AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **45** (2009), Pages 251–261

Some results on decompositions of low degree circulant graphs

Darryn Bryant Geoffrey Martin

The University of Queensland
Department of Mathematics
Qld 4072
Australia

Abstract

The circulant graph of order n with connection set S is denoted by $\mathrm{Circ}(n, S)$. Several results on decompositions of $\mathrm{Circ}(n, \{1, 2\})$ and $\mathrm{Circ}(n, \{1, 2, 3\})$ are proved here. The existence problems for decompositions into paths of arbitrary specified lengths and for decompositions into cycles of arbitrary specified lengths are completely solved for $\mathrm{Circ}(n, \{1, 2\})$. For all $m \geq 3$, we prove that $\mathrm{Circ}(n, \{1, 2, 3\})$ has an m-cycle decomposition if and only if the obvious necessary conditions are satisfied. We also prove that there exists a decomposition of $\mathrm{Circ}(n, \{1, 2, 3\})$ into t circuits (connected subgraphs in which each vertex has even degree) of sizes $m_1, m_2, \ldots, m_t$ if and only if each $m_i \geq 3$ and $m_1 + m_2 + \cdots + m_t = 3n$. This settles the problem of decomposing $\mathrm{Circ}(n, \{1, 2, 3\})$ into specified numbers of 3-cycles, 4-cycles and 5-cycles.

1 Introduction

A *decomposition* of a graph K is a set $\{G_1, G_2, \ldots, G_t\}$ of subgraphs of K such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_t) = E(K)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. Here we examine decompositions of some low degree circulant graphs into cycles, into paths, and into circuits. The *circulant graph* of order n with *connection set* $S \subseteq \mathbb{Z}_n \setminus \{0\}$ is denoted by $\mathrm{Circ}(n, S)$. It has vertex set $\mathbb{Z}_n$ and edge set given by joining x to $x + s$ for each $x \in \mathbb{Z}_n$ and each $s \in S$. We will assume $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and define the *length* of an edge $\{x, y\}$ to be the unique $s \in S$ such that $s = x - y$ or $s = y - x$ (working modulo n). Of course, circulant graphs are *Cayley graphs* on cyclic groups. We will be interested almost exclusively in decompositions of $\mathrm{Circ}(n, \{1, 2\})$ and $\mathrm{Circ}(n, \{1, 2, 3\})$.

Work on decompositions of circulant graphs has focused on decompositions into perfect matchings (decompositions into perfect matchings are 1-*factorisations*), or into Hamilton cycles. The graph $\mathrm{Circ}(n, S)$ has a 1-factorisation if and only if S has an element of even order [21]. In [2], Alspach asks whether every connected $2k$-regular Cayley graph on a finite abelian group has a decomposition into k Hamilton cycles. A lot of results have been obtained on this problem, see [8, 14, 15, 17, 18], but the general problem is unsolved, even in the case of circulant graphs. Further results on decompositions of circulant graphs into isomorphic subgraphs are obtained in [3].

Here we consider the existence of decompositions of circulant graphs into cycles of arbitrary specified lengths, focusing in particular on $\mathrm{Circ}(n, \{1, 2\})$ (see Theorem 5) and $\mathrm{Circ}(n, \{1, 2, 3\})$ (see Theorems 7 and 8). We also examine decompositions of $\mathrm{Circ}(n, \{1, 2\})$ into paths (see Theorem 2), and decompositions of $\mathrm{Circ}(n, \{1, 2, 3\})$ into circuits (see Theorem 1). A *circuit* is a connected graph in which each vertex has even degree. Our result on circuit decompositions of $\mathrm{Circ}(n, \{1, 2, 3\})$ settles a question posed by Billington and Cavenagh in [9]. They ask whether there exist infinitely many 6-regular graphs which are arbitrarily decomposable into closed trails; that is, can be decomposed into closed trails of specified sizes $m_1, m_2, \dots, m_t$ whenever $m_1 + m_2 + \cdots + m_t$ is the size of the graph in question. Theorem 1 says that $\{\mathrm{Circ}(n, \{1, 2, 3\}) : n \geq 7\}$ is one such infinite family.

2 Circuit and path decompositions

The following theorem says that $\mathrm{Circ}(n, \{1, 2, 3\})$ can be decomposed into circuits of arbitrary specified sizes $m_1, m_2, \dots, m_t$ whenever the obvious necessary numerical conditions are satisfied. A similar result has been proven for all sufficiently dense graphs by Balister in [7], and for various families of graphs in [6, 9, 16].

