then $d \leq \frac{\delta(N-\delta)}{k-1}$.*

Lemma 4.4. *If G is an (a,d)-distance antimagic graph of order N and if G has k vertices of degree Δ, then $a \geq \frac{\Delta(\Delta+1)}{2} - (N-k)d$*

Proof. Let G be an (a,d)-distance antimagic graph of order N for some a and d. Then there exists a bijection $f : V(G) \to \{1,2,\ldots,N\}$ such that the set of all vertex weights is $\{a, a+d, \ldots, a+(N-1)d\}$. Let $\{u_1, u_2, \ldots, u_k\}$, $1 \le k \le N$ be a set of vertices of G of degree Δ. Then for $1 \le i \le k$,

$$\begin{aligned} w(u_i) &\ge 1+2+\cdots+\Delta. \\ &= \tfrac{\Delta(\Delta+1)}{2}. \end{aligned} \tag{4.5}$$

Now due to pigeonhole principle

$$min\{w(u_i) : 1 \le i \le k\} \le a + (N-k)d. \tag{4.6}$$

Therefore in view of (4.5) and (4.6), we get

$$a \ge \frac{\Delta(\Delta+1)}{2} - (N-k)d$$

The helm graph H_n is the graph obtained from the wheel graph $W_n = C_n + K_1$ by attaching the pendant edge at each vertex of the cycle C_n. Using the lemmas above, we prove that it is not (a,d)-distance antimagic.

Theorem 4.5. *The helm graph H_n is not (a,d)-distance antimagic for any n.*

Proof. Let u_0 be a center vertex, $\{u_1, u_2, \ldots, u_n\}$ be a set of consecutive vertices of cycle C_n and $\{u'_1, u'_2, \ldots, u'_n\}$ be a set of consecutive pendant vertices such that u_i and u'_i are adjacent for $1 \le i \le n$. Suppose H_n is (a,d)-distance antimagic graph for some a and d. Then there exists a bijection $f : V(H_n) \to \{1,2,\ldots,2n+1\}$ such that the set of all vertex weights is $\{a, a+d, \ldots, a+2nd\}$. Since for $1 \le i \le n$, $f(u_i) = w(u'_i) \in \{a, a+d, \ldots, a+2nd\}$, we see that

$$\begin{aligned} w(u_0) &= f(u_1)+f(u_2)+\cdots+f(u_n) \\ &\ge a+(a+d)+\cdots+(a+(n-1)d) \\ &= na + \tfrac{(n-1)n}{2}d. \end{aligned}$$

But $a + 2nd \ge w(u_0)$ and hence

$$a + 2nd \ge na + \frac{(n-1)n}{2}d. \tag{4.7}$$

Consequently

$$\begin{aligned} 0 &\geq (n-1)a + \left(\tfrac{(n-1)n}{2} - 2n\right) d \\ &= (n-1)a + \tfrac{n^2-5n}{2} d \\ &= (n-1)a + \tfrac{n(n-5)}{2} d \end{aligned}$$

which is not possible if $n \geq 5$ because a and d are positive integers. Thus H_n is not (a, d)-distance antimagic for $n \geq 5$. Now we prove the same when $n = 3$ and $n = 4$.

It follows from Corollary 4.3 that

$$\begin{aligned} d &\leq \tfrac{\delta(N-\delta)}{k-1} \\ &= \tfrac{1((2n+1)-1)}{n-1} \\ &= \tfrac{2n}{n-1} \\ &= 2 + \tfrac{2}{n-1}. \end{aligned}$$

So $d \leq 3$ for $n = 3$ and $d \leq 2$ for $n = 4$.

Case I: $n = 3$.

<u>Sub-case 1</u>: $d = 1$.
Observe that when $n = 3$, order of the graph H_n is 7, $\Delta = 4$ and k (i.e. the number of vertices of degree $\Delta = 4$) is equal to 3. Therefore in view of Lemma 4.4, we get $a \geq 6$. On the other hand when $n = 3$ and $d = 1$, equation (4.7) reduces to $2a \leq 3$ and so H_3 is not $(a, 1)$ distance antimagic.

