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Some Results on Comultiplication Modules

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Abstract

Let M be a faithful multiplication and comultiplication module over a commutative ring R. In this paper we investigate some results on such modules.

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1 Introduction

Throughout this work, R denotes a commutative ring with identity and M denotes a unital R-module. We investigate here the question of those conditions under which a factor module M/N of an R-module M is comultiplication module as R-module and also as R/I-module for some ideal I of R, and some properties of comultiplication submodules, [Theorem 3.1]. For $N \leq M$, the set $[N :_R M] = \{r \in R \,|\, rM \leq N\}$ is called colon of N and it is an ideal of R. Let I be an ideal of R, the submodule $[N :_M I]$ of M is defined by $[N :_M I] = \{m \in M \,|\, Im \leq N\}$. In particular, if $I = (a_1, \dots , a_k)$ be a finitely generated ideal of R, then $[N :_M I] = \bigcap_{s=1}^{k} [N :_M a_s]$. We will obtain some results of faithful multiplication and comultiplication modules, [Theorem 3.3].

2 Preliminary Notes

An R-modules M is called multiplication if for every submodule N of M, there exists an ideal I of R such that $N = IM$. It is clear that every homomorphic

image of a multiplication module M is also multiplication. An R-module M is called cancellation module, if for ideals I and J of R, $IM = JM$ implies that $I = J$. Also M is faithful if $ann_R(M) = 0$. We note that $I \subseteq [N : M]$ and hence $N = IM \leq [N : M]M \leq N$, so $N = IM = [N : M]M$ and if M be a cancellation module, then $I = [N : M] = [IM : M]$, see [1], [3], [4].
A submodule N of M is said to be pure if $IN = N \cap IM$ for every ideal I of R. Moreover, N is said to be copure if $[N :_M I] = N + [0 :_M I]$ for every ideal I of R. The R-module M is said to be fully pure (resp. fully copure) if every submodule of M is pure (resp. copure).

Definition 2.1. *Let M is an R-module. A submodule N of M is called* comultiplication submodule *of M and we denote this concept by $N \leq_c M$, whenever there exists an ideal I of R such that $N = [0 :_M I] = ann_M(I)$.*
In particular, if all submodules of M be comultiplication submodules, then M is called a comultiplication module. *Also, M is a comultiplication module if and only if $N = [0 :_M ann_R(N)]$ for each submodule N of M.*

Example 2.2. *$\mathbb{Z}_4$ is a comultiplication $\mathbb{Z}$-module. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/4\mathbb{Z}$ and set $N = 2\mathbb{Z}/4\mathbb{Z}$. Then N and M/N are comultiplication $\mathbb{Z}$-modules.*

Remark 2.3. *Let M be a faithful multiplication R-module and $IM = JM$ for some ideals I and J of R, then $I - J \subseteq ann_R(M) = 0$, hence $I = J$, therefore M is cancellation module. Let $N \leq M$ and I be an ideal of R, if $Ir \in I[N :_R M]$ with $r \in [N :_R M]$, then $rM \leq N$. Therefore*
$(Ir)M = I(rM) \leq IN \Rightarrow Ir \in [IN :_R M] \Rightarrow I[N :_R M] \leq [IN :_R M]$
Conversely, $r \in [IN :_R M] \Rightarrow rM \in [IN :_R M]M \leq IN = I[N :_R M]M$, hence $[IN :_R M]M \leq I[N :_R M]M$. Since M is a cancellation module, hence $[IN :_R M] \leq I[N :_R M]$. It follows that $[IN :_R M] = I[N :_R M]$.

Theorem 2.4. *Every faithful multiplication module M is finitely generated.*
Proof: See ([2], Theorem 2.6).

It follows that if M be faithful multiplication module, then for every proper ideal I of R, $M \neq IM$.

3 Main Results

Theorem 3.1. *Let M be an R-module and $N \leq L \leq M$, then*
i) L/N is a comultiplication submodule of M/N if and only if there exists an ideal I of R such that $L = [N :_M I]$. If M be a semisimple R-module, then M is fully copure and $L/N \cong ann_M(I)/ann_N(I)$.
Moreover, if we consider M/N as an R/I-module, then M/N is comultiplication R/I-module if and only if for every submodule L of M such that

$N \leq L \leq M$, there exists an ideal $J \supseteq I$ of R such that $L = [N :_M J]$.
ii) if $K_1 \leq_c L$ and $K_2 \leq_c M$, then $K_1 \cap K_2 \leq_c L$.
iii) if $T_1 \leq_c L/N$ and $T_2 \leq_c M/N$, then $T_1 \cap T_2 \leq_c L/N$.
iv) if R be a Noetherian ring, then every comultiplication submodule N of M is a finite intersection of comultiplication submodules of M.
v) if $f : M \to M'$ be an isomorphism of R-modules and $K \leq_c M$, then we have $f(K) \leq_c M'$. Moreover, if $L \leq_c M'$, then $f^{-1}(L) \leq_c M$.

Proof: i) Since $IN \leq N$ for every ideal I of R, hence $N \leq L = [N :_M I]$. We consider M/N as an R-module. If $L/N \leq_c M/N$, then there exists an ideal I of R such that
$L/N = [N :_{M/N} I] = \{m + N \in M/N \,|\, I(m + N) = Im + N = N\}$
$= \{m + N \in \frac{M}{N} \,|\, Im \leq N\} = \{m + N \in \frac{M}{N} \,|\, m \in [N :_M I]\} = [N :_M I]/N$.
Therefore $L = [N :_M I]$. The converse is clearly true.
Moreover, if N be copure, then $[N :_M I] = N + [0 :_M I]$. Therefore

$$\frac{L}{N} = \frac{N + [0 :_M I]}{N} \cong \frac{[0 :_M I]}{N \cap [0 :_M I]} = \frac{[0 :_M I]}{[0 :_N I]} = \frac{ann_M(I)}{ann_N(I)}$$

In particular, if M be a semisimple R-module, then there exists $K \leq M$ such that $M = N \oplus K$. Therefore
$[N :_M I] = [N :_K I] + [N :_N I] = [0 :_K I] + N \leq [0 :_M I] + N$. Conversely, it is clear that $[0 :_M I] + N \leq [N :_M I]$, hence N is copure.
Similarly, if we consider M/N as an R/I-module, then $L/N \leq_c M/N$ iff for some ideal J/I of R/I we have $L/N = [0_{M/N} :_{M/N} J/I] = [N :_M J]/N$.
ii) Let $K_1 = [0 :_L I]$ and $K_2 = [0 :_M J]$ for some ideals I, J of R, then
$K_1 \cap K_2 = [0 :_L I] \cap [0 :_M J] = [0 :_L I + J] \Rightarrow K_1 \cap K_2 \leq_c L$.
iii) By (i), $T_1 = [N :_L I]$ and $T_2 = [N :_M J]$ for some ideals I, J of R, then
$T_1 \cap T_2 = [N :_L I] \cap [N :_M J] = [N :_L I + J] \Rightarrow T_1 \cap T_2 \leq_c L/N$.
iv) Since R is Noetherian ring, hence every ideal of R is f.g. say $I = \sum_{i=1}^n Ra_i$.
Let $N \leq_c M$, then there exists an ideal I of R such that $N = [0 :_M I]$, hence
$N = [0 :_M I] = [0 :_M \sum_{i=1}^n Ra_i] = \bigcap_{i=1}^n [0 :_M Ra_i] = \bigcap_{i=1}^n N_i$;
where $N_i = [0 :_M Ra_i] \leq_c M$.
Moreover, if N be a completely irreducible submodule of M, then there exists $1 \leq k \leq n$, such that $N = [0 :_M I] = N_k = [0 :_M Ra_k]$ or equivalently, $ann_M(I) = ann_M(Ra_k)$.
v) Let $K = [0 :_M I]$ for some ideal I of R, then

$$f(K) = f([0 :_M I]) = \{f(x) | Ix = 0\} \leq [0 :_{M'} I];$$

because $f(x) \in M'$ and $f(Ix) = If(x) = 0$, hence $f(x) \in [0 :_{M'} I]$.
Conversely, let $y \in [0 :_{M'} I]$, then there exists $x \in M$ such that $y = f(x)$ and $Iy = If(x) = f(Ix) = 0$. Since f is monomorphism, hence $Ix = 0$, then $x \in [0 :_M I]$ and $y = f(x) \in f(K)$. Therefore $f(K) = [0 :_{M'} I] \leq_c M'$.
Similarly, let $L = [0 :_{M'} I]$, then clearly $K = f^{-1}(L) = [0 :_M I] \leq_c M$.

Corollary 3.2. *Let M be an R-module and $N \leq L \leq M$. If $N \leq_c M$ and*

M/N be a comultiplication R-module, then $L \leq_c M$.

Proof: We suppose that $N = [0 :_M J]$. Since $L/N \leq_c M/N$ by (i), we have $L = [N :_M I]$ for some ideal I of R. Therefore $L = [[0 :_M J] :_M I] = [0 :_M IJ]$.

Theorem 3.3. *Let M be a faithful multiplication and comultiplication R-module, then the following assertions hold.*
i) for every submodule N of M, $[N :_R M]$ is a comultiplication ideal of R,
ii) R is comultiplication as an R-module.

Proof: Let $N \leq M$, then since M is comultiplication module, there exists an ideal I of R such that $N = [0 :_M I]$, then
$[N :_R M] = [[0 :_M I] :_R M] = [0 :_R IM] = ann_R(IM)$.
Since M is faithful multiplication R-module, then $ann_R(IM) = [0 :_R IM] = ann_R(I) = [0 :_R I]$. Therefore $[N :_R M] = [0 :_R I]$ is a comultiplication ideal.
ii) Let I be an ideal of R, then we set $N = IM \leq M$. Since M is faithful multiplication R-module, hence it is cancellation module and hence finitely generated. Therefore for every $N \leq M$ and every ideal I of R, $[IN :_R M] = I[N :_R M]$. In particular, $[N :_R M] = [IM :_R M] = I[M :_R M] = IR = I$. By (i), I is a comultiplication ideal.

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