Theorem 1 *Let n and $m_1, m_2, \dots, m_t$ be integers with $n \geq 7$ and $m_i \geq 3$ for $i = 1, 2, \dots, t$. There exists a decomposition $\mathcal{D} = \{G_1, G_2, \dots, G_t\}$ of the circulant graph $\mathrm{Circ}(n, \{1, 2, 3\})$ where G_i is a circuit of size m_i for $i = 1, 2, \dots, t$ if and only if $m_1 + m_2 + \cdots + m_t = 3n$.*

Proof The conditions are clearly necessary for the existence of such a decomposition. To prove sufficiency, we will actually prove a slightly stronger result from which the theorem follows easily. For any $x \geq 1$ let T_x be the 3-cycle $(x, x+1, x+3)$ and define the graph J_n by $V(J_n) = \{1, 2, \dots, n+3\}$ and $E(J_n) = E(T_1) \cup E(T_2) \cup \cdots \cup E(T_n)$. Note that T_i and T_j are edge-disjoint for $i \neq j$. So J_n has $3n$ edges and for $n \geq 7$, one can obtain a graph isomorphic to $\mathrm{Circ}(n, \{1, 2, 3\})$ from J_n by identifying vertex i with vertex $n+i$ for $i = 1, 2, 3$. We will show that for any sequence $m_1, m_2, \dots, m_t$ satisfying $m_i \geq 3$ for $i = 1, 2, \dots, t$ and $m_1 + m_2 + \cdots + m_t = 3n$, there exists a decomposition $\mathcal{D} = \{G_1, G_2, \dots, G_t\}$ of J_n where G_i is a circuit of size m_i for $i = 1, 2, \dots, t$. Moreover, we will show that there is such a decomposition with the additional property that the vertex $n + 3$ is in G_1.

The proof is by induction on n. The result clearly holds for $n = 1$ and $n = 2$, so assume $n \geq 3$ and that the result holds for $J_1, J_2, \dots, J_{n-1}$. The proof splits into the following four cases.

(a) $m_1 = 3$.

(b) $m_1 = 4$.

(c) $m_1 = 5$.

(d) $m_1 \geq 6$.

(a) Take a decomposition of J_{n-1} into circuits of sizes $m_2, m_3, \ldots, m_t$ (which exists by the inductive assumption) and add the 3-cycle $T_n = (n, n+1, n+3)$ to obtain the required decomposition of J_n.

(b) First consider the case $m_1, m_2, \ldots, m_t \in \{3, 4\}$. Since J_n has $3n$ edges, the required number of 4-cycles is $3k$ for some $k \geq 1$. We obtain the required decomposition of J_n by combining a decomposition of J_{n-4} into $n - 4k$ cycles of length 3 and $3(k-1)$ cycles of length 4 (which exists by the inductive assumption) with the following decomposition of $T_{n-3} \cup T_{n-2} \cup T_{n-1} \cup T_n$ into three 4-cycles.

$$\{(n-3, n-2, n+1, n), (n-2, n-1, n+2, n), (n-1, n, n+3, n+1)\}$$

We can now assume $m_i \geq 5$ for some $i \in \{2, 3, \ldots, t\}$. Without loss of generality suppose $m_2 \geq 5$. By the inductive assumption we have a decomposition of J_{n-2} into $t-1$ circuits of sizes $m_2 - 2, m_3, m_4, \ldots, m_t$ where vertex $n+1$ is in a circuit of size $m_2 - 2$. If we take such a decomposition, replace the edge $\{n-1, n+1\}$ of the circuit of size $m_2 - 2$ with the path $[n-1, n+2, n, n+1]$, and add the 4-cycle $(n-1, n, n+3, n+1)$, then we obtain the required decomposition of J_n.

(c) Note that $m_1 = 5$ implies there is some $i \in \{2, 3, \ldots, t\}$ such that $m_i \geq 4$ (as $5 + 3 + 3 + \cdots + 3$ is not divisible by 3). Without loss of generality suppose $m_2 \geq 4$. By the inductive assumption we have a decomposition of J_{n-2} into $t-1$ circuits of sizes $m_2 - 1, m_3, m_4, \ldots, m_t$ where vertex $n+1$ is in a circuit of size $m_2 - 1$. If we take such a decomposition, replace the edge $\{n-1, n+1\}$ of the circuit of size $m_2 - 1$ with the path $[n-1, n, n+1]$, and add the 5-cycle $(n-1, n+1, n+3, n, n+2)$, then we obtain the required decomposition of J_n.