<u>Sub-case 2</u>: $d = 2$.
In this case equation (4.7) reduces to $a \leq 3$. Also using Lemma 4.4, we get $a \geq 2$ and so the only possible values of a are 2 and 3. Now when $a = 2$, the set of all vertex weights is $\{2, 4, 6, 8, 10, 12, 14\}$. For $1 \leq i \leq 3$, since $\deg(u_i) = 4$,

$$w(u_i) \geq 1 + 2 + 3 + 4 = 10.$$

Therefore the set of vertex weights of vertices u_0, u_1', u_2' and u_3' is $\{2, 4, 6, 8\}$. But this is not possible because $w(u_0)=f(u_1) + f(u_2) + f(u_3)=w(u_1') + w(u_2') + w(u_3')$, where as none of the integers in the set $\{2, 4, 6, 8\}$ is the sum of other three integers. Further, if $a = 3$, the set of all vertex weights is $\{3, 5, 7, 9, 11, 13, 15\}$ and so a similar argument rules out this possibility also. Therefore H_3 is not $(a, 2)$ distance antimagic.

Sub-Case 3: $d = 3$.
In this case the set of all vertex weights is $\{a, a+3, \dots, a+18\}$ and also for $1 \leq i \leq 3$, $f(u_i) = w(u'_i) \in \{a, a+3, \dots, a+18\}$. Therefore

$$max\{f(u_i) : 1 \leq i \leq 3\} \geq a + 6.$$

But $f(u_i) \leq 7$, for $1 \leq i \leq 3$ and hence $7 \geq a + 6$, which gives only one possibility $a = 1$. Once again it may now be verified that for $1 \leq i \leq 3$, $w(u'_i) \in \{1, 4, 7\}$. But then $w(u_0) = w(u'_1) + w(u'_2) + w(u'_3) = 12$, which does not belong to the set of vertex weights. So H_3 is not $(a, 3)$ distance antimagic.

Case II: $n = 4$.

Sub-case 1: $d = 1$.
Here the set of all vertex weights is $\{a, a+1, \dots, a+8\}$. Since for $1 \leq i \leq 4$, $f(u_i) = w(u'_i) \in \{a, a+1, \dots, a+8\}$ and $w(u_0) = f(u_1) + f(u_2) + f(u_3) + f(u_4)$, we must have

$$a + (a+1) + (a+2) + (a+3) \leq a + 8.$$

Consequently $a \leq \frac{2}{3}$, which is not possible.

Sub-case 2: $d = 2$.
Here the set of all vertex weights is $\{a, a+2, \dots, a+16\}$ and so arguing as above, we must have

$$a + (a+2) + (a+4) + (a+6) \leq a + 16.$$

This gives $a \leq \frac{4}{3}$ and consequently $a = 1$. In this case the set of vertex weights is $\{1, 3, 5, 7, 9, 11, 13, 15, 17\}$. But then $w(u_0) = w(u'_1) + w(u'_2) + w(u'_3) + w(u'_4)$ is an even number which does not belong to the set of vertex weights. So H_4 is not $(a, 2)$ distance antimagic.
Our next result is about the book graph which is defined with the help of Cartesian product of graphs and so we introduce it here.

Definition 4.6. *The Cartesian product $G \times H$ of graphs G and H is a graph such that its vertex set is $V(G \times H) = \{(u, v) : u \in V(G), v \in V(H)\}$; and any two vertices (u, v) and (u', v') are adjacent in $G \times H$ if and only if either $u = u'$ and v is adjacent to v' in H, or $v = v'$ and u is adjacent to u' in G.*

With this definition of Cartesian product, we define the book graph B_n as the Cartesian product of the star graph S_n and the path P_2 and we show that it is not (a,d)-distance antimagic

Theorem 4.7. *The book graph $B_n = S_n \times P_2$ of order $2n+2$ is not (a,d)-distance antimagic for any n.*

