The Annals of Mathematical Statistics
1971, Vol. 42, No. 5, 1562–1568

SOME RESULTS ON INVARIANT SETS FOR TRANSLATION PARAMETER FAMILY OF PROBABILITY MEASURES—I

BY P. K. PATHAK[1] AND NEIL W. RICKERT[2]

University of New Mexico and University of Illinois, Chicago

1. Preliminaries. Let μ be a given probability measure on $(\mathbf{X}, \mathbf{B})$, where $\mathbf{X}$ is some finite dimensional Euclidean space and $\mathbf{B}$ is the class of Borel sets on $\mathbf{X}$. For each $\theta \in \mathbf{X}$, let $\mu_\theta(A) = \mu(A-\theta)$, where $A \in \mathbf{B}$ and $A-\theta = \{x: x+\theta \in A\}$. A set '$A$' is called *$\mu$-invariant* if $\mu_\theta(A) = \mu(A) \ \ \forall\, \theta \in \mathbf{X}$. For example the null set $\varnothing$ and the whole space $\mathbf{X}$ are trivially μ-invariant. The class of all μ-invariant sets is denoted by $\mathbf{A}(\mu)$. A set 'A' is called "non-trivial" if $0 < \mu(A) < 1$. That "non-trivial" μ-invariant sets exist is seen by noting that if μ assigns probability $\frac{1}{2}$ each to $\{0\}$ and $\{1\}$, then A is μ-invariant if and only if $A' = A+1$, e.g., $A = \bigcup_{-\infty}^{\infty} (2n, 2n+1]$ is μ-invariant with $\mu(A) = \frac{1}{2}$. It is easily seen that $\mathbf{A}(\mu)$ is a monotone class, is closed for complementation and disjoint unions, and is not necessarily a σ-algebra. The probability measure μ is called weakly incomplete (weakly-complete) if it has (or does not have) non-trivial μ-invariant sets.

The results we present here have originated from a paper of Basu and Ghosh (1969) on μ-invariant sets. Our main object is to make a careful study of some of the conjectures contained in their paper. A brief account of the results contained in the paper is as follows:

(i) Let $\hat{\mu}(t) = \int \exp\{i(t, x)\} d\mu(x)$ denote the Fourier transform of μ. Basu and Ghosh show that if $S(\mu) = \{t: \hat{\mu}(t) = 0\}$ consists of finitely many elements, then μ is weakly complete. We shall show that if $S(\mu)$ is compact, or contained in a certain coset of a closed subgroup, then μ is weakly complete.

(ii) Now let μ be a probability measure on $(\mathbf{R}^1, \mathbf{B})$ and suppose that $S(\mu) = \{\pm 1, \pm 2, \cdots\}$. In this case Basu and Ghosh show that $\mathbf{A}(\mu)$ consists of all Borel sets of period 2π, i.e. a Borel set $A \in \mathbf{A}(\mu)$ iff $A+2\pi = A$ a.e. We strengthen this result by showing that the same assertion concerning $\mathbf{A}(\mu)$ is true if $S(\mu) = \{\pm 1, \pm 2, \cdots\} \cup K \cup J$, where K is compact and J is contained in a certain coset of a closed subgroup. We thus prove a conjecture mentioned in Basu and Ghosh ((1969) Theorem 8, page 168). Basu and Blum, in a personal communication, have noted that, in this case, μ must necessarily be absolutely continuous.

(iii) Let $A \in \mathbf{A}(\mu)$. Then $\forall\, \theta \in \mathbf{X}$, $A-\theta \in \mathbf{A}(\mu)$. Thus $\mathbf{A}(\mu)$ is translation invariant. If $\mathbf{A}(\mu)$ is also a σ-algebra, then $\mathbf{A}(\mu)$ becomes a translation invariant σ-algebra. Now let H be a closed subgroup of $\mathbf{X}$ and $\mathbf{B}_H$ be the class of Borel sets E with the property that $E+h = E$ for every $h \in H$. Clearly $\mathbf{B}_H$ is translation invariant. We show that every translation invariant σ-algebra is of the $\mathbf{B}_H$ kind, i.e. the σ-algebra is the σ-algebra of Borel sets that are invariant for some closed subgroup H. An immedi-

Received August 6, 1970.

[1] Research supported by the National Science Foundation Grant GP 14786.

