SOME RESULTS ON GENERALIZED MULTIPLICATIVE PERFECT NUMBERS

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Abstract. In this article, based on ideas and results by J. Sándor [5, 6], we define k-multiplicatively e-perfect numbers and k-multiplicatively e-superperfect numbers and prove some results on them. We also characterize the k-T_0T^*-perfect numbers defined by Das and Saikia [2] in details.

1. Introduction

A natural number n is said to be **perfect** (A000396) if the sum of all proper divisors of n is equal to n. Or equivalently, $\sigma(n) = 2n$, where $\sigma(k)$ is the sum of the divisors of k. It is a well known result of Euler-Euclid that the form of even perfect numbers is $n = 2^k p$, where $p = 2^{k+1} - 1$ is a Mersenne prime and $k \geq 1$. Till date, no odd perfect number is known, and it is believed that none exists. Moreover n is said to be **super-perfect** if $\sigma(\sigma(n)) = 2n$. It was proved by Suryanarayana-Kanold [3, 12] that the general form of such super-perfect numbers are $n = 2^k$, where $2^{k+1} - 1$ is a Mersenne prime and $k \geq 1$. No odd super-perfect numbers are known till date. Unless, otherwise mentioned all n considered in this paper will be a natural number and $d(n)$ will be the number of divisors of n. We also denote by $[\![1, n]\!]$, the set $\{1, 2, \ldots, n\}$ and by $\mathbb{N}^*$, the set $\mathbb{N} \cup \{0\}$.

Let $T(n)$ denote the product of all the divisors of n. A **multiplicatively perfect number** (A007422) is a number n such that $T(n) = n^2$ and n is called **multiplicatively super-perfect** if $T(T(n)) = n^2$. Sándor [5] characterized such numbers and also numbers called **k-multiplicatively perfect numbers**, which are numbers n, such that $T(n) = n^k$ for $k \geq 2$. In this article we shall give some results on other classes of perfect numbers defined by various authors as well as by us. For a general introduction to such numbers, we refer the readers to subsection 1.11 in Sándor and Crstici's book [9, p. 55 – 58] and to the article [2], which contains various references to the existing literature.

2. k-multiplicatively e-perfect and superperfect numbers

Sándor [6] studied the multiplicatively e-perfect numbers defined below. If $n = p_1^{a_1} \cdots p_r^{a_r}$ is the prime factorization of $n > 1$, a divisor d of n is called an **exponential divisor** (or, e-divisor for short) if $d = p_1^{b_1} \cdots p_r^{b_r}$ with $b_i \mid a_i$ for $i = 1, \ldots, r$. This notion is due to Straus and Subbarao [11]. Let $\sigma_e(n)$ denote the sum of e-divisors of n, then Straus and Subbarao

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define n as **exponentially perfect** (or, e-perfect for short) (A054979) if $\sigma_e(n) = 2n$. They proved that there are no odd e-perfect numbers, and for each r, such numbers with r prime factors are finite. We refer the reader to the article [6] for some historical comments and results related to such e-perfect numbers. In [8], Sándor also studied some type of e-harmonic numbers. An integer n is called **e-harmonic of type** 1 if $\sigma_e(n)|nd_e(n)$, where $\sigma_e(n)$ (resp. $d_e(n)$) is the sum (resp. number) of e-divisors of n. It is easy to check that

$$d_e(n) = d(a_1) \cdots d(a_r).$$

Sándor [8] also defined n to be **e-harmonic of type** 2 if $S_e(n)|nd_e(n)$, where

$$S_e(n) = \prod_{i=1}^{r} \left(\sum_{d_i|a_i} p_i^{a_i - d_i} \right).$$

Let $T_e(n)$ denote the product of all the e-divisors of n. Then n is called **multiplicatively e-perfect** if $T_e(n) = n^2$ and **multiplicatively e-superperfect** if $T_e(T_e(n)) = n^2$. The main result of Sándor [6] is the following.

Theorem 1 (Sándor, [6] Theorem 2.1)**.** *n is multiplicatively e-perfect if and only if $n = p^a$, where p is a prime and a is an ordinary perfect number. n is multiplicatively e-superperfect if and only if $n = p^a$, where p is a prime and a is an ordinary superperfect number, that is $\sigma(\sigma(a)) = 2a$.*

We now give two result on e-harmonic numbers below, before we proceed to the main goal of our paper, that is to define and characterize some other classes of numbers.

Theorem 2. *If n is multiplicatively e-perfect or multiplicatively e-superperfect, then n is e-harmonic of type 1 if and only if $\sigma_e(n)/p|d_e(n)$, where p is the prime as described in Theorem 1.*

Proof. We prove the result for the case when n is multiplicatively e-perfect, the other case is similar. By Theorem 1, $n = p^a$ where p is a prime and a is an ordinary perfect number. So, for n to be e-harmonic of type 1, we must have $\sigma_e(n)|nd_e$, which is enough to verify our claim.

□

Theorem 3. *If n is multiplicatively e-perfect or multiplicatively e-superperfect, then n is e-harmonic of type 2 if and only if $S_e(n)|d_e(n)$.*

We skip the proof of Theorem 3, as it is similar to the proof of Theorem 2 and uses Theorem 1 in a similar way.

Inspired by the work of others in introducing generalized multiplicative perfect numbers, we now introduce the following two classes of numbers which are a natural generalization to the concept of e-perfect numbers..

Definition 4. A natural number n is called **k-multiplicatively e-perfect** if $T_e(n) = n^k$, where $k \geq 2$.

Definition 5. A natural number n is called **k-multiplicatively e-superperfect number** if $T_e(T_e(n)) = n^k$, where $k \geq 2$.

Before proceeding, we note from Sándor [6] that

$$T_e(n) = p_1^{\sigma(a_1)d(a_2)\cdots d(a_r)} \cdots p_r^{\sigma(a_r)d(a_1)\cdots d(a_{r-1})}. \tag{2.1}$$

Sándor [7] also gave an alternate expression for $T_e(n)$ in terms of the arithmetical function $t(n)$ defined as

$$t(n) = p_1^{2^{\frac{\sigma(a_1)}{d(a_1)}}} \cdots p_r^{2^{\frac{\sigma(a_r)}{d(a_r)}}}$$

with $t(1) = 1$. We have from Sándor [7]

$$T_e(n) = (t(n))^{d_e(n)/2}. \tag{2.2}$$

We however, do not use (2.2) in this note as we are only interested in the canonical forms of the numbers we have so far defined.

Before we characterize these classes of numbers we work on a few examples.