Proof. Clearly B_1 is same as the cycle graph C_4, which is not (a,d)-distance antimagic as shown in [1]. So we assume that $n > 1$. Let $\{u_0, u_1, \ldots, u_n\}$ and $\{v_1, v_2\}$ be the sets of vertices of the star graph S_n and the path P_2 respectively, where u_0 is the vertex of the star graph S_n with degree n. Suppose B_n is (a,d)-distance antimagic graph for some a and d. Then there exists a bijection $f : V(B_n) \to \{1, 2, \ldots, 2n+2\}$ such that the set of all vertex weights is $\{a, a+d, \ldots, a+(2n+1)d\}$.
Now observe that $N((u_0,v_1)) \bigcup N((u_0,v_2)) = V(B_n)$ and $N((u_0,v_1)) \bigcap N((u_0,v_2)) = \emptyset$. So

$$\begin{aligned} w((u_0,v_1)) + w((u_0,v_2)) &= 1+2+3+\cdots+(2n+2) \\ &= 2n^2+5n+3. \end{aligned} \tag{4.8}$$

Also for B_n,

$$\begin{aligned} &(n+1)(1+2)+2(3+4+\cdots+(2n+2)) \\ \leq \sum_{i=0}^{n}\sum_{j=1}^{2} w((u_i,v_j)) &= (2n+2)a + \tfrac{(2n+1)(2n+2)d}{2} \\ &\leq (n+1)((2n+1)+(2n+2)) + 2(1+2+\cdots+2n). \end{aligned}$$

This implies that

$$4n^2+13n+3 \leq (2n+2)a + (2n^2+3n+1)d \leq 8n^2+9n+3. \tag{4.9}$$

Now in view of Corollary 4.3, we have

$$\begin{aligned} d &\leq \tfrac{\delta(N-\delta)}{k-1} \\ &= \tfrac{2((2n+2)-2)}{2n-1} \\ &= \tfrac{4n}{2n-1} \\ &= 2 + \tfrac{2}{2n-1}. \end{aligned}$$

Therefore

$$d \leq 2 \text{ for } n > 1. \tag{4.10}$$

Case I: $d = 1$.
Here (4.9) reduces to

$$4n^2 + 13n + 3 \leq (2n+2)a + (2n^2 + 3n + 1)(1) \leq 8n^2 + 9n + 3.$$

Since a is an integer

$$\left\lceil \frac{n^2 + 5n + 1}{n+1} \right\rceil \leq a \leq \left\lfloor \frac{3n^2 + 3n + 1}{n+1} \right\rfloor;$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$ and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. This implies that

$$n + 4 - \left\lfloor \frac{3}{n+1} \right\rfloor \leq a \leq 3n + \left\lfloor \frac{1}{n+1} \right\rfloor. \tag{4.11}$$

Now

$$\begin{aligned} w((u_0, v_1)) + w((u_0, v_2)) &\leq (a + 2nd) + (a + (2n+1)d) \\ &= 2a + (4n+1)d \\ &\leq 2(3n) + (4n+1)(1) \\ &= 10n + 1. \end{aligned}$$

But then due to (4.8), $2n^2 + 5n + 3 \leq 10n + 1$, which is not possible for $n \geq 3$. Thus B_n is not $(a, 1)$-distance antimagic for $n \geq 3$. Now we show that B_n is not $(a, 1)$-distance antimagic for $n = 2$. When $n = 2$, (4.8) implies that

$$w((u_0, v_1)) + w((u_0, v_2)) = 2(2)^2 + 5(2) + 3 = 21. \tag{4.12}$$

Hence we also have

$$f((u_0, v_2)) + f((u_1, v_1)) + f((u_2, v_1)) + f((u_0, v_1)) + f((u_1, v_2)) + f((u_2, v_2)) = 21. \tag{4.13}$$

Also by (4.11), $5 \leq a \leq 6$.

Sub-case 1: $a = 5$.