[2] Research supported by the National Science Foundation Grant GP 127–60–346–32–54–327.

ate consequence of this result is that if μ is a probability measure on $(\mathbf{R}^1, \mathbf{B})$ and if $A(\mu)$ is a σ-algebra, then $\mathbf{A}(\mu)$ consists of all periodic Borel sets of some period. We also furnish an example to show that μ-invariant sets in general are not necessarily periodic.

(iv) Let E be a set on the plane. Let μ be the normalized restriction of the Lebesgue measure on E. Basu and Ghosh noted that μ is weakly incomplete if E is a parallelogram, and raised as an open problem the question of weak incompleteness of μ when E is, for example, a disk or a triangle. In this paper we show that, at least in one case, μ is weakly incomplete when E is a triangle.

Before we present these results, it is considered worthwhile to describe a few useful results from harmonic analysis. Consider $L^1(\mathbf{X}, \mathbf{B}, \lambda)$, where λ denotes the Lebesgue measure. For every $f \in L^1(\mathbf{X}, \mathbf{B}, \lambda)$, or simply L^1, let $I(f)$ denote the ideal generated by f and $S(f) = \{t : \hat{f}(t) = 0\}$ where $\hat{f}$ denotes the Fourier transform of f. The following theorems are then well known (see, e.g. Rudin (1962), page 160).

THEOREM 1.1. *If $f, g \in L^1$ and $S(f) \subset S(g)$ and if the intersection of the boundaries of $S(f)$ and $S(g)$ is countable, then $g \in I(f)$.*

THEOREM 1.2. *Let K be a compact set. Then there exists a non-trivial bounded function $h \in L^1$ such that $\hat{h}(x) = 1$ if $x \in K$.*

THEOREM 1.3. *Let $f \in L^1$, let $u \in L^\infty$ and suppose that $u * f = 0$, where $*$ denotes the convolution operation. Then $u * g = 0$ $\forall g \in I(f)$.*

THEOREM 1.4 (Basu–Ghosh). *Let $f \in L^1$ and $S(f) = \{\pm c, \pm 2c, \cdots\}$. Let $g \in L^\infty$ and suppose that $g * f = \alpha$, where α is a constant. Then $g(x+2\pi/c) = g(x)$* a.e.

THEOREM 1.5 (Basu–Ghosh). *Let μ and ν be two probability measures. Suppose that $A \in \mathbf{A}(\mu)$. Then $A \in \mathbf{A}(\mu * \nu)$ and $\mu * \nu(A+\theta) = \mu(A)$ $\forall \theta \in \mathbf{X}$.*

2. A useful theorem. The following theorem and its corollaries will be found particularly useful in our work.

THEOREM 2.1. *Let $f \in L^1$ and $S(f)$ be compact. Let $g \in L^\infty$ and suppose g assumes finitely many values. Let $f * g = 0$. Then $g = 0$* a.e.

PROOF. Let K be a compact set such that int $(K) \supset S(f)$. Let h be a non-trivial bounded function in L^1 such that $\hat{h}(x) = 1$ $\forall x \in K$. Let $k \in L^1$ and consider $l = k * (1-h)$ so that $S(l) \supset K \supset \text{int}(K) \supset S(f)$. From Theorem 1.1 it follows that $l \in I(f)$. Consequently $g * l = 0$ so that $(g - g * h) * k = 0$ $\forall k \in L^1$. Hence $g = g * h$ a.e. Thus g and $g * h$ have the same essential range. Since $\mathbf{X}$ is connected and $g * h$ continuous, it follows that the essential range of $g * h$, and consequently of g, is a connected set. The hypothesis that g assumes finitely many values and $g * f = 0$ now imply that $g = 0$ a.e. □

COROLLARY 2.1.1. *Let $f \in L^1(\mathbf{R}^1, \mathbf{B}, \lambda)$* and *$S(f) \subset \{\pm c, \pm 2c, \cdots\} \cup K$, where K is a compact set. Let $g \in L^\infty$ and suppose that g assumes finitely many values. Let $g * f = 0$. Then $g(x+2\pi/c) = g(x)$* a.e.