(d) By the inductive assumption we have a decomposition of J_{n-1} into t circuits of sizes $m_1 - 3, m_2, m_3, \ldots, m_t$ where vertex $n+2$, and hence also vertex n, is in a circuit of size $m_1 - 3$. If we take such a decomposition and add the three edges of the 3-cycle $(n, n+1, n+3)$ to the circuit of size $m_1 - 3$, then we obtain the required decomposition of J_n. □

We now consider the problem of decomposing $\text{Circ}(n, \{1, 2\})$ into paths of specified lengths $m_1, m_2, \ldots, m_t$. The following theorem completely settles this problem. Strong results on the path decomposition problem for complete graphs were proven by Tarsi [22].

Theorem 2 *Let n and $m_1, m_2, \ldots, m_t$ be integers with $n \geq 5$. There exists a decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_t\}$ of the circulant graph $\text{Circ}(n, \{1, 2\})$ where G_i is a path with m_i edges for $i = 1, 2, \ldots, t$ if and only if $m_1 + m_2 + \cdots + m_t = 2n$ and $m_i \leq n - 1$ for $i = 1, 2, \ldots, t$.*

Proof The conditions are clearly necessary. We now prove that they are also sufficient. Without loss of generality we can assume $m_1 \leq m_2 \leq \cdots \leq m_t$. Moreover, we can assume $m_1+m_2 \geq n$ (as we can obtain any decomposition with $m_1+m_2 \leq n-1$ from a decomposition into $t-1$ paths of lengths $m_1+m_2, m_3, m_4, \ldots, m_t$). It is easy to see that the conditions imply $t \geq 3$.

First suppose $t = 3$. Let G_3 be the path with vertices $0, 1, \ldots, m_3$ and edges $\{\{0,1\}\} \cup \{\{i, i+2\} : i = 0, 1, \ldots, m_3 - 2\}$. Let G_2 be the path with vertices $\{0, n-1, n-2, \ldots, n-m_2\}$ and edges $\{\{0, n-1\}\} \cup \{\{i, i-2\} : i = 0, n-1, n-2, \ldots, m_3+1\} \cup \{\{i, i-1\} : i = m_3-1, m_3-2, \ldots, n-m_2+1\}$. Let G_1 be the path with edges $E(\mathrm{Circ}(n, \{1,2\})) \setminus (E(G_3) \cup E(G_2))$. So G_1 has vertices $1, 2, \ldots, n-m_2$ and $n-1, n-2, \ldots, m_3-1$ and edges $\{\{1, n-1\}\} \cup \{\{i, i+1\} : i = 1, 2, \ldots, n-m_2-1\} \cup \{\{i, i-1\} : i = n-1, n-2, \ldots, m_3\}$. It is straightforward to check that $\{G_1, G_2, G_3\}$ is the required path decomposition of $\mathrm{Circ}(n, \{1,2\})$.

Now suppose $t \geq 4$. Since we can assume $m_1 + m_2 \geq n$ and $m_1 \leq m_2 \leq \cdots \leq m_t$, we have $m_1 + m_2 + m_3 + m_4 \geq 2n$. But $m_1 + m_2 + \cdots + m_t = 2n$ and thus it follows that $t = 4$, n is even, and $m_1 = m_2 = m_3 = m_4 = \frac{n}{2}$. The required decomposition is given by

- $G_1 = [0, 2, 1, 3, 4, 5, 6, \ldots, \frac{n}{2}]$;
- $G_2 = [\frac{n}{2}, \frac{n}{2}+1, \ldots, n-1, 0]$;
- $G_3 = [2, 4, 6, \ldots, n-2, 0, 1]$;
- $G_4 = [2, 3, 5, 7, \ldots, n-1, 1]$.

□

3 Cycle decompositions

In this section we examine cycle decompositions of $\mathrm{Circ}(n, \{1,2\})$ and $\mathrm{Circ}(n, \{1, 2, 3\})$. If $m_1, m_2, \ldots, m_t$ is a list of cycle lengths (possibly containing repeated elements), then an $(m_1, m_2, \ldots, m_t)$-*cycle decomposition* is a decomposition $\{G_1, G_2, \ldots, G_t\}$ where G_i is an m_i-cycle for $i = 1, 2, \ldots, t$. Obvious necessary conditions for the existence of an $(m_1, m_2, \ldots, m_t)$-cycle decomposition of a graph K are

- $3 \leq m_i \leq |V(K)|$ for $i = 1, 2, \ldots, t$,
- each vertex of K has even degree, and
- $m_1 + m_2 + \cdots + m_t = |E(K)|$.

We say that a list $m_1, m_2, \ldots, m_t$ is *admissible* for a graph K if these three conditions are satisfied. The *cycle decomposition problem* for a graph K (or a family $\mathcal{K}$ of graphs) involves proving the existence or otherwise of an (M)-cycle decomposition of K (or of K for each $K \in \mathcal{K}$) for each admissible list M.