Now,

$$\begin{aligned} w((u_0,v_1))+w((u_0,v_2)) &\leq 2a+(4n+1)d \\ &= 2(5)+(4(2)+1)(1) \\ &= 19. \end{aligned}$$

But this is not possible due to (4.12).
Sub-case 2: $a = 6$.

$$\begin{aligned} \textstyle\sum_{i=0}^{2}\sum_{j=1}^{2} w((u_i,v_j)) &= 6a+15d \\ &= 6(6)+15(1) = 51. \end{aligned}$$

Therefore since $deg((u_0,v_j)) = 3$ for $j \in \{1,2\}$, $deg((u_i,v_j)) = 2$ for $i,j \in \{1,2\}$ and due to (4.13), we have

$$3(f((u_0,v_1))+f((u_0,v_2)))+2(21-(f((u_0,v_1))+f((u_0,v_2)))) = 51.$$

This gives $f((u_0,v_1))+f((u_0,v_2)) = 9$.
Also for $i \in \{1,2\}$, $w(u_i,v_1) = f(u_0,v_1)+f(u_i,v_2)$, $w(u_i,v_2) = f(u_0,v_2)+f(u_i,v_1)$. Therefore $w(u_i,v_j) \neq 9$, for $i,j \in \{1,2\}\}$. Hence either $w(u_0,v_1) = 9$ or $w(u_0,v_2) = 9$. Consequently in view of (4.12), we have either $w(u_0,v_1) = 12$ or $w(u_0,v_2) = 12$, which is not possible because in this case the set of vertex weights is $\{6,7,8,9,10,11\}$.

Case II: $d = 2$.
Here (4.9) reduces to

$$4n^2+13n+3 \leq (2n+2)a+(2n^2+3n+1)(2) \leq 8n^2+9n+3.$$

Since a is an integer, this gives

$$\left\lceil \frac{7n+1}{2n+2} \right\rceil \leq a \leq \left\lfloor \frac{4n^2+3n+1}{2n+2} \right\rfloor,$$

which is same as

$$3+\left\lceil \tfrac{n-5}{2n+2} \right\rceil \leq a \leq 2n-1+\left\lfloor \tfrac{n+3}{2n+2} \right\rfloor. \tag{4.14}$$

Now

$$\begin{aligned} w((u_0,v_1))+w((u_0,v_2)) &\leq (a+2nd)+(a+(2n+1)d) \\ &= 2a+(4n+1)d \\ &\leq 2(2n-1)+(4n+1)(2) \\ &= 12n. \end{aligned}$$

But then due to (4.8), $2n^2+5n+3 \leq 12n$, which is not possible for $n \geq 4$. Thus B_n is not $(a,2)$-distance antimagic for $n \geq 4$.

Sub-case 1: $n=2$.

In view of (4.14), $a=3$. So we have

$$\sum_{i=0}^{2}\sum_{j=1}^{2} w((u_i,v_j)) = 6a+15d$$

$$= 6(3)+15(2) = 48.$$

As $deg((u_0,v_j)) = 3$ for $j \in \{1,2\}$, $deg((u_i,v_j)) = 2$ for $i,j \in \{1,2\}$; in this case (4.13) gives

$$3(f((u_0,v_1))+f((u_0,v_2)))+2(21-(f((u_0,v_1))+f((u_0,v_2)))) = 48.$$

Therefore $f((u_0,v_1))+f((u_0,v_2)) = 6$ and so $f((u_0,v_1))$ and $f((u_0,v_2))$ are of same parity. Now since the set of vertex weights contains only odd numbers, $w((u_i,v_j))$ is odd for $i,j \in \{1,2\}$. Further for $i,j \in \{1,2\}$, $w((u_i,v_1)) = f((u_0,v_1))+f((u_i,v_2))$ and $w((u_i,v_2)) = f((u_0,v_2))+f((u_i,v_1))$. Now using the fact that $w((u_i,v_j))$ is odd for $i,j \in \{1,2\}$ and that $f((u_0,v_1))$ and $f((u_0,v_2))$ are of same parity, it follows that $f((u_i,v_j))$ are of same parity for $i,j \in \{1,2\}$. This is not possible because the range of f is $\{1,2,3,4,5,6\}$, which contains only three members with the same parity.