PROOF. Let $u(x) = 1$ if $0 \leqq x \leqq 2\pi/c$ and $= 0$ otherwise. Let $h \in L^1$ be such that $S(h)$ is compact and int $[S(h)] \supset K$. Clearly $S(u * h) \supset \{\pm c, \pm 2c, \cdots\} \cup K$; and from Theorem 1.1, $h * u \in I(f)$. Consequently, $g * h * u = 0$. Since $g * h \in L^\infty$, it follows from Theorem 1.4 that $g * h(x+2\pi/c) = g * h(x)$. Therefore $(g_{2\pi/c}-g) * h = 0$, where $g_{2\pi/c}(x) = g(x+2\pi/c)$. Since $S(h)$ is compact and $(g_{2\pi/c}-g)$ assumes only finitely many values, we have, from Theorem 2.1, $g_{2\pi/c}(x)-g(x) = g(x+2\pi/c) - g(x) = 0$ a.e. □

COROLLARY 2.1.2. *Let* $f \in L^1(\mathbf{R}^1, \mathbf{B}, \lambda)$ *and* $S(f) \subset cZ+b$, where $cZ = \{0, \pm c, \pm 2c, \cdots\}$ *and* $2b \in' cZ$. *Let* $g \in L^\infty$ *and suppose that* g *is real-valued. Let* $g * f = 0$. *Then* $g(x) = 0$ a.e.

PROOF. The equation $g * f = 0$ implies that

$$g * f_1 = \int g(y-x) \exp\{ib(y-x)\} f_1(x)dx = 0$$

where $f_1(x) = f(x) \exp\{ibx\}$.

Since $S(f_1) = S(f)-b \subset cZ$, it follows from Theorem 1.4 that $g(x+2\pi/c) \exp\{ib(x+2\pi/c)\} = g(x) \exp\{ibx\}$ a.e. Thus $g(x+2\pi/c) \exp\{i\, 2\pi b/c\} = g(x)$ a.e. Since the left side of this last equation is complex and the right side real, it follows that $g(x) = 0$ a.e. □

The following corollaries can be proved in a similar fashion.

COROLLARY 2.1.3. *Let* $f \in L^1(\mathbf{R}^1, \mathbf{B}, \lambda)$ *and* $S(f) \subset H \cup J \cup K$, *where* $H = aZ$, $J = cZ+b$ *with* $2b \in' cZ$ *and* K *a compact set. Let* $g \in L^\infty$ *and suppose that* g *assumes finitely many real values. Let* $g * f = 0$. *Then* $g(x+2\pi/a) = g(x)$ a.e.

3. Main results. We now state and prove the results outlined in Section 1.

3.1. *Existence of* μ-*invariant sets.* Given a p.m. μ, it is perhaps natural to ask if μ admits non-trivial μ-invariant sets. An answer to this question, as the following theorem shows, depends more on the set $S(\mu)$ rather than the p.m. μ itself.

THEOREM 3.1. *Let* μ_1 and μ_2 *be two absolutely continuous* p.m.'*s with* $S(\mu_1) = S(\mu_2)$. *If the boundary of* $S(\mu_i)$ *is countable, then* $\mathbf{A}(\mu_1) = \mathbf{A}(\mu_2)$.

PROOF. Let $f_i = d\mu_i/d\lambda$. Let $A \in \mathbf{A}(\mu_1)$. The proof follows easily on noting that $A \in \mathbf{A}(\mu_1)$ iff $[I_{-A}-c] * f_1 = 0$, where $c = \mu_1(A)$. Since $S(f_1) = S(f_2)$ $[\equiv S(\mu_1) = S(\mu_2)]$ and the boundary of $S(f_1)$ is countable, it follows from Theorem 1.1 that $f_2 \in I(f_1)$ so that $[I_{-A}-c] * f_2 = 0$. Consequently, $A \in \mathbf{A}(\mu_2)$ and $\mu_2(A) = \mu_1(A) = c$. □

COROLLARY 3.1.1. *If, in the above theorem* μ_2 *is absolutely continuous and* μ_1 *any* p.m., *then* $\mathbf{A}(\mu_1) \subset \mathbf{A}(\mu_2)$.

PROOF. The above theorem yields $\mathbf{A}(\mu_1 * \mu_2) = \mathbf{A}(\mu_2)$. By virtue of Theorem 1.5, we have $\mathbf{A}(\mu_1) \subset \mathbf{A}(\mu_1 * \mu_2) = \mathbf{A}(\mu_2)$.

That the strict inequality $\mathbf{A}(\mu_1) \subset A(\mu_2)$ does indeed hold is seen by observing that if $A \in \mathbf{A}(\mu_1)$, then every B, with $\lambda(A \Delta B) = 0$, belongs to $\mathbf{A}(\mu_2)$. The converse, however, is not true.