In 1981 Alspach [1] conjectured that for any admissible list $m_1, m_2, \ldots, m_t$, an $(m_1, m_2, \ldots, m_t)$-cycle decomposition of K_n (the complete graph) or of $K_n - I$ (the complete graph of even order with the edges of a perfect matching removed) exists.

Numerous results have been obtained on this conjecture but it remains unsolved. The conjecture has been proven by Alspach, Gavlas and Šajna for the case where all the cycles are of uniform length [4, 19, 20], and by Balister for cases where n is sufficiently large and the longest cycle has length at most about $n/20$ [5]. A recent result [11] proves the existence of about 10% of all admissible cycle decompositions of K_n. See [10] for a survey on Alspach's cycle decomposition problem, and [12] for a survey of cycle decompositions generally.

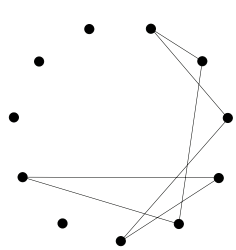
In Theorem 5, we give a complete solution to the cycle decomposition problem in the case of $\mathrm{Circ}(n, \{1, 2\})$. For $\mathrm{Circ}(n, \{1, 2, 3\})$, we settle the problem for decompositions into cycles of uniform length m in Theorem 7, and for decompositions into cycles of length at most 5 in Theorem 8.

We shall see that there are numerous admissible lists $m_1, m_2, \ldots, m_t$ for which $(m_1, m_2, \ldots, m_t)$-cycle decompositions of $\mathrm{Circ}(n, \{1, 2\})$ do not exist. However, we have found no such lists for $\mathrm{Circ}(n, \{1, 2, 3\})$ and we thus pose the following problem.

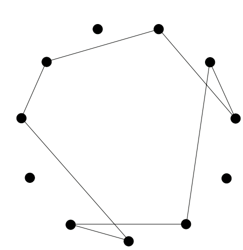
Problem 3 *Let $n \geq 7$. Does every admissible cycle decomposition of the circulant graph* $\mathrm{Circ}(n, \{1, 2, 3\})$ *exist* ?

One might ask whether an $(m_1, m_2, \ldots, m_t)$-cycle decomposition of the graph $\mathrm{Circ}(n, \{1, 2, \ldots, k\})$ exists for each admissible list $m_1, m_2, \ldots, m_t$ whenever $k \geq 3$ and $n \geq 2k + 1$. The following result answers this question in the negative. For example, it shows that there is no 3-cycle decomposition of $\mathrm{Circ}(n, \{1, 2, 3, 4, 5, 6\})$ for $n \geq 19$.

We now give a few definitions that will be used later. Let $C = (v_1, v_2, \ldots, v_m)$ be a cycle in the graph $\mathrm{Circ}(n, \{d_1, d_2, \ldots, d_k\})$, and for $i = 1, 2, \ldots, m$, let e_i be the integer in the set $\{-d_1, -d_2, \ldots, -d_k, d_1, d_2, \ldots, d_k\}$ such that $e_i \equiv v_{i+1} - v_i \pmod{n}$ for $i = 1, 2, \ldots, m - 1$, and $e_m \equiv v_1 - v_m \pmod{n}$. Then clearly we have $e_1 + e_2 + \cdots + e_m \equiv 0 \pmod{n}$. If we have $e_1 + e_2 + \cdots + e_m = 0$ (in $\mathbb{Z}$ not just in $\mathbb{Z}_n$), then we call C a *short cycle*, and otherwise we call C a *long cycle*. The figure below shows a short cycle and a long cycle in a circulant graph.



A "short" cycle.

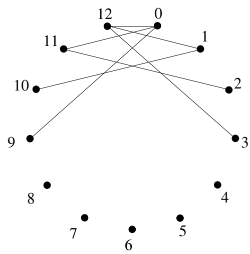


A "long" cycle.

Consider the set E of edges of $\mathrm{Circ}(n, \{d_1, d_2, \ldots, d_k\})$ defined by

$$E = \{\{-d_i, 0\}, \{-d_i + 1, 1\}, \ldots, \{-1, d_i - 1\} : i = 1, 2, \ldots k\}$$

and note that $|E| = d_1 + d_2 + \cdots + d_k$. Informally, E contains the edges crossing an imaginary line between 0 and $n-1$. The set E for the graph $\mathrm{Circ}(13, \{1, 2, 4\})$ is shown in the figure below.