Sub-case 2: $n=3$.
In view of (4.8), we have

$$w((u_0,v_1))+w((u_0,v_2)) = 2(3)^2+5(3)+3 = 36. \tag{4.15}$$

Therefore

$$f((u_0,v_2))+\sum_{i=1}^{3} f((u_i,v_1))+f((u_0,v_1))+\sum_{i=1}^{3} f((u_i,v_2)) = 36. \tag{4.16}$$

Also due to (4.14), $a=3,4,5$.

But if $a=3$ or $a=4$ then we obtain

$$\begin{aligned} w((u_0,v_1))+w((u_0,v_2)) &\leq 2a+(4n+1)d \\ &\leq 2(4)+(4(3)+1)(2) \\ &= 34. \end{aligned}$$

Thich contradicts (4.15). Finally, if $a=5$, we have

$$\begin{aligned} \textstyle\sum_{i=0}^{3}\sum_{j=1}^{2} w((u_i,v_j)) &= 8a+28d \\ &= 8(5)+28(2)=96. \end{aligned}$$

As $deg((u_0,v_j))=4$ for $j\in\{1,2\}$, $deg((u_i,v_j))=2$ for $i\in\{1,2,3\}$,$j\in\{1,2\}$; in this case (4.16) gives

$$4(f((u_0,v_1))+f((u_0,v_2)))+2(36-(f((u_0,v_1))+f((u_0,v_2))))=96.$$

Therefore $f((u_0,v_1))+f((u_0,v_2))=12$ and so $f((u_0,v_1))$ and $f((u_0,v_2))$ are of same parity. Since $w((u_i,v_j))$ are odd for $i\in\{1,2,3\}$ and $j\in\{1,2\}$, once again it can be shown that $f((u_i,v_j))$ are of same parity for $i\in\{1,2,3\}$ and $j\in\{1,2\}$. But this is not possible because there are only four members in the range of f with the same parity.

The corona $G\odot H$ of two graphs G and H is formed from one copy of G and $|V(G)|$ (i.e. the cardinality of the vertex set of G) copies of H in which the i^{th} vertex of G is joined to every vertex in the i^{th} copy of H, where $1\leq i\leq |V(G)|$. Sometimes $G\odot H$ is also known as the corona product of graphs G and H. Our next result is interesting especially from the perspective that every complete graph is (a,d)-distance antimagic.

Theorem 4.8. *The corona $K_n\odot K_1$ is not (a,d)-distance antimagic for $n>1$.*

Proof. We note that when $n=2$, $K_n\odot K_1$ is the graph P_4. Arumugam and Kamatchi [1] proved that if either $a,d\geq 2$ or $a=1$, then P_4 is not (a,d)-distance antimagic. Also Nalliah [7] proved that P_4 is not $(a,1)$-distance antimagic. Hence P_4 is not (a,d)-distance antimagic for any a and d. Consequently, while proving the theorem we assume that $n\geq 3$. Now if $K_n\odot K_1$ is (a,d)-distance antimagic then by Corollary 4.3,

$$\begin{aligned} d &\leq \tfrac{\delta(N-\delta)}{k-1} \\ &= \tfrac{1(2n-1)}{n-1} \\ &= 2+\tfrac{1}{n-1} \\ &\leq 2. \end{aligned}$$

Let $\{u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_n'\}$ be the vertex set of the graph $K_n \odot K_1$ in which $deg(u_i) = n$ and $deg(u_i') = 1$. As $deg(u_i) = n$ and $deg(u_i') = 1$, it follows that $w(u_i) \geq 1 + 2 + \cdots + n$ and $w(u_i') \leq 2n$ for all i. Therefore for $i, j \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned} w(u_i) - w(u_j') &\geq (1 + 2 + \cdots + n) - 2n \\ &= \tfrac{n^2 - 3n}{2}. \end{aligned}$$

But it is easy to see that there exists at least one pair of vertices (u_{i_0}, u_{j_0}') such that $w(u_{i_0}) - w(u_{j_0}') = d$, and so we have

$$d \geq \frac{n^2 - 3n}{2}.$$

Since $d \leq 2$, this is not possible for $n \geq 5$. This proves that $K_n \odot K_1$ is not (a,d)-distance antimagic for $n \geq 5$. We now prove the same when $n = 3, 4$.