Basu and Ghosh show that if μ is a p.m., $S(\mu)$ consists of finitely many elements, then μ is weakly complete. Our next result is a strengthening of this theorem. We show that if $S(\mu)$ is compact, then μ is weakly complete.

THEOREM 3.2. *Let μ be a* p.m. *and suppose that $S(\mu)$ is compact. Then μ is weakly complete.*

PROOF. First let $\mu \ll \lambda$ and $f = d\mu/d\lambda$. Let $A \in \mathbf{A}(\mu)$ with $\mu(A) = c$. Then $(I_{-A} - c) * f = 0$. Since $S(f)$ is compact, it follows from Theorem 2.1 that $I_{-A} = c$ a.e. Hence $c = 0$ or 1. In the general case let v be an absolutely continuous p.m. with $S(v) = \varnothing$. Then $A \in \mathbf{A}(\mu)$ implies that $A \in \mathbf{A}(\mu * v)$ with $\mu(A) = \mu * v(A)$. But $\mu * v$ is absolutely continuous with $S(\mu * v)$ [$\equiv S(\mu)$] compact. Consequently $\mu(A) = \mu * v(A) = 0$ or 1. Thus μ is weakly complete. □

THEOREM 3.3. *Let μ be a* p.m. *on* $(\mathbf{R}^1, \mathbf{B})$ *and suppose that $S(\mu) \subset J \cup K$, where $J = cZ + b$ with $2b \in' cZ$ and K compact. Then μ is weakly complete.*

PROOF. This is clear from Corollary 2.1.3.

THEOREM 3.4. *Let μ be a* p.m. *on* $(\mathbf{R}^1, \mathbf{B})$ *and suppose that $S(\mu) \subset H \cup J \cup K$, where $H = aZ$, $J = cZ + b$ with $2b \in' cZ$ and K a compact set. Let $A \in \mathbf{A}(\mu)$. Then* $I_A(x + 2\pi/a) = I_A(x)$ a.e.

PROOF. This is clear from Corollary 2.1.4.

It is worthwhile to point out that the above theorem originated in an attempt to establish a conjecture made by the referee of the Basu–Ghosh paper ((1969) page 168). The theorem shows that the referee's conjecture is indeed true.

3.2. *Periodicity of μ-invariant sets.* In all our standard examples on $(\mathbf{R}^1, \mathbf{B})$ we noted that all μ-invariant sets are periodic sets of some period. Basu and Ghosh established that if $\mathbf{A}(\mu)$ is a separable σ-field, then every $A \in \mathbf{A}(\mu)$ is periodic. We were then tempted to conjecture that the class $\mathbf{A}(\mu)$ consists entirely of sets that are periodic of some period. This conjecture is valid with certain reservations. We first present an example to show that μ-invariant sets in general need not be periodic.

3.3. *An example.* Let $g(x) = x - [x]$, where $[x]$ denotes the greatest integer less than x. Let a be a positive real number. Define $h(x) = g(x) + g(ax) - (g(1+a)x)$. It is easy to verify that h takes only the values 0 and 1, and hence is the indicator function of a set. Now let $\mu = \mu_1 * \mu_2 * \mu_3$, where μ_1 is the uniform p.m. on [0, 1], μ_2 the uniform p.m. on $[0, 1/a]$ and μ_3 the uniform p.m. on $[0, 1/(1+a)]$. Since $g(x)$ is periodic of period 1, $\forall\, \theta$, $\int_0^1 g(x+\theta)dx = 1/2$ so that $(g - 1/2) * \mu_1 = 0$. Similarly $(g(ax) - 1/2) * \mu_2 = 0$ and $(g(1+a)x) - 1/2) * \mu_3 = 0$. Consequently $(h(x) - 1/2) * \mu = 0$. It thus follows that the set A with $I_{-A}(x) = h(x)$ is a μ-invariant set.

If a is irrational then A is not periodic.

This example shows that on $(\mathbf{R}^1, \mathbf{B})$ not all μ-invariant sets can be periodic. It therefore seems natural to investigate conditions under which the class $\mathbf{A}(\mu)$ consists of periodic sets. As an attempt in this direction, we show here that if $\mathbf{A}(\mu)$ is a

σ-algebra, then $\mathbf{A}(\mu)$ consists entirely of periodic sets of a given period. We shall need the following definitions and results in this connection.