Example 6. There exist $6k$-multiplicatively e-perfect numbers with $k \in \mathbb{N}^\star$, which have the form $(p_1 \cdot p_2 \cdot p_3)^\alpha$ provided

$$\sigma(\alpha)d(\alpha)^2 = 6k\alpha$$

Thus, by (2.1), we have

$$T_e((p_1 \cdot p_2 \cdot p_3)^\alpha) = (p_1 \cdot p_2 \cdot p_3)^{\sigma(\alpha)d(\alpha)^2}.$$

Example 7. There exist k-multiplicatively e-superperfect numbers for a nonzero positive even number k. For instance, by a routine application of (2.1) it can be verified that there exist

- 20-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^3$;
- 24-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^2$;
- 32-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^4$;
- 48-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^{16}$;
- 64-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^{64}$;
- 110-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^{93}$;
- 168-multiplicatively e-superperfect numbers which have the form $p_1^6 \cdot p_2^{16}$ or $(p_1 \cdot p_2)^{27}$;
- 216-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^{14}$;
- 234-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^{10}$;
- 252-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^8$.

In the following, we explain some of these examples. By analysis of the cases above, we notice that there exist for nonzero positive integers m such that $2^{m+1} - 1$ is prime (and so $m + 1$ is prime), $8(m + 2)$-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^{2^m}$. Indeed, according to (2.1), we have

$$T_e((p_1 \cdot p_2)^{2^m}) = (p_1 \cdot p_2)^{\sigma(2^m)\cdot d(2^m)} = (p_1 \cdot p_2)^{(2^{m+1}-1)\cdot(m+1)}$$

and since we assume that $m + 1$ and $2^{m+1} - 1$ are prime, we have

$$\begin{aligned} T_e(T_e((p_1 \cdot p_2)^{2^m})) &= (p_1 \cdot p_2)^{\sigma((2^{m+1}-1)\cdot(m+1))\cdot d((2^{m+1}-1)\cdot(m+1))} \\ &= (p_1 \cdot p_2)^{2^{m+1}\cdot(m+2)\cdot 4} = ((p_1 \cdot p_2)^{2^m})^{8(m+2)}. \end{aligned}$$

Moreover, by analysis of the cases above, we can notice that there exist for some odd prime number p, $2p$-multiplicatively e-superperfect numbers which have the form $(p_1 \cdot p_2)^{2p}$. To see this, by (2.1), we have

$$T_e((p_1 \cdot p_2)^{2p}) = (p_1 \cdot p_2)^{\sigma(2p)\cdot d(2p)} = (p_1 \cdot p_2)^{3\cdot(p+1)\cdot 4} = (p_1 \cdot p_2)^{12\cdot(p+1)}.$$

If we can represent p as $p = 2^m q - 1$ with m a nonzero positive integer and q an odd positive integer, then we have

$$T_e((p_1 \cdot p_2)^{2p}) = (p_1 \cdot p_2)^{2^{m+2} q \cdot 3}.$$

If $q = 3$, we have

$$T_e(T_e((p_1 \cdot p_2)^{2p})) = (p_1 \cdot p_2)^{\sigma(2^{m+2}\cdot 3^2)\cdot d(2^{m+2}\cdot 3^2)} = (p_1 \cdot p_2)^{(2^{m+3}-1)\cdot 13\cdot(m+3)\cdot 3}.$$

We search a solution such that $2^{m+3} - 1$ is divisible by $p = 2^m \cdot 3 - 1$. That is, an integer $x \geq 1$ such that

$$2^{m+3} - 1 = xp = x(2^m \cdot 3 - 1).$$

This gives

$$\begin{aligned} 2^m \cdot 8 - 1 &= xp \\ \Rightarrow 3 \cdot 2^m \cdot 8 - 3 &= 3xp \\ \Rightarrow (p+1) \cdot 8 - 3 &= 3xp \\ \Rightarrow (3x - 8) \cdot p &= 5. \end{aligned}$$

Since p is a prime this implies that $p = 5$ and $x = 3$. Therefore, $m = 1$ and we recover that

$$T_e(T_e((p_1 \cdot p_2)^{10})) = (p_1 \cdot p_2)^{15\cdot 13\cdot 12} = ((p_1 \cdot p_2)^{10})^{234}.$$

If q is not divisible by 3, then we have

$$\begin{aligned} T_e(T_e((p_1 \cdot p_2)^{2p})) &= (p_1 \cdot p_2)^{\sigma(2^{m+2}q\cdot 3)\cdot d(2^{m+2}q\cdot 3)} = (p_1 \cdot p_2)^{(2^{m+3}-1)\cdot\sigma(q)4\cdot(m+3)\cdot d(q)\cdot 2} \\ &= (p_1 \cdot p_2)^{8(m+3)\cdot\sigma(q)\cdot d(q)\cdot(2^{m+3}-1)}. \end{aligned}$$

We search a solution such that $2^{m+3} - 1$ is divisible by p. That is, an integer $y \geq 1$ such that

$$2^{m+3} - 1 = yp = y(2^m q - 1).$$

This will reduce to

$$(qy - 8) \cdot p = 8 - q.$$

If $q > 7$, then $8 - q < 0$ where as $(qy - 8) \cdot p > 0$. We reach a contradiction meaning that the only possible values of q are $1, 5, 7$. If $q = 1$, then we have $(y - 8) \cdot p = 7$. This implies that $p = 7, y = 9$ and $m = 3$ and so on

$$T_e(T_e((p_1 \cdot p_2)^{2p})) = (p_1 \cdot p_2)^{8\cdot 6\cdot 63} = ((p_1 \cdot p_2)^{14})^{216}.$$

If $q = 5, 7$, then we get no integral solutions for y.

Now, we characterize some of these classes of numbers in the following theorems. Note that Theorem 8 is analogous to Theorem 1 of Sándor.

Theorem 8. *If $n = p^a$, where p is a prime and a is a k-perfect number, then n is k-multiplicatively e-perfect. If $n = p^a$, where p is a prime and a is a k-superperfect number, then n is k-multiplicatively e-superperfect.*

Proof. We know from Sándor [6], that if $n = p^a$ where p is a prime and a is a non-zero positive integer, then we have

$$T_e(n) = p^{\sigma(a)}.$$

So, if $n = p^a$ where p is a prime and a is a k-perfect number, then

$$\sigma(a) = ka,$$

and so

$$T_e(n) = p^{ka} = n^k.$$

Moreover, if $n = p^a$ where p is a prime and a is a non-zero positive integer, then we have

$$T_e(T_e(n)) = T_e(p^{\sigma(a)}) = p^{\sigma(\sigma(a))}.$$

So, if $n = p^a$ where p is a prime and a is a k-perfect number, then

$$\sigma(\sigma(a)) = ka,$$

and so

$$T_e(T_e(n)) = p^{ka} = n^k.$$

□

Theorem 9. *Let p be a prime number and let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, with $r \in \mathbb{N}^*$ be the prime factorisation of an integer $n > 1$ where p_i with $i \in [\![1, r]\!]$ are prime numbers and $\alpha_i \in \mathbb{N}^*$ for all $i \in [\![1, r]\!]$. n is p-multiplicatively e-perfect if and only if for each $i \in [\![1, r]\!]$, we have*

$$\sigma(\alpha_i) = \gcd(\alpha_i, \sigma(\alpha_i))p$$

and

$$\alpha_i = \gcd(\alpha_i, \sigma(\alpha_i)) \prod_{j \in [\![1,r]\!] \setminus \{i\}} d(\alpha_j),$$

with

$$2^{r-1} \leq \prod_{j \in [\![1,r]\!] \setminus \{i\}} d(\alpha_j) < p,$$

where we set

$$\prod_{j \in \emptyset} d(\alpha_j) = 1.$$

In particular, if $r = 1$, then $\alpha_1 | \sigma(\alpha_1)$ and we have

$$\sigma(\alpha_1) = \alpha_1 p.$$

Proof. If $r = 1$, then using Theorem 8, $n = p_1^{\alpha_1}$ is p-multiplicatively e-perfect if and only if α_1 is a p-perfect number.