When $n = 4$, the set of vertex weights is $\{a, a+d, a+2d \ldots, a+7d\}$. Also $f(u_i) = w(u_i') \in \{a, a+d, a+2d \ldots, a+7d\}$, for $i \in \{1, 2, 3, 4\}$. Since $deg(u_i) = 4$ and $|N(u_i) \bigcap \{u_1, u_2, u_3, u_4\}| = 3$, for $i \in \{1, 2, 3, 4\}$, we have $w(u_{i_0}) \geq (a+d) + (a+2d) + (a+3d) + 1$, for some $i_0 \in \{1, 2, 3, 4\}$. Therefore $a + 7d \geq (a+d) + (a+2d) + (a+3d) + 1$. As $d \leq 2$, this inequality does not give any (positive) integer value of a and so $K_4 \odot K_1$ is not (a,d) distance antimagic. Finally, we show that $K_3 \odot K_1$ is not (a,d) distance antimagic by considering the following two cases on d.

Case I: $d = 1$.
Here the set of vertex weights is $\{a, a+1, \ldots, a+5\}$ and for $i \in \{1, 2, 3\}$, $f(u_i) = w(u_i') \in \{a, a+1, \ldots, a+5\}$. Since $deg(u_i) = 3$ and $|N(u_i) \bigcap \{u_1, u_2, u_3\}| = 2$ for $i \in \{1, 2, 3\}$, we have $w(u_{i_0}) \geq (a+1) + (a+2) + 1$ for some $i_0 \in \{1, 2, 3\}$. Therefore $a + 5 \geq (a+1) + (a+2) + 1$ which implies that $a = 1$ and hence $w(u_i) \leq 6$, for $i \in \{1, 2, 3\}$. But this is not possible because $\max\{w(u_i) : i \in \{1, 2, 3\}\} \geq (1 + 2 + 3) + 2 = 8$.

Case II: $d = 2$.
In this case the set of vertex weights is $\{a, a+2, \ldots, a+10\}$ and so arguing as in the case $d = 1$, we first derive $a \leq 3$. Now consider the following sub-cases.

Sub-case 1: $a = 1$.
Here the set of vertex weights is $\{1, 3, 5, 7, 9, 11\}$. Since for $i \in \{1, 2, 3\}$, $f(u_i) = w(u_i') \in \{1, 3, 5, 7, 9, 11\}$ and also $f(u_i) \leq 6$, we have $1, 3, 5 \in \{f(u_i) : i \in \{1, 2, 3\}\}$. Therefore $2, 4, 6 \in \{f(u_i') : i \in \{1, 2, 3\}\}$. But then $w(u_i)$ will be even for $i \in \{1, 2, 3\}$, which is not possible.

Sub-case 2: $a = 2$.
Here the set of vertex weights is $\{2, 4, 6, 8, 10, 12\}$ and so a similar argument as above rules out this possibility.

Sub-case 3: $a = 3$.
In this case for $i \in \{1, 2, 3\}$, $f(u_i) = w(u_i') \in \{3, 5, 7, 9, 11, 13\}$ which is not feasible because the range set of f is $\{1, 2, 3, 4, 5, 6\}$.
Conclusion and Future scope: We have shown that the circulant graph $Circ(2n, \{1, n\})$ admits $(a, d)-$distance antimagic labeling for all even n. Interested readers may think of this problem for odd n. The circulant graphs $Circ(n, S)$, for more general sets S are wide open for investigation of $(a, d)-$distance antimagic labeling. Identifying the full set of values of m and n so that mK_{2n+1} admits $(a, d)-$distance antimagic labeling is another problem to look at. We have obtained upper bound on d and lower bound on a for the existence of $(a, d)-$distance antimagic labeling of a given graph. These bounds are mainly used here to show that certain graph families do not admit $(a, d)-$distance antimagic labeling. A similar investigation can be carried out for some other graph families.

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