Proposition 4 *Let $d_1, d_2, \ldots, d_k$ be positive integers such that $d_1 < d_2 < \cdots < d_k$ and $d_1 + d_2 + \cdots + d_k$ is odd, and let $n \geq 2d_k + 1$. Then any cycle decomposition of $\mathrm{Circ}(n, \{d_1, d_2, \ldots, d_k\})$ contains a cycle of length at least $\lceil \frac{n}{d_k} \rceil$.*

Proof We shall show that under the given conditions, any cycle decomposition of the graph $\mathrm{Circ}(n, \{d_1, d_2, \ldots, d_k\})$ contains a long cycle. It is clear that any long cycle has length at least $\lceil \frac{n}{d_k} \rceil$ and the result will thus follow.

Since the edges of a short cycle must cross the imaginary line between 0 and $n-1$ an even number of times, the number of edges of E in each short cycle in the graph $\mathrm{Circ}(n, \{d_1, d_2, \ldots, d_k\})$ is even. Thus, if $|E|$ is odd then there is at least one long cycle in any cycle decomposition of $\mathrm{Circ}(n, \{d_1, d_2, \ldots, d_k\})$. Since $|E| = d_1 + d_2 + \cdots + d_k$, the result follows. □

Theorem 5 *Let n be an integer with $n \geq 5$ and let $m_1, m_2, \ldots, m_t$ be a sequence of integers with $m_i \geq 3$ for $i = 1, 2, \ldots, t$. There exists an $(m_1, m_2, \ldots, m_t)$-cycle decomposition of $\mathrm{Circ}(n, \{1, 2\})$ if and only if each of the following conditions hold.*

(1) $m_1 + m_2 + \cdots + m_t = 2n$.

(2) $m_i \leq n$ *for* $i = 1, 2, \ldots, t$.

(3) Either

(i) $t = 3$ *and* $\frac{n}{2} \leq m_1, m_2, m_3 \leq n$, *or*

(ii) there exists a $k \in \{1, 2, \ldots, t\}$ *such that* $m_k \geq n - t + 1$.

Proof It is clear that Conditions (1) and (2) are necessary. We will now prove the necessity of Condition (3). In this proof we shall use the definitions of *short* and *long* cycles given above. It follows from Proposition 4 that any cycle decomposition of $\mathrm{Circ}(n, \{1, 2\})$ contains a long cycle. We now show that if a cycle decomposition of $\mathrm{Circ}(n, \{1, 2\})$ contains a short cycle, then it contains at most one long cycle. To see this, observe that any short cycle, of length m say, has vertices $x+1, x+2, \ldots, x+m$

and edges $\{x+1,x+2\}$, $\{x+m-1,x+m\}$ and $\{i,i+2\}$ for $i=x+1,x+2,\ldots,x+m-2$ where x is some element of $\mathbb{Z}_n$. In particular, this short cycle contains the edges $\{x,x+1\}$ and $\{x,x+2\}$. But then it is clear that any long cycle must contain the edge $\{x-1,x+1\}$. Hence either the decomposition contains only long cycles, or it contains exactly one long cycle.

Suppose first that we have only long cycles. Then, since the length of a long cycle is at least $\frac{n}{2}$, we have $t \leq 4$. Clearly, $t=1$ is not possible. Also, $t=4$ is not possible as $t=4$ implies $m_1=m_2=m_3=m_4=\frac{n}{2}$ and any long cycle of length $\frac{n}{2}$ contains only edges of length 2. Thus we have $t=2$ or $t=3$. If $t=2$ then $m_1=m_2=n$ and Condition 3(ii) is satisfied. If $t=3$ then Condition (3)(i) is satisfied.

Now suppose that we have exactly one long cycle and $t-1$ short cycles. It follows from the arguments concerning the structure of short cycles given in the first paragraph of the proof, that any short cycle has at most two vertices in common with other short cycles. Thus, the number of vertices which occur in short cycles is at least $\Sigma-(t-1)$ where Σ is the sum of the lengths of the short cycles. Since this number is at most n, we have $\Sigma \leq n+t-1$. From this it follows that the long cycle has length at least $2n-(n+t-1)=n-t+1$. Thus Condition (3)(ii) is satisfied. We have shown that Condition (3) is necessary.

We now prove the sufficiency of Conditions (1)–(3). Suppose first that Conditions (1), (2) and (3)(i) are satisfied. For $\frac{n}{2} \leq m \leq n$ and $x \in \mathbb{Z}_n$, define the m-cycle $C(x,m)$ to be the cycle containing $2m-n$ consecutive edges of length 1 starting at x followed by $n-m$ consecutive edges of length 2. That is, $C(x,m)$ contains the edges

- $\{i,i+1\}$ for $i=x,x+1,x+2,\ldots,x+2m-n-1$; and
- $\{i,i+2\}$ for $i=x+2m-n,x+2m-n+2,x+2m-n+4,\ldots,x-2$.