We now assume that $r \geq 2$. We have

$$T_e(n) = n^p \Leftrightarrow \begin{cases} \sigma(\alpha_1)d(\alpha_2)\cdots d(\alpha_r) = p\alpha_1; \\ \vdots \\ \sigma(\alpha_r)d(\alpha_1)\cdots d(\alpha_{r-1}) = p\alpha_r. \end{cases} \tag{2.3}$$

Clearly, if for each $i \in [\![1, r]\!]$, we have

$$\sigma(\alpha_i) = \gcd(\alpha_i, \sigma(\alpha_i))p$$

and

$$\alpha_i = \gcd(\alpha_i, \sigma(\alpha_i)) \prod_{j \in [\![1,r]\!] \setminus \{i\}} d(\alpha_j)$$

then it can be easily verified that (2.3) is satisfied.

Now we notice that if all α_i ($i \in [\![1, r]\!]$) are equal to 1, then (2.3) is consistent only if $p = 1$. It is impossible since p is a prime number. If at least an α_i is equal to 1, say $\alpha_1 = 1$, then from (2.3), we have

$$d(\alpha_2)\cdots d(\alpha_r) = p.$$

Then one of $d(\alpha_2), \ldots, d(\alpha_r)$ is equal to the prime p and the others are equal to 1. Say

$$d(\alpha_2) = p,$$

and

$$d(\alpha_3) = \cdots = d(\alpha_r) = 1.$$

It implies that

$$\alpha_1 = \alpha_3 = \cdots = \alpha_r = 1.$$

Then the equation $\sigma(\alpha_2)d(\alpha_1)d(\alpha_3)\cdots d(\alpha_r) = p\alpha_2$ of (2.3) gives $\sigma(\alpha_2) = p\alpha_2$. But, for all $i \in [\![1, r]\!] \setminus \{1, 2\}$, the equation $\sigma(\alpha_i)d(\alpha_1)d(\alpha_2)\cdots d(\alpha_r) = p\alpha_i$ of (2.3) gives $\sigma(\alpha_i) = \alpha_i$ which is not possible since we know that $\sigma(\alpha_i) \geq \alpha_i + 1$. So, we must have $\alpha_i \geq 2$ for all $i \in [\![1, r]\!]$. Thus,

$$\prod_{j \in [\![1,r]\!] \setminus \{i\}} d(\alpha_j) \geq 2^{r-1}.$$

We now prove that if (2.3) is true, then for each $i \in [\![1, r]\!]$, we have

$$\sigma(\alpha_i) = \gcd(\alpha_i, \sigma(\alpha_i))p$$

and

$$\alpha_i = \gcd(\alpha_i, \sigma(\alpha_i)) \prod_{j \in [\![1,r]\!] \setminus \{i\}} d(\alpha_j).$$

Let $g_i = \gcd(\alpha_i, \sigma(\alpha_i))$ for all $i \in [\![1, r]\!]$. So, for each $i \in [\![1, r]\!]$, there exists two non-zero positive integer a_i, s_i such that $\alpha_i = g_i a_i$ and $\sigma(\alpha_i) = g_i s_i$ with $\gcd(a_i, s_i) = 1$. Notice that $s_i > 1$. Otherwise, we would have $\sigma(\alpha_i) | \alpha_i$, a contradiction.

For each $i \in [\![1,r]\!]$, the equation

$$\sigma(\alpha_i) \prod_{j\in[\![1,r]\!]\setminus\{i\}} d(\alpha_j) = p\alpha_i$$

of (2.3) now gives

$$s_i \prod_{j\in[\![1,r]\!]\setminus\{i\}} d(\alpha_j) = pa_i.$$

Since $\gcd(a_i, s_i) = 1$, then from Euclid's lemma we get $a_i | \prod_{j\in[\![1,r]\!]\setminus\{i\}} d(\alpha_j)$, and $s_i | p$. So, there exists an integer k such that $\prod_{j\in[\![1,r]\!]\setminus\{i\}} d(\alpha_j) = ka_i$, and $p = ks_i$.

Using the fact that p is a prime and $s_i > 1$, we have that $k = 1$ and we get $\prod_{j\in[\![1,r]\!]\setminus\{i\}} d(\alpha_j) = a_i$, and $p = s_i$. Therefore $\sigma(\alpha_i) = g_i p$ and $\gcd(p, a_i) = 1$. Moreover, since $\sigma(\alpha_i) \geq \alpha_i + 1$, we have

$$g_i(p - a_i) \geq 1$$

implying that $p > a_i$. $\square$

Example 10. Let $n = p_1^\alpha \cdot p_2^\alpha$ be the prime factorisation of an integer $n > 1$ where p_1 and p_2 are prime numbers. Let $\alpha = 18$. So, we have

$$d(\alpha) = d(2 \cdot 3^2) = 6 \qquad \sigma(\alpha) = 1 + 2 + 3 + 6 + 9 + 18 = 39 = 3 \cdot 13 = 3p$$

where $p = 13$. Moreover, we have also

$$\gcd(\alpha, \sigma(\alpha)) = \gcd(18, 39) = \gcd(3 \cdot 6, 3 \cdot 13) = 3 \cdot \gcd(6, 13) = 3 \neq \alpha$$

and

$$\gcd(\alpha, \sigma(\alpha)) \cdot d(\alpha) = 3 \cdot 6 = 18 = \alpha.$$

At this stage, we notice that Theorem 9 can be applied. Let us verify that it is the case. We have

$$T_e(n) = p_1^{\sigma(18)\cdot d(18)} \cdot p_2^{\sigma(18)\cdot d(18)} = p_1^{39\cdot 6} \cdot p_2^{39\cdot 6} = p_1^{3\cdot 13\cdot 6} \cdot p_2^{3\cdot 13\cdot 6} = (p_1^{18} \cdot p_2^{18})^{13} = n^p.$$

So, for all primes p_1, p_2, integers of the form $p_1^{18} \cdot p_2^{18}$ are 13-multiplicatively e-perfect numbers.

