

SOME RESULTS ON $(k, \mu)'$ -ALMOST KENMOTSU MANIFOLDS *

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Abstract. In this paper, we study the quasi-conformal curvature tensor \tilde{C} and projective curvature tensor P on a $(k, \mu)'$ -almost Kenmotsu manifold M^{2n+1} of dimension greater than 3. We obtain that if M^{2n+1} is non-Kenmotsu and satisfies $R \cdot \tilde{C} = 0$ or $P \cdot P = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Keywords: Almost Kenmotsu manifold, $(k, \mu)'$ -nullity condition, quasi-conformal curvature tensor, projective curvature tensor.

1. Introduction

In 1972, K. Kenmotsu introduced a new class of almost contact metric manifolds, nowadays known as Kenmotsu manifolds [8]. The concept of almost Kenmotsu manifolds, regarded as a generalization of Kenmotsu manifolds, was studied by Janssens and Vanhecke (see [4]). In 2007, Pitiş [7] published a book containing many systematic studies related to Kenmotsu manifolds. Some geometric properties and fundamental formulas of almost Kenmotsu manifolds were obtained by Kim and Pak [11] and Pastore et al. [5, 6]. Several authors studied almost Kenmotsu manifolds considering some curvature conditions (see [12, 13, 14]). Recently, some curvature properties of some types of almost Kenmotsu manifolds were obtained by Wang and Liu in [15, 16, 17, 18].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n + 1)$ -dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat (see [2]). For $n \geq 1$, M is locally projectively flat if and

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only if the projective curvature tensor P vanishes. Here P is defined by

$$(1.1) \quad P(X, Y)U = R(X, Y)U - \frac{1}{2n}[S(Y, U)X - S(X, U)Y]$$

for any vector fields $X, Y, U \in \mathfrak{X}(M)$, where S is the Ricci tensor of M .

The Weyl conformal curvature tensor C on a $(2n + 1)$ -dimensional manifold M is defined by [20]

$$(1.2) \quad \begin{aligned} C(X, Y)Z = & R(X, Y)Z + \frac{r}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y] \\ & - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \end{aligned}$$

for any vector fields X, Y, Z on M , where S , Q and r denote the Ricci curvature tensor, the Ricci operator with respect to the metric g and the scalar curvature, respectively. Note that the Weyl conformal curvature tensor on any three dimension Riemannian manifold vanishes.

For a $(2n + 1)$ -dimensional manifold M , the quasi-conformal curvature tensor \tilde{C} is defined by [21]

$$(1.3) \quad \begin{aligned} \tilde{C}(X, Y)Z = & aR(X, Y)Z - \frac{r}{2n+1}[\frac{a}{2n} + 2b][g(Y, Z)X - g(X, Z)Y] \\ & + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], \end{aligned}$$

where a and b are two constants. If $a = 1$ and $b = -\frac{1}{2n-1}$, then the quasi-conformal curvature tensor reduces to the Weyl conformal curvature tensor.

In this paper, we aim to extend some known results regarding the projective and quasi-conformal curvature tensor on Kenmotsu manifolds (see [1, 2, 9, 10]) to a class of almost Kenmotsu manifolds. In Section 2, we recall some basic formulas and properties of almost Kenmotsu manifolds and the notion of $(k, \mu)'$ -almost Kenmotsu manifolds. In Section 3, we introduce some properties of such manifolds used to prove our main results. In Section 4 and 5, we classify almost Kenmotsu manifolds satisfying $R \cdot \tilde{C} = 0$ and $P \cdot P = 0$, respectively.

2. Almost Kenmotsu manifolds

Let M^{2n+1} be an almost contact metric manifold of dimension $2n + 1$, equipped with an almost contact metric structure (ϕ, ξ, η, g) (see [3]) satisfying

$$(2.1) \quad \phi^2 = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \xi = 0, \quad \phi\xi = 0,$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any $X, Y \in \mathfrak{X}(M)$, where ϕ, ξ, η, g and $\mathfrak{X}(M)$ denote a $(1, 1)$ -tensor field, a vector field, a 1-form, the Riemannian metric and the Lie algebra of all differentiable vector fields on M^{2n+1} , respectively.

The fundamental 2-form Φ of an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any fields $X, Y \in \mathfrak{X}(M)$. M^{2n+1} is called an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. The almost contact metric manifold is said to be normal if the Nijenhuis tensor of ϕ is given by $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold [4].

On an almost Kenmotsu manifold M^{2n+1} , the two $(1, 1)$ -type tensor fields $l = R(\cdot, \xi)\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi\phi$ are symmetric, where R is the Riemannian curvature tensor of g and \mathcal{L} is the Lie differentiation. Then we get

$$(2.3) \quad h\xi = 0, \quad l\xi = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0.$$

We also have the following formulas presented in [5, 6]:

$$(2.4) \quad \nabla_X \xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi \xi = 0),$$

$$(2.5) \quad \phi l \phi - l = 2(h^2 - \phi^2),$$

$$(2.6) \quad tr l = S(\xi, \xi) = g(Q\xi, \xi) = -2n - tr h^2,$$

$$(2.7) \quad R(X, Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X$$

for any $X, Y \in \mathfrak{X}(M)$, where $h' = h \circ \phi$ and $S, Q, \nabla, \mathfrak{X}(M)$ denote the Ricci tensor, the Ricci operator with respect to g , the Levi-Civita connection of g and the Lie algebra of all vector fields on M^{2n+1} , respectively.