DEFINITION 3.1. Let m be a positive measure on $(\mathbf{X}, \mathbf{B})$. Let $\mathbf{B}_1$ be a sub-σ-field. An element f on $L^\infty(\mathbf{X}, \mathbf{B}, m)$ will be called $\mathbf{B}_1$-measurable if there is a $\mathbf{B}_1$-measurable function in the equivalence class determined by f.

The $\mathbf{B}_1$-measurable elements of L^∞ evidently form a uniformly closed subspace of L^∞, and this subspace obviously determines the measure algebra of $\mathbf{B}_1$.

DEFINITION 3.2. A function f in $L^1(\mathbf{X}, \mathbf{B}, m)$ will be said to have the conditional expectation zero iff, $\forall\ B \in \mathbf{B}_1$, $\int I_B f\, dm = 0$.

The following lemma is now easy to establish.

LEMMA 3.1. *Let m be σ-finite. A member g of L^∞ is $\mathbf{B}_1$-measurable* iff $\int fg\, dm = 0$ *for every f in L^1 with conditional expectation zero.*

PROOF. We first assume that m is a probability measure and denote by 'E' the expectation operator with respect to m. Now if g is $\mathbf{B}_1$-measurable and f has conditional expectation zero, we have $E[fg] = E[gE[f \mid \mathbf{B}_1]] = 0$. Now suppose that $E[f \cdot g] = 0$ for every f with $E[f \mid \mathbf{B}_1] = 0$. Let $h \in L^1$. Let $f = h - E[h \mid \mathbf{B}_1]$. Then $E[f \cdot g] = 0$ by hypothesis. Also, by what we have already seen, $E[f \cdot \hat{g}] = 0$ where $\hat{g} = E[g \mid \mathbf{B}_1]$. Thus $E[f \cdot (g - \hat{g})] = 0$ so that $E[h(g-\hat{g})] = E[\hat{h}(g-\hat{g})] = 0$. Therefore $E[h(g-\hat{g})] = 0\ \ \forall\ h \in L^1$. Hence $g = \hat{g}$ a.e. Consequently g is $\mathbf{B}_1$-measurable. This completes the proof when m is a probability measure. For the general case we can find a probability measure p equivalent to m. A function f is in $L^1(m)$ iff $f \cdot (dm/dp)$ is in $L^1(p)$. Likewise f has conditional expectation zero with respect to m iff $f \cdot (dm/dp)$ has conditional expectation zero with respect to p. It is now a simple matter to translate the proof for a probability measure to the general case.

LEMMA 3.2. *The space of $\mathbf{B}_1$-measurable elements of L^∞ is a weak*-closed subspace of L^∞.*

PROOF. This is clear from Lemma 3.1.

THEOREM 3.5. *Let $\mathbf{X}$ be a finite dimensional Euclidean space. Let $\mathbf{B}$ be the class of Borel sets on $\mathbf{X}$. Let $\mathbf{B}_1$ be a translation invariant σ-field of Borel sets. Then there is a closed subgroup H such that $\mathbf{B}_1$ is, modulo null sets, the σ-field of H-invariant Borel sets* (i.e. $A \in \mathbf{B}_1$ iff $\forall\ h \in H,\ A + h = A$ a.e.).

PROOF. Let U be the subspace of L^∞ consisting of $\mathbf{B}_1$-measurable functions and let V be the continuous functions in U. It is clear that U and V are uniformly closed subspaces. By Lemma 3.2, U is a weak*-closed subspace of L^∞. Let f be an element of L^1 and let $g \in U$. It can then be seen that, by writing the integral in full and interchanging the order of integration, $\int (f * g(x))h(x) = 0$ for every h in L^1 with conditional expectation zero. By Lemma 3.1, $f * g$ is in U. However $f * g$ is continuous and so in V. Also we can approximate g by functions of the form $f * g$, in the weak*-topology, by allowing f to be an element of an approximate identity for $L^1(\mathbf{X})$. It thus follows that V is a weak*-dense subspace of U.

Let H be the subset of X consisting of all y such that $f(x+y) = f(x)$ for *all* x and for *all* f in V. H is clearly a closed subgroup of X. If $a-b \in H$, it follows from the definition of H that $f(a) = f(b)$ for all $f \in V$. Conversely if $f(a) = f(b)$ for all $f \in V$, we have $f(x+a-b) = f(x)$ for all x and f since $f_{x-b}(\cdot) = f(\cdot + x - b)$ is also an element of V. Thus $f(a) = f(b)$ for all $f \in V$ iff $(a-b) \in H$. It is now evident that V is a self-adjoint algebra which separates points in $X \mid H$.