Example 11. Let $n = p_1^\alpha \cdot p_2^\alpha \cdot p_3^\alpha$ be the prime factorisation of an integer $n > 1$ where p_1, p_3 and p_2 are prime numbers. Let $\alpha = 9$. So, we have

$$d(\alpha) = d(3^2) = 3 \qquad \sigma(\alpha) = 1 + 3 + 9 = 13 = 1 \cdot 13 = 1 \cdot p.$$

where $p = 13$. Moreover, we have also

$$\gcd(\alpha, \sigma(\alpha)) = \gcd(9, 13) = 1 \neq \alpha$$

and

$$\gcd(\alpha, \sigma(\alpha)) \cdot d(\alpha)^2 = 1 \cdot 3^2 = 9 = \alpha$$

At this stage, we notice that Theorem 9 can be applied. Let us verify that it is well the case. We have

$$T_e(n) = p_1^{\sigma(9)\cdot d(9)^2} \cdot p_2^{\sigma(9)\cdot d(9)^2} \cdot p_3^{\sigma(9)\cdot d(9)^2} = p_1^{13\cdot 9} \cdot p_2^{13\cdot 9} \cdot p_3^{13\cdot 9} = (p_1^9 \cdot p_2^9 \cdot p_3^9)^{13} = n^p$$

So, for all primes p_1, p_2 and p_3, integers of the form $p_1^9 \cdot p_2^9 \cdot p_3^9$ are 13-multiplicatively e-perfect numbers.

Remark 12. Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $r \in \mathbb{N}^*$ be the prime factorisation of an integer $n > 1$ where p_i with $i \in [\![1, r]\!]$ are prime numbers and $\alpha_i \in \mathbb{N}^*$ for all $i \in [\![1, r]\!]$. If all α_i are primes, then we have

$$\sigma(\alpha_i) = \alpha_i + 1,$$

and

$$d(\alpha_i) = 2.$$

Notice that in such a case, Theorem 9 cannot be applied since $\sigma(\alpha_i)$ is not divisible by α_i for all $i \in [\![1, r]\!]$. From (2.1) we have,

$$T_e(n) = p_1^{(\alpha_1+1)\cdot 2^{r-1}} \cdots p_r^{(\alpha_r+1)\cdot 2^{r-1}},$$

and so

$$T_e(n) = n \prod_{i=1}^{r} p_i^{2^{r-1}}.$$

If $r = 1$, then $T_e(n) = np_1 = p^{1+\alpha_1}$. If $r \geq 2$ and if $\alpha_i = 2$ for all $i \in [\![1, r]\!]$, then n is a perfect square and we have

$$n = (p_1 \cdots p_r)^2,$$

and

$$T_e(n) = n^{1+2^{r-2}}.$$

In particular, if $r = 2$, then $T_e(n) = n^2$ meaning that n is multiplicatively e-perfect number.

We now prove a result related to the bounds on the prime p in Theorem 9. For that we will need the following results.

Theorem 13 (Nicolas and Robin [4])**.** *For $n \geq 3$,*

$$\frac{\log d(n)}{\log 2} \leq C_1 \frac{\log n}{\log \log n}$$

where $C_1 = 1.5379\cdots$ with equality for $n = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

Theorem 14 ([10], p. 77)**.** *For any natural number $n \geq 3$, $\sigma(n) < n\sqrt{n}$.*

Theorem 15. *Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the prime factorization of integer n, where $p_i, a_i, i \in \mathbb{N}, a_i \geq 3$ and let n be a k-multiplicatively-e-perfect number. Then, we have*

$$2^{r-1} < k < \prod_{i=1}^{r} \left(a_i^{0.5 + \frac{C(r-1)}{\log \log a_i}} \right)^{\frac{1}{r}},$$

where $C = C_1 \log 2$ and $C_1 = 1.5379\cdots$.

Proof. As n is a k-multiplicatively-e-perfect number, so $n^k = p_1^{a_1 k} p_2^{a_2 k} \cdots p_r^{a_r k} = T_e(n)$. By using (2.1) we get

$$\begin{aligned} \sigma(a_1)d(a_2)\cdots d(a_r) &= a_1 k, \\ &\vdots \\ d(a_1)d(a_2)\cdots \sigma(a_r) &= a_r k. \end{aligned}$$

Multiplying these r equalities we get,

$$\sigma(a_1)\sigma(a_2)\cdots\sigma(a_r)d(a_1)^{r-1}d(a_2)^{r-1}\cdots d(a_r)^{r-1} = a_1 a_2 \cdots a_r k^r. \tag{2.4}$$

Now we proceed to prove that $k > 2^{r-1}$. For that notice, for any a_i, $\sigma(a_i) \geq (a_i + 1)$ and $d(a_i) \geq 2$. Thus we get the following inequality,

$$\sigma(a_1)\cdots\sigma(a_r)d(a_1)^{r-1}\cdots d(a_r)^{r-1} \geq (a_1+1)(a_2+1)\cdots(a_r+1)2^{r-1}\cdots 2^{r-1}$$

Substituting the right hand side of (2.4) in the above inequality we get,

$$a_1 a_2 \cdots a_r k^r \geq (a_1+1)(a_2+1)\cdots(a_r+1)(2^{r-1})^r$$

which implies

$$(k/2^{r-1})^r \geq \frac{(a_1+1)(a_2+1)\cdots(a_r+1)}{a_1 a_2 \cdots a_r} > 1.$$

This gives us $k > 2^{r-1}$.

Now we proceed to set the upper bound. By Theorem 13 we have $d(a_i) \leq a_i^{\frac{C}{\log\log a_i}}$, where $C = C_1 \log 2$. Hence

$$\prod_{i=1}^{r} d(a_i)^{r-1} \leq \prod_{i=1}^{r} \left(a_i^{\frac{C}{\log\log a_i}}\right)^{r-1}. \tag{2.5}$$

Again, by an application of Theorem 14 we get

$$\prod_{i=1}^{r} \sigma(a_i) < \prod_{i=1}^{r} a_i\sqrt{a_i}. \tag{2.6}$$

Now, using (2.4), (2.5) and (2.6) we get

$$k < \prod_{i=1}^{r} \left(a_i^{0.5+\frac{C(r-1)}{\log\log a_i}}\right)^{\frac{1}{r}}.$$

□

Example 16. Let $m = p^6$, where p is a prime. Now consider another arbitrary prime q. Then n can be q-multiplicatively-e-perfect only for the primes in the interval $2^{1-1} < q \leq \left(6^{0.5+\frac{C_1 \log 2\cdot(1-1)}{\log\log 6}}\right)^{\frac{1}{1}}$. That is $1 < q < \sqrt{6}$.

The bounds on p mentioned in Theorem 15 are not tight and there is further scope to work on such bounds of primes. Moreover notice that the bounds mentioned here requires complete prime factorization of integer m. Hence, bounds that doesn't requires prime factorization of m would be more efficient. But we do not discuss this direction in the present paper.