3. Some properties of $(k, \mu)'$ -almost Kenmotsu manifolds

If the characteristic vector field ξ of an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfies the $(k, \mu)'$ -nullity condition (see [6]), then it is called a $(k, \mu)'$ -almost Kenmotsu manifold. The $(k, \mu)'$ -nullity condition is defined as follows:

$$(3.1) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y]$$

for any vector fields X, Y , where both k and μ are constant on M^{2n+1} . M^{2n+1} is said to be a (k, μ) -almost manifold Kenmotsu manifold if there holds $R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]$ for any vector fields X, Y and $k, \mu \in \mathbb{R}$. A (k, μ) -almost Kenmotsu manifold satisfies $k = -1$ and $h = 0$ (see [6]). A (k, μ) -almost Kenmotsu manifold is a special case of $(k, \mu)'$ -almost Kenmotsu manifolds. Following [6], on any $(k, \mu)'$ -almost Kenmotsu manifold M^{2n+1} , we have

$$(3.2) \quad h'^2 X = -(k + 1)X + (k + 1)\eta(X)\xi$$

for any vector field $X \in \mathfrak{X}(M)$ and $\mu = -2$. From (3.2), we know that $h' = 0$ is equivalent to $k = -1$ and $h' \neq 0$ everywhere if and only if $k < -1$. Furthermore,

by (3.1) and the symmetry of the Riemannian curvature tensor R , it is easy to see that

$$(3.3) \quad R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] - 2[g(h'X, Y)\xi - \eta(Y)h'X]$$

for any $X, Y \in \mathfrak{X}(M)$. In case of $k < -1$, we denote by $[\lambda]'$ and $[-\lambda]'$ the eigenspaces of h' corresponding two eigenvalues $\lambda > 0$ and $-\lambda$, respectively. Obviously, by (3.2), we have

$$(3.4) \quad \lambda = \sqrt{-k-1} > 0.$$

Before presenting one of our main results, we give the following two lemmas.

Lemma 3.1. [6, Proposition 4.2] *Let M^{2n+1} be a $(k, \mu)'$ -almost Kenmotsu manifold such that $h' = 0$. Then, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies*

$$(3.5) \quad R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0,$$

$$(3.6) \quad R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0,$$

$$(3.7) \quad R(X_\lambda, Y_{-\lambda})Z_\lambda = (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda},$$

$$(3.8) \quad R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,$$

$$(3.9) \quad R(X_\lambda, Y_\lambda)Z_\lambda = (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$

$$(3.10) \quad R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].$$

Lemma 3.2. [18, Lemma 3.2] *Let M^{2n+1} be a $(k, \mu)'$ -almost Kenmotsu manifold such that $h' \neq 0$. Then the Ricci operator of M^{2n+1} is given by*

$$(3.11) \quad Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'.$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Proof. See the proof of [19, Lemma 3.2]. \square

4. $(k, \mu)'$ -almost Kenmotsu manifolds satisfying $R(X, Y) \cdot \tilde{C} = 0$

In this section, we consider a non-Kenmotsu $(k, \mu)'$ -almost Kenmotsu manifold M^{2n+1} satisfying the condition

$$(4.1) \quad R(X, Y) \cdot \tilde{C} = 0,$$

or equivalently

$$(4.2) \quad \begin{aligned} (R(X, Y) \cdot \tilde{C})(U, V)W &= R(X, Y)\tilde{C}(U, V)W - \tilde{C}(R(X, Y)U, V)W \\ &\quad - \tilde{C}(U, R(X, Y)V)W - \tilde{C}(U, V)R(X, Y)W \\ &= 0 \end{aligned}$$

for any $X, Y, U, V, W \in \mathfrak{X}(M)$.

From the definition of \tilde{C} (see (1.3)), we have

$$\begin{aligned}
 \tilde{C}(\xi, Y)Z &= [ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]g(Y, Z)\xi \\
 &\quad - [ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]\eta(Z)Y \\
 &\quad - (-a\mu + 2nb)g(h'Y, Z)\xi + (-a\mu + 2nb)\eta(Z)h'Y,
 \end{aligned}
 \tag{4.3}$$

$$\begin{aligned}
 \tilde{C}(\xi, Y)\xi &= [ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]\eta(Y)\xi \\
 &\quad - [ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]Y \\
 &\quad + (-a\mu + 2nb)h'Y,
 \end{aligned}
 \tag{4.4}$$

where r, a and b denote the scalar curvature and two constants, respectively. Let us denote by $A = [ak - \frac{r}{2n+1}(\frac{a}{2n} + 2b) + 2nkb - 2nb]$, $B = -A$, $D = (-a\mu + 2nb)$ and $E = -D$.

Substituting $X = U = \xi$ in (4.2) we have

$$\begin{aligned}
 (R(\xi, Y) \cdot \tilde{C})(\xi, V)W &= R(\xi, Y)\tilde{C}(\xi, V)W - \tilde{C}(R(\xi, Y)\xi, V)W \\
 &\quad - \tilde{C}(\xi, R(\xi, Y)V)W - \tilde{C}(\xi, V)R(\xi, Y)W \\
 &= 0
 \end{aligned}
 \tag{4.5}$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Making use of (3.3), (4.3) and (4.4) we calculate every term in equation (4.5