Now let g be a bounded continuous function satisfying the condition $g(x) = g(y)$ whenever $x-y \in H$. Let $h_1, \cdots, h_n$ be in L^1 and $\varepsilon > 0$ be given. Let M be the supremum of $|g(x)|$ and $\int |h_i|$. Choose a compact set K such that $\int_{K'} |h_i| < \varepsilon/8m$. Since V is an algebra and g satisfies the relation $g(x) = g(y)$ whenever $(x-y) \in H$, it follows from the Stone–Weierstrass theorem that we can find an f in V such that $|f(x) - g(x)| < \varepsilon/4M$ if $x \in K$. Let M_1 be the supremum of $|f|$. By the classical Weierstrass theorem there is a polynomial P such that $|t - P(t)| < \varepsilon/4M$ if $|t| < 2M$, and $|P(t)| < 3M$ if $|t| < M_1$. Set $f_1 = P(f)$. Then $|f_1 - g(x)| \leqq 4M$ for all x. But now $|\int h_i(x)(g(x) - f_1(x))| < \varepsilon$ for all $i = 1, \cdots, n$. This means that g is in the weak*-closure of V so g is in U and hence g is in V. It follows that V consists of all bounded continuous functions which are invariant under H, and that U, the weak*-closure of V, consists of all L^∞ functions which are measurable with respect to the σ-field of H-invariant Borel sets. □

COROLLARY 3.5.1. *Let μ be a probability measure on* $(\mathbf{R}^1, \mathbf{B})$ *and suppose that* $\mathbf{A}(\mu)$ *is a "non-trivial" sub-σ-field of* $\mathbf{B}$. *Then there exists a 'c' such that* $A \in \mathbf{A}(\mu)$ iff $A + c = A$ a.e.

PROOF. This is clear from the above theorem on noting that $\mathbf{A}(\mu)$ is translation invariant and a non-trivial closed subgroup H of $\mathbf{R}^1$ is of the form $H = \{nc : n = 0, \pm 1, \cdots\}$ for some c. Consequently if $\mathbf{A}(\mu)$ consists of sets invariant with respect to H, then $A \in \mathbf{A}(\mu)$ implies that $A + c = A$ a.e. The converse part is trivial. □

REMARK. In our proof of the above corollary we have tacitly excluded from consideration the case when H is a trivial subgroup, i.e. $H = \{0\}$ or $H = \mathbf{R}^1$. It can be easily seen that $\mathbf{A}(\mu)$ coincides with $\mathbf{B}$ when $H = \{0\}$ and consists solely of sets that are (within sets of Lebesgue measure zero) equal to the null set or the whole space when $H = \mathbf{R}^1$.

3.4. *Weak incompleteness of a* p.m. Let E be a triangle on the plane and μ the normalized restriction of the Lebesgue measure to E. Is μ weakly incomplete? This is one of the questions raised in the paper of Basu–Ghosh ((1969) page 173). We provide here an affirmative answer to this question. For simplicity we consider the probability measure with the following density $f(x, y) = 2$ if $x > 0$, $y > 0$ and $x + y \leqq 1$, and $= 0$ otherwise. The characteristic function of f is $\hat{f}(s, t) = 2[\{\exp(is) - 1\}/st - \{\exp(is) - \exp(it)\}/(s-t)t]$ so that

$$S(f) = \{(2m\pi, 2n\pi) : m \neq n = \pm 1, \pm 2, \cdots\}.$$

That f does indeed have non-trivial measure invariant sets can be proved as follows. Let (X, Y) denote random variables with the joint density function f. Let g denote the density function of $(X+2Y)$. Then $S(g) \supset \{2n\pi: n = \pm 1, \pm 2, \cdots\}$. Thus every Borel set of period 1 on $\mathbf{R}^1$ is measure-invariant with respect to the measure induced by $X+2Y$. Let $B^* = \{(x, y): x+2y \in B\}$, where B is a Borel set of unit period. It is now easy to see that B^* is measure-invariant with respect to the density function f. Hence the measure induced by f is weakly incomplete.

REFERENCES

BASU, D. and GHOSH, J. K. (1969). Invariant sets for translation-parameter families of measures. *Ann. Math. Statist.* **40** 162–174.

RUDIN, H. (1962). *Fourier Analysis on Groups*. Interscience, New York.