3. k-T_0T^*-perfect numbers

A divisor d of n is said to be **unitary** if $\gcd(d, n/d) = 1$. Let $T^*(n)$ be the product of unitary divisors of n. Bege [1] has studied the multiplicatively unitary perfect numbers and proved results very similar to Sándor. Das and Saikia [2] introduced the concept of **T^*T-perfect numbers** which are numbers n such that $T^*(n)T(n) = n^2$. They also introduced k-T^*T-perfect numbers and characterized both these classes of numbers. They further introduced the concept of T_0^*T-superperfect and k-T_0^*T-perfect numbers. A number n is called a **T_0^*T-superperfect number** if $T^*(T(n)) = n^2$ and it is called a **k-T_0^*T-perfect number** if $T^*(T(n)) = n^k$ for $k \geq 2$. Das and Saikia [2] characterized these classes of numbers. They also introduced the **k-T_0T^*-perfect numbers** as the numbers n such that $T(T^*(n)) = n^k$ for $k \geq 2$. It is our aim in this section to characterize these k-T_0T^*-perfect numbers.

Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorization of $n > 1$. Then the number of unitary divisors of n, $\tau^*(n) = 2^r$ and $T^*(n) = n^{2^{r-1}}$. Das and Saikia [2] mentioned that for k-T_0T^*-perfect number we must have

$$2^r(\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 4k \tag{3.1}$$

for $k \geq 2$. In the following results we characterize these class of numbers. Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $r \in \mathbb{N}^*$ be the prime factorization of an integer $n > 1$ where p_i with $i \in [\![1, r]\!]$ are prime numbers and $\alpha_i \in \mathbb{N}^*$ for all $i \in [\![1, r]\!]$.

Theorem 17.
(1) *All* 2-T_0T^**-perfect numbers have the form* $n = p_1^3$*;*
(2) *All* 3-T_0T^**-perfect numbers have the form* $n = p_1^5$*;*
(3) *All* 4-T_0T^**-perfect numbers have the form* $n = p_1^7$*;*
(4) *All* 5-T_0T^**-perfect numbers have the form* $n = p_1^9$*;*
(5) *All* 6-T_0T^**-perfect numbers have the form* $n = p_1^{11}$*;*
(6) *All* 7-T_0T^**-perfect numbers have the form* $n = p_1^{13}$*;*
(7) *All* 8-T_0T^**-perfect numbers have the form* $n = p_1^{15}$*;*
(8) *All* 9-T_0T^**-perfect numbers have the form* $n = p_1^{17}$ *or* $n = p_1p_2$*;*
(9) *All* 10-T_0T^**-perfect numbers have the form* $n = p_1^{19}$*;*

Proof. We will examine in detail the cases where $k = 2, 3, 9$ leaving to the reader the task to verify the other statements as the proofs are similar.

We first prove that all 2-T_0T^*-perfect numbers have the form $n = p_1^3$. In the following, we will investigate the different subcases beginning from $r = 1$.

- $r = 1$: (3.1) becomes $2 \cdot (\alpha_1 + 1) = 8$; it gives $\alpha_1 = 3$.
- $r = 2$: (3.1) becomes $4 \cdot (2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 8$ which is equivalent to $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 2$; it is not possible since $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1)$ is odd whereas 2 is even.

- $r = 3$: (3.1) becomes $8 \cdot (4\alpha_1 + 1) \cdot (4\alpha_2 + 1) \cdot (4\alpha_3 + 1) = 8$ which is equivalent to $(4\alpha_1 + 1) \cdot (4\alpha_2 + 1) \cdot (4\alpha_3 + 1) = 1$; it is not possible since $\alpha_1, \alpha_2, \alpha_3 \geq 1$ imply that $(4\alpha_1 + 1) \cdot (4\alpha_2 + 1) \cdot (4\alpha_3 + 1) \geq 125$.
- $r \geq 4$: (3.1) becomes $2^{r-3} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 1$ which is not possible since $2^{r-3} \not| 1$ for $r \geq 4$.

So, only the subcase where $r = 1$ is valid, which means that if $k = 2$, then $n = p_1^3$.

Secondly, we now prove that all 3-T_0T^*-perfect numbers have the form $n = p_1^5$. In the following, we will investigate the different subcases beginning from $r = 1$.

- $r = 1$: (3.1) becomes $2 \cdot (\alpha_1 + 1) = 12$; it gives $\alpha_1 = 5$.
- $r = 2$: (3.1) becomes $4 \cdot (2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 12$ which is equivalent to $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 3$; it is not possible since for $\alpha_1, \alpha_2 \geq 1$, we have $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) \geq 9$.
- $r \geq 3$: (3.1) becomes $2^{r-2} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 3$ which is not possible since $2^{r-2} \not| 3$ for $r \geq 3$.

So, only the subcase where $r = 1$ is valid, which means that if $k = 3$, then $n = p_1^5$.

Third, we prove that all 9-T_0T^*-perfect numbers have the form $n = p_1^{17}$ or $n = p_1p_2$. In the following, we will investigate the different subcases beginning from $r = 1$.

- $r = 1$: (3.1) becomes $2 \cdot (\alpha_1 + 1) = 36$; it gives $\alpha_1 = 17$.
- $r = 2$: (3.1) becomes $4 \cdot (2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 36$ which is equivalent to $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 9$; there is a trivial solution which is obtained when $\alpha_1 = \alpha_2 = 1$, if at least one of the integers among the integers α_1 and α_2 is greater than 2, then $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) > 9$ implying that there is no other solution.
- $r \geq 3$: (3.1) becomes $2^{r-2} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 9$ which is not possible since $2^{r-2} \not| 9$ for $r \geq 3$.

So, only the subcases where $r = 1$ and $r = 2$ with $\alpha_1 = \alpha_2 = 1$ is valid, which means that if $k = 9$, then either $n = p_1^{17}$ or $n = p_1p_2$. $\square$

Theorem 18. *All p-T_0T^*-perfect numbers for a prime p have the form $n = p_1^{2p-1}$.*

Proof. According to (3.1), we must solve the equation

$$2^r \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 4p. \tag{3.2}$$

In the following, we will investigate the different cases beginning from $r = 1$.

- $r = 1$: (3.2) becomes $2 \cdot (\alpha_1 + 1) = 4p$; it gives $\alpha_1 = 2p - 1$.
- $r = 2$: (3.2) becomes $4 \cdot (2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 4p$; it gives $(2\alpha_1 + 1)(2\alpha_2 + 1) = p$; notice that the conditions $\alpha_1, \alpha_2 \geq 1$ imply that $(2\alpha_1 + 1)(2\alpha_2 + 1) \geq 9$ and so in such a case, p must be necessarily an odd prime number; notice also that if one of the numbers among the numbers $2\alpha_1 + 1, 2\alpha_2 + 1$ is equal to p, it implies that the other number among the numbers $2\alpha_1 + 1, 2\alpha_2 + 1$ is equal to 1 implying that one of the numbers among the numbers α_1, α_2 would be zero, which is not compatible with the conditions $\alpha_1, \alpha_2 \geq 1$; therefore, this case is not possible due to the fact that p is a prime number which is squarefree.
- $r \geq 3$: (3.2) becomes $2^{r-2} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = p$ which is not possible since $2^{r-2} \not| p$ for $r \geq 3$ for odd prime p.