If n is odd, then the (m_1,m_2,m_3)-decomposition of $\mathrm{Circ}(n,\{1,2\})$ is

$$\{C(0,m_1),C(2m_1-n,m_2),C(2m_1+2m_2-2n,m_3)\}.$$

If n is even, then the (m_1,m_2,m_3)-decomposition of $\mathrm{Circ}(n,\{1,2\})$ is

$$\{C(0,m_1),C(2m_1-n+1,m_2),C\}$$

where C is the cycle containing the edges

- $\{2m_1-n,2m_1-n+1\}$,
- $\{i,i+2\}$ for $i=2m_1-n+1,2m_1-n+3,\ldots,2m_1+2m_2-2n-1$,
- $\{i,i+1\}$ for $i=2m_1+2m_2-2n+1,2m_1+2m_2-2n+2,\ldots,n-1$, and
- $\{i,i+2\}$ for $i=0,2,\ldots,2m_1-n-2$.

Now suppose that Conditions (1), (2) and (3)(ii) are satisfied. In this case our decomposition will have exactly one long cycle, and this long cycle will have length

m_k. Without loss of generality assume $k = t$. Define g by $g = m_t - n + t - 1$, and for $3 \leq m \leq n$ define $I(x, m)$ by $I(x, m) = \{x, x+1, x+2, \ldots, x+m-1\}$ (working modulo n) for each $x \in \mathbb{Z}_n$. Since $n - t + 1 \leq m_t \leq n$ it follows immediately that $0 \leq g \leq t - 1$. If $g = 0$ then define subsets $S_1, S_2, \ldots, S_{t-1}$ of $\mathbb{Z}_n$ by $S_1 = I(0, m_1)$ and $S_{i+1} = I(\max(S_i), m_{i+1})$ for $i = 1, 2, \ldots, t-2$. Otherwise, $g \geq 1$ and we define subsets $S_1, S_2, \ldots, S_{t-1}$ of $\mathbb{Z}_n$ by $S_1 = I(0, m_1)$, $S_{i+1} = I(\max(S_i) + 1, m_{i+1})$ for $i = 1, 2, \ldots, g$, and $S_{i+1} = I(\max(S_i), m_{i+1})$ for $i = g+1, g+2, \ldots, t-2$. It is straightforward to check that $S_1 \cup S_2 \cup \cdots \cup S_{t-1} = \mathbb{Z}_n$.

Now recall from the first paragraph of the proof that there is a unique short cycle on the vertices of $I(x, m)$. Thus we have a unique short cycle (of length m_i) on the vertices of S_i for $i = 1, 2, \ldots, t-1$. The edges not contained in these short cycles form a long cycle, C say, of length m_t. In detail, if we define, for $i = 1, 2, \ldots, t-1$, P_i to be the path

$$\min(S_i) + 1, \min(S_i) + 2, \ldots, \max(S_i) - 1, \min(S_{i+1}) + 1$$

if $\min(S_{i+1}) = \max(S_i)$ and to be the path

$$\min(S_i) + 1, \min(S_i) + 2, \ldots, \max(S_i) - 1, \min(S_{i+1}), \max(S_i), \min(S_{i+1}) + 1$$

if $\min(S_{i+1}) = \max(S_i) + 1$, then C is the cycle $P_1 \cup P_2 \cup \cdots \cup P_{t-1}$. □

To prove the next theorem we use the following lemma from [13].

Lemma 6 [13] *If $n \geq 7$ and F is a 2-regular graph of order n with no 3-cycles then there is a 2-factorisation of $\mathrm{Circ}(n, \{1, 2, 3\})$ in which each 2-factor is isomorphic to F.*

Theorem 7 *Let $n \geq 7$ and $m \geq 3$. There exists an m-cycle decomposition of $\mathrm{Circ}(n, \{1, 2, 3\})$ if and only if $m \leq n$ and m divides $3n$.*

Proof The conditions are clearly necessary for existence. For all $n \geq 7$, a 3-cycle decomposition of $\mathrm{Circ}(n, \{1, 2, 3\})$ is given by $\{(i, i+1, i+3) : i \in \mathbb{Z}_n\}$. Thus we assume $m \geq 4$. By Lemma 6, there is an m-cycle decomposition of $\mathrm{Circ}(mx, \{1, 2, 3\})$ whenever $m \geq 4$, $x \geq 1$ and $mx \geq 7$. Thus we may assume that m does not divide n. This implies that 3 divides m and $\frac{m}{3}$ divides n. The proof that there is an m-cycle decomposition of $\mathrm{Circ}(n, \{1, 2, 3\})$ when 3 divides m, $\frac{m}{3}$ divides n and $m \leq n$ splits into four cases depending on the congruence class of m modulo 12. For each value of m we define a sequence $D_m = d_1, d_2, \ldots, d_m$ with $d_1 + d_2 + \cdots + d_m = 0$ and $|d_i| \in \{1, 2, 3\}$ for $i \in \{1, 2, \ldots, m\}$ as follows. The subscript on each bracket indicates the number of integers enclosed by that bracket.

For $m \equiv 0 \pmod{12}$ with $m \geq 12$, let $m = 12x$ and let D be the following sequence.

$$\underbrace{1, 3, 1, 3, \ldots, 1, 3,}_{2x} -1, 3, \underbrace{3, 1, 3, 1, \ldots, 3, 1,}_{2x-2} 2, 3, \underbrace{2, 2, \ldots, 2,}_{2x-2} -1, \underbrace{-2, -2, \ldots, -2,}_{2x-2}$$

$$-1, -2, \underbrace{-3, -1, -3, -1, \ldots, -3, -1,}_{2x-2} -2, -3, \underbrace{-3, -1, -3, -1, \ldots, -3, -1,}_{2x-2} -2.$$

For $m \equiv 3 \pmod{12}$ with $m \geq 15$, let $m = 12x + 3$ and let D be the following sequence.

$$1, \underbrace{1, 3, 1, 3, \ldots, 1, 3,}_{2x} -1, \underbrace{3, 1, 3, 1, \ldots, 3, 1,}_{2x} \underbrace{2, 2, \ldots, 2,}_{2x+1} -1, \underbrace{-2, -2, \ldots, -2,}_{2x}$$
$$\underbrace{-3, -1, -3, -1, \ldots, -3, -1, -3,}_{2x-1} -3, \underbrace{-3, -1, -3, -1, \ldots, -3, -1, -3.}_{2x-1}$$

For $m \equiv 6 \pmod{12}$ with $m \geq 6$, let $m = 12x + 6$, let D be the sequence

$$3, 2, -1, -3, 1, -2$$

for $m = 6$, and let D be following sequence for $m \geq 18$.

$$\underbrace{1, 3, 1, 3, \ldots, 1, 3,}_{2x} 3, 2, \underbrace{1, 3, 1, 3, \ldots, 1,}_{2x-1} 2, 3, \underbrace{2, 2, \ldots, 2,}_{2x-1} -1, \underbrace{-2, -2, \ldots, -2,}_{2x-1}$$
$$-1, -2, \underbrace{-3, -1, -3, -1, \ldots, -3, -1, -3,}_{2x-1} -3, 1, \underbrace{-3, -1, -3, -1, \ldots, -3, -1,}_{2x} -2.$$

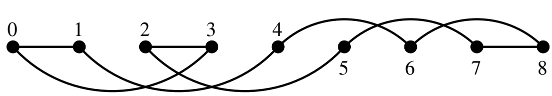
For $m \equiv 9 \pmod{12}$ with $m \geq 9$, let $m = 12x + 9$ and let D be the following sequence.

$$1, \underbrace{1, 3, 1, 3, \ldots, 1, 3,}_{2x} 3, \underbrace{3, 1, 3, 1, \ldots, 3, 1,}_{2x} \underbrace{2, 2, \ldots, 2,}_{2x+2} -1, \underbrace{-2, -2, \ldots, -2,}_{2x+1}$$
$$\underbrace{-3, -1, -3, -1, \ldots, -3, -1, -3,}_{2x+1} 1, \underbrace{-3, -1, -3, -1, \ldots, -3, -1, -3.}_{2x+1}$$

In each case we define an m-cycle C as follows.

$$C = (0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{m-1})$$

It is straightforward to verify that the orbit of C under the permutation $x \mapsto x + \frac{m}{3} \pmod{n}$ is an m-cycle decomposition of $\mathrm{Circ}(n, \{1, 2, 3\})$. For example, when $m = 9$, the cycle C is shown below.



The orbit of C under the permutation $x \mapsto x + 3 \pmod{9}$ is indeed a 9-cycle decomposition of $\mathrm{Circ}(n, \{1, 2, 3\})$ since for each part Q of the partition

$$P = \{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}\}$$

of the vertices of C, we have

$$\bigcup_{a \in Q} \{x - a : \{a, x\} \in E(C)\} = \{-3, -2, -1, 1, 2, 3\}.$$

In general, one can verify that for each part Q of the partition

$$P = \{\{0, \tfrac{m}{3}, \tfrac{2m}{3}\}, \{1, \tfrac{m}{3}+1, \tfrac{2m}{3}+1\}, \ldots, \{\tfrac{m}{3}-1, \tfrac{m}{3}+\tfrac{m}{3}-1, \tfrac{2m}{3}+\tfrac{m}{3}-1\}\}$$

of the vertices of C, we have

$$\bigcup_{a\in Q}\{x-a : \{a,x\} \in E(C)\} = \{-3,-2,-1,1,2,3\}.$$

□

Since any circuit of length m is necessarily an m-cycle for $m \le 5$, we have the following Theorem as an immediate corollary of Theorem 1.

Theorem 8 *Let $n \ge 7$ and let $m_1, m_2, \ldots, m_t$ be any sequence of integers with $3 \le m_i \le 5$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = 3n$. Then there exists an $(m_1, m_2, \ldots, m_t)$-cycle decomposition of* $\mathrm{Circ}(n, \{1, 2, 3\})$.

Acknowledgement

Research supported by the Australian Research Council.

References

[1] B. Alspach, Research problems, Problem 3, *Discrete Math.*, **36** (1981), 333.

[2] B. Alspach, Research problems, Problem 59, *Discrete Math.*, **50** (1984), 115.

[3] B. Alspach, D. Dyer and D. Kreher, On isomorphic factorizations of circulant graphs, *J. Combin. Des.*, **14** (2006), no. 5, 406–414.

[4] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory Ser. B*, **81** (2001), 77–99.

[5] P. Balister, On the Alspach conjecture, *Combin. Probab. Comput.*, **10** (2001), 95–125.

[6] P. Balister, Packing circuits in K_n, *Combin. Probab. Comput.*, **10** (2001), 463–499.

[7] P. Balister, Packing closed trails into dense graphs, *J. Combin. Theory Ser. B*, **88** (2003), 107-118.

[8] J-C. Bermond, O. Favaron and M. Mahéo, Hamiltonian decomposition of Cayley graphs of degree 4, *J. Combin. Theory Ser. B*, **46** (1989), 142–153.

[9] E. J. Billington and N. J. Cavenagh, Sparse graphs which decompose into closed trails of arbitrary lengths, *Graphs and Combinatorics*, **24** (2008), 129–147.

[10] D. Bryant, Cycle decompositions of complete graphs, in *Surveys in Combinatorics 2007*, A. Hilton and J. Talbot (Editors), London Mathematical Society Lecture Note Series **346**, Proceedings of the 21st British Combinatorial Conference, Cambridge University Press, 2007, pp. 67–97.

[11] D. Bryant and D. Horsley, Packing cycles in complete graphs, *J. Combin. Theory Ser. B*, **98** (2008), 1014–1037.

[12] D. Bryant and C. A. Rodger, Cycle decompositions, in *The CRC Handbook of Combinatorial Designs,* 2*nd edition* (Eds. C. J. Colbourn, J. H. Dinitz), CRC Press, Boca Raton (2007), 373–382.

[13] D. Bryant and V. Scharaschkin, Results on the Oberwolfach problem from 2-factorisations of circulant graphs, (preprint).

[14] M. Dean, Hamilton cycle decomposition of 6-regular circulants of odd order, *J. Combin. Des.*, **15** (2007), 91–97.

[15] M. Dean, On Hamilton cycle decomposition of 6-regular circulant graphs, *Graphs Combin.*, **22** (2006), 331–340.

[16] M. Horňák and Z. Kocková, On complete tripartite graphs arbitrarily decomposable into closed trails, *Tatra Mt. Math. Publ.*, **36** (2007), 71–107.

[17] J. Liu, Hamiltonian decompositions of Cayley graphs on abelian groups of odd order, *J. Combin. Theory Ser. B*, **66** (1996), 75–86.

[18] J. Liu, Hamiltonian decompositions of Cayley graphs on abelian groups of even order, *J. Combin. Theory Ser. B*, **88** (2003), 305–321.

[19] M. Šajna, Cycle decompositions of K_n and $K_n - I$, *Ph.D. Thesis*, Simon Fraser University, July 1999.

[20] M. Šajna, Cycle Decompositions III: Complete graphs and fixed length cycles, *J. Combin. Des.*, **10** (2002), 27–78.

[21] G. Stern and H. Lenz, Steiner triple systems with given subspaces; another proof of the Doyen-Wilson-theorem, *Boll. Un. Mat. Ital. A (5)* **17** (1980), 109–114.

[22] M. Tarsi, Decomposition of a complete multigraph into simple paths: Nonbalanced handcuffed designs, *J. Combin. Theory Ser. A* **34** (1983), 60–70.

(Received 25 Sep 2008; revised 1 Apr 2009)