So, only the case where $r = 1$ is valid for which $n = p_1^{2p-1}$. $\square$

Theorem 19. *Let k be a non-zero positive integer. Then we have the following:*

1. *For any prime p, the integers whose prime decompositions have the form $p_1^{2p^k-1}$ are p^k-T_0T^*-perfect numbers.*
2. *If $k \geq 2$, then for any odd prime p, the integers whose prime decompositions have the form $p_1^{(p^{a_1}-1)/2} \cdot p_2^{(p^{a_2}-1)/2}$ where $a_1, a_2 \in \mathbb{N}^*$ such that $a_1 + a_2 = k$, are p^k-T_0T^*-perfect numbers.*
3. *Any integer which doesn't have prime decomposition as $p_1^{2p^k-1}$ for prime p or $p_1^{(p^{a_1}-1)/2} \cdot p_2^{(p^{a_2}-1)/2}$ for odd prime p, where $a_1, a_2 \in \mathbb{N}^*$ such that $a_1 + a_2 = k$, are not p^k-T_0T^*-perfect numbers.*

Proof. According to Equation (3.1), we must solve

$$2^r \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 4p^k \tag{3.3}$$

Next, we will examine the different cases beginning from $r = 1$.

- $r = 1$: (3.3) becomes $2 \cdot (\alpha_1 + 1) = 4p^k$; it gives $\alpha_1 = 2p^k - 1$, which proves the first statement.
- $r = 2$: (3.3) becomes $4 \cdot (2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 4p^k$; it gives $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = p^k$; in this case, since $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1)$ is odd whatever nonzero positive integers α_1, α_2 are, the prime p must be odd; if $k = 1$, then we have $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = p$ which is not possible for $\alpha_1, \alpha_2 \geq 1$ since p is a prime number which is squarefee; if $k \geq 2$, then according to the fundamental theorem of arithmetic, since $2\alpha_i + 1 \geq 3$ with $\alpha_i \geq 1$ for $i = 1, 2$, the equation $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = p^k$ implies that there exists two nonzero positive integers a_1, a_2 such that $2\alpha_1 + 1 = p^{a_1}$ and $2\alpha_2 + 1 = p^{a_2}$ and $a_1 + a_2 = k$, which proves the second statement.
- $r \geq 3$: (3.3) becomes $2^{r-2} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = p^k$; in this case, p^k must be divisible by 2 and since p is prime, it entails that $p = 2$; but then $\alpha_i \cdot 2^{r-1} + 1$ for all $i \in [\![1, r]\!]$ is divisible by 2; that is impossible since $\alpha_i \cdot 2^{r-1} + 1$ for all $i \in [\![1, r]\!]$ is odd, which proves the third statement.

□

Remark 20. Let k be an integer which is greater than 2 and let p be an odd prime number. According to Theorem 19, the integers of the form $p_1^{(p^a-1)/2} \cdot p_2^{(p^{k-a}-1)/2}$ with $a \in [\![1, k-1]\!]$ are p^k-T_0T^*-perfect numbers.

Corollary 21.

1. *For any prime p, the integers whose prime decompositions have the form $p_1^{2p^2-1}$, are p^2-T_0T^*-perfect numbers.*
2. *If p is an odd prime number, then the integers whose prime decompositions have the form $p_1^{(p-1)/2} \cdot p_2^{(p-1)/2}$, are p^2-T_0T^*-perfect numbers.*
3. *Any integer which doesn't have prime decomposition as $p_1^{2p^2-1}$ for a prime p or $p_1^{(p-1)/2} \cdot p_2^{(p-1)/2}$ for an odd prime p, are not p^2-T_0T^*-perfect numbers.*

The first statement of Corollary 21, follows directly from Theorem 19. The second statement of Corollary 21, follows from Remark 20 when $k = 2$ and so $n = p_1^{(p^a-1)/2} \cdot p_2^{(p^{2-a}-1)/2}$

with a a nonzero positive integers which verifies the condition $1 \leq a < 2$. In this case, there is only one possibility, namely $a = 1$.

Corollary 22. *Let k be a nonzero positive integer. All 2^k-T_0T^*-perfect numbers have the form $p_1^{2^{k+1}-1}$.*

Corollary 22 follows directly from Theorem 19.

Theorem 23. *Let k be an odd positive integer which is not prime. All k-T_0T^*-perfect numbers have the form p_1^{2k-1} or $p_1^{(d-1)/2} \cdot p_2^{(k/d-1)/2}$ where d is a positive proper divisor of k which satisfies the inequalities $2 < d < k$.*

Proof. According to Equation (3.1), we must solve

$$2^r \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 4k. \tag{3.4}$$

Next, we will examine the different cases beginning from $r = 1$.

- $r = 1$: (3.4) becomes $2 \cdot (\alpha_1 + 1) = 4k$; it gives $\alpha_1 = 2k - 1$.
- $r = 2$: (3.4) becomes $4 \cdot (2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 4k$; it gives $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = k$; in this case, let $k = q_1^{\beta_1} \cdots q_s^{\beta_s}$ be the prime decomposition of the odd integer k where q_i is an odd prime for all $i \in [\![1, s]\!]$ with $s \in \mathbb{N}^*$ and $\beta_i \in \mathbb{N}^*$ for all $i \in [\![1, s]\!]$ with $s \in \mathbb{N}^*$; according to the fundamental theorem of arithmetic, we have $2\alpha_1 + 1 = q_1^{\gamma_1} \cdots q_s^{\gamma_s}$ and $2\alpha_2 + 1 = q_1^{\delta_1} \cdots q_s^{\delta_s}$ where for all $i \in [\![1, s]\!]$ with $s \in \mathbb{N}^*$, $\gamma_i, \delta_i \in \mathbb{N}$ such that $\gamma_i + \delta_i = \beta_i$; since $\alpha_1, \alpha_2 \geq 1$ ($r = 2$), there exists at least one integer among the integers γ_i ($i = 1, \ldots, s$) which is greater than 1 and there exists at least one integer among the integers δ_i ($i = 1, \ldots, s$) which is greater than 1; we must have $2 < q_1^{\gamma_1} \cdots q_s^{\gamma_s} < k$ and $2 < q_1^{\delta_1} \cdots q_s^{\delta_s} < k$; notice that we have $q_1^{\delta_1} \cdots q_s^{\delta_s} = k/(q_1^{\gamma_1} \cdots q_s^{\gamma_s})$ which follows from the equation $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = k$; in the following, we set $d = q_1^{\gamma_1} \cdots q_s^{\gamma_s}$ and so $q_1^{\delta_1} \cdots q_s^{\delta_s} = k/d$; accordingly, d is a positive proper divisor of k, $2\alpha_1 + 1 = d$ and $2\alpha_2 + 1 = k/d$; it gives $\alpha_1 = (d-1)/2$ and $\alpha_2 = (k/d - 1)/2$.
- $r \geq 3$: (3.4) becomes $2^{r-2} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = k$; in this case, k must be divisible by 2; which is impossible since k is odd.

So, only the cases where $r = 1$ and $r = 2$ are possible. This completes the proof. $\square$

Remark 24. We can notice that in Theorem 23, if d is a positive proper divisor of k, then k/d is also a positive proper divisor of k. For instance, if $k = p^2$ where p is an odd prime, then there is only one possibility for d which is consistent with Theorem 23, namely $d = k/d = p$. Notice that in this case, using Theorem 23, the two results stated in Corollary 21 can be recovered. Another case which illustrates the fact mentioned at the beginning of this remark, is when $k = pq$ wher p and q are odd prime numbers. Then the positive proper divisors of k are p and q. In this case, Theorem 23 implies that all pq-T_0T^*-perfect number for odd prime numbers p and q, have the form p_1^{2pq-1} or $p_1^{(p-1)/2} \cdot p_2^{(q-1)/2}$.

Definition 25. Let n be an integer which is greater than 1. A **multiplicative partition** (A001055) or **unordered factorization** of n is a decomposition of n into a product of integers which belong to $[\![1, n]\!]$, where the order of terms is irrelevant.

Theorem 26. *Let k be an odd positive integer and let m be a nonzero positive integer. Then we have the following:*

(1) *The integers whose prime decompositions have the form $p_1^{2^{m+1}k-1}$, are $2^m k$-T_0T^*-perfect numbers.*
(2) *For $k > 1$, if there exist integers $d_1, \ldots, d_{m+2}$ which form a multiplicative partition of k such that for all $i \in [\![1, m+2]\!]$, $d_i \equiv 1 \pmod{2^{m+1}}$, then the integers whose prime decompositions have the form $p_1^{(d_1-1)/2^{m+1}} \cdots p_{m+2}^{(d_{m+2}-1)/2^{m+1}}$, are $2^m k$-T_0T^*-perfect numbers.*
(3) *Any integer which doesn't have prime decomposition as $p_1^{2^{m+1}k-1}$ or $p_1^{(d_1-1)/2^{m+1}} \cdots p_{m+2}^{(d_{m+2}-1)/2^{m+1}}$, are not $2^m k$-T_0T^*-perfect numbers.*

Proof. According to Equation (3.1), we must solve

$$2^r \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 2^{m+2}k \tag{3.5}$$

Next, we will examine the different cases beginning from $r = 1$.

- $r = 1$: (3.5) becomes $2 \cdot (\alpha_1 + 1) = 2^{m+2}k$; it gives $\alpha_1 = 2^{m+1}k - 1$.
- $2 \leq r < m+2$: (3.5) becomes $2^r \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 2^{m+2}k$; it gives $(\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 2^{m+2-r}k$; which is impossible since $(\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1)$ is odd whereas $2^{m+2-r}k$ for $2 \leq r < m+2$ is even.
- $r = m+2$: (3.5) becomes $(\alpha_1 \cdot 2^{m+1} + 1) \cdots (\alpha_{m+2} \cdot 2^{m+1} + 1) = k$; if $k = 1$, then it is not possible since $(\alpha_1 \cdot 2^{m+1} + 1) \cdots (\alpha_{m+2} \cdot 2^{m+1} + 1) > 2^{(m+2)(m+1)} > 1$ for $m \in \mathbb{N}^*$; so, $k > 1$; in this case, let $k = q_1^{\beta_1} \cdots q_s^{\beta_s}$ be the prime decomposition of the odd integer k where q_i is an odd prime for all $i \in [\![1, s]\!]$ with $s \in \mathbb{N}^*$ and $\beta_i \in \mathbb{N}^*$ for all $i \in [\![1, s]\!]$ with $s \in \mathbb{N}^*$; then according to the fundamental theorem of arithmetic, for all $i \in [\![1, m+2]\!]$, we have $2\alpha_i + 1 = q_1^{\gamma_{i,1}} \cdots q_s^{\gamma_{i,s}}$ and $\beta_i = \sum_{j=1}^{s} \gamma_{i,j}$; since for all $i \in [\![1, m+2]\!]$, $\alpha_i \geq 1$, for given $i \in [\![1, m+2]\!]$, there exists at least one integer among the integers $\gamma_{i,j}$ $(j = 1, \ldots, s)$ which is greater than 1; we must have $2 < q_1^{\gamma_{i,1}} \cdots q_s^{\gamma_{i,s}} < k$; in the following, for all $i \in [\![1, m+2]\!]$, we set $d_i = q_1^{\gamma_{i,1}} \cdots q_s^{\gamma_{i,s}}$; accordingly, for all $i \in [\![1, m+2]\!]$, d_i is a positive proper divisor of k; we have also $k = d_1 \cdots d_{m+2}$ and for all $i \in [\![1, m+2]\!]$, $\alpha_i \cdot 2^{m+1} + 1 = d_i$; provided for all $i \in [\![1, m+2]\!]$, $d_i - 1$ is divisible by 2^{m+1}, it gives $\alpha_i = (d_i - 1)/2^{m+1}$.
- $r > m+2$: (3.5) becomes $2^{r-(m+2)} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = k$; which is impossible since $2^{r-(m+2)} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1)$ for $r > m+2$ is even whereas k is assumed to be odd.

So, only the case where $r = 1$ for odd positive integer k and for nonzero positive integer m and the case where $r = m+2$ for odd positive integer k which has a multiplicative partition whose members are congruent to 1 modulo 2^{m+1} for $m \in \mathbb{N}^*$, are possible. This completes the proof. $\square$

Corollary 27. *Let m be a nonzero positive integer. All $2^m(2^{m+1}+1)^{m+2}$-T_0T^*-perfect numbers have the form $p_1^{2^{m+1}(2^{m+1}+1)^{m+2}-1}$ or $p_1 \cdots p_{m+2}$.*

Corollary 27 follows directly from Theorem 26 by taking $k = (2^{m+1}+1)^{m+2}$ for $m \in \mathbb{N}^*$. Notice that k is odd here.

Theorem 28. *Let p be a prime, with $2^p - 1$ being a Mersenne prime. Then $2^{p-1} \cdot (2^p - 1)$ is the only even perfect number which is a $3 \cdot (2p-1)$-T_0T^*-perfect number.*

Proof. According to (3.1), we must solve the equation

$$2^r \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 12 \cdot (2p - 1) \tag{3.6}$$

where p is a prime such that $2^p - 1$ is a Mersenne prime.

First we find the possible forms of all $3 \cdot (2p-1)$-T_0T^*-perfect numbers. In the following, we will investigate the different cases beginning from $r = 1$.

- $r = 1$: (3.6) becomes $2 \cdot (\alpha_1 + 1) = 12 \cdot (2p - 1)$; it gives $\alpha_1 = 12p - 5$.
- $r = 2$: (3.6) becomes $4 \cdot (2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 12 \cdot (2p - 1)$ which is equivalent to $(2\alpha_1 + 1) \cdot (2\alpha_2 + 1) = 3 \cdot (2p - 1)$; a trivial solution is given by $\alpha_1 = 1$ and $\alpha_2 = p - 1$ (notice that a particular subcase of this trivial solution is $\alpha_1 = \alpha_2 = 1$ which is consistent with (3.6) only if $p = 2$); more generally, we must have either $2\alpha_1 + 1 \equiv 0 \pmod 3$ or $2\alpha_2 + 1 \equiv 0 \pmod 3$; without loss of generality, let us take $2\alpha_2 + 1 = 3k$ with k an odd positive integer which divides $2p - 1$ so that (3.6) is satisfied; then it is not difficult to see that $2\alpha_1 + 1 = \frac{2p-1}{k}$ and we obtain for α_1 and α_2, the parametrisation $\alpha_1(k) = \frac{2p-(k+1)}{2k}$ and $\alpha_2(k) = \frac{3k-1}{2}$; notice that the subcase $\alpha_1 = p - 1$ and $\alpha_2 = 1$ is recovered when $k = 1$.
- $r \geq 3$: (3.6) becomes $2^{r-2} \cdot (\alpha_1 \cdot 2^{r-1} + 1) \cdots (\alpha_r \cdot 2^{r-1} + 1) = 3 \cdot (2p - 1)$ which is not possible since $2^{r-2} \not| 3 \cdot (2p - 1)$ for $r \geq 3$.

So, only the case where $r = 1$ for which $n = p_1^{12p-5}$ and the case where $r = 2$ for which $n = p_1^{\frac{2p-(k+1)}{2k}} \cdot p_2^{\frac{3k-1}{2}}$ such that k is an odd positive integer which divides $2p - 1$ are valid.

In particular, when $r = 2$, if $k = 1$, then we get $n = p_1^{p-1} \cdot p_2$. Conversely, when $r = 2$, if $\alpha_1(k) = p-1$ and $\alpha_2(k) = 1$, then using the parametrisation given above for α_1 and α_2 when $r = 2$, we have $\dfrac{2p = (k+1)}{2k} = p - 1$ and $\dfrac{3k-1}{2} = 1$. This implies that $k = 1$. Therefore, when $r = 2$

$$n = p_1^{p-1} \cdot p_2 \Leftrightarrow \begin{cases} \alpha_1(k) = p - 1; \\ \alpha_2(k) = 1. \end{cases} \Leftrightarrow k = 1.$$

Secondly we see if there exists an even perfect number which is $3 \cdot (2p-1)$-T_0T^*-perfect number. According to the Euclid-Euler theorem, an even perfect number takes the form $2^{q-1} \cdot (2^q - 1)$, where q is a prime such that $2^q - 1$ is a Mersenne prime. Accordingly, an even perfect number corresponds to the case where $r = 2$ which was investigated above. So, $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2}$ is an even perfect number if and only if

$$p_1^{\alpha_1} \cdot p_2^{\alpha_2} = 2^{q-1} \cdot (2^q - 1).$$

Since the prime factorization of an integer is unique up to the order of the prime factors, without loss of generality, taking $1 \leq \alpha_2 \leq \alpha_1$, since 2 and $2^q - 1$ are prime, we will have $p_1 = 2$, $p_2 = 2^q - 1$ with $\alpha_1(k) = q - 1$ and $\alpha_2(k) = 1$. Using the parametrization introduced above when $r = 2$, this system is equivalent to $\dfrac{2p - (k+1)}{2k} = q - 1$ and $\dfrac{3k-1}{2} = 1$. Solving this system for (k, p), it results that $k = 1$ and $p = q$. Since when $r = 2$

$$n = p_1^{p-1} \cdot p_2 \Leftrightarrow \begin{cases} \alpha_1(k) = p-1; \\ \alpha_2(k) = 1. \end{cases} \Leftrightarrow k = 1.$$

Let us finish the proof by studying the subcase where both $2\alpha_1+1$ and $2\alpha_2+1$ are divisible by 3. If both $2\alpha_1+1$ and $2\alpha_2+1$ are divisible by 3, then there exist two odd positive integers k_1 and k_2 such that $2\alpha_1+1 = 3k_1$ and $2\alpha_2+1 = 3k_2$. It gives $\alpha_1 = \dfrac{3k_1-1}{2}$ and $\alpha_2 = \dfrac{3k_2-1}{2}$. Accordingly, the equation $(2\alpha_1+1)\cdot(2\alpha_2+1) = 3\cdot(2p-1)$ becomes $9k_1k_2 = 3\cdot(2p-1)$. After simplification, we get $3k_1k_2 = 2p-1$.

If n is an even perfect number, then $n = 2^{p-1}\cdot(2^p-1)$ and without loss of generality, (since for $r = 2$, $n = p_1^{\alpha_1}\cdot p_2^{\alpha_2}$ and since 2^p-1 is a Mersenne prime) we can set $p_1 = 2$ and $p_2 = 2^p-1$. Then we have

$$\alpha_1 = \frac{3k_1-1}{2} = p-1 \Rightarrow 3k_1 = 2p-1$$

and

$$\alpha_2 = \frac{3k_2-1}{2} = 1 \Rightarrow k_2 = 1.$$

It is consistent with the equation $3k_1k_2 = 2p-1$ and the fact that k_1, k_2 are odd positive integers. It means that $n = p_1^{p-1}p_2$ as expected. In fact, using the parametrisation

$$\alpha_1(k) = \frac{2p-(k+1)}{2k}$$
$$\alpha_2(k) = \frac{3k-1}{2}$$

and taking by identification $k = k_2$ and $k_1 = \frac{2p-1}{3k}$, because $3k_1k_2 = 2p-1$ when both $2\alpha_1+1$ and $2\alpha_2+1$ are divisible by 3, this subcase can be recovered. In particular, if $k = 1$, then it is consistent with the statement of Theorem 28. Notice that if both $2\alpha_1+1$ and $2\alpha_2+1$ are divisible by 3, then $p \equiv 2 \pmod 3$. Indeed, since k_1 is an odd positive integer, there exists an integer m_1 such that $k_1 = 2m_1+1$. Or, $3k_1 = 2p-1$. So, $2p = 3k_1+1 = 6m_1+4$. It gives $p = 3m_1+2$. Notice also that m_1 shall be an odd positive integer since p is necessarily prime so that 2^p-1 be a Mersenne prime.

We conclude that $2^{p-1}\cdot(2^p-1)$ is the only even perfect number which is a $3\cdot(2p-1)$-T_0T^*-perfect number. $\square$

In this section, we have presented results only on the canonical representation of k-T_0T^* perfect numbers. Other techniques may be used to derive results on the bounds of such numbers, but here we do not proceed in that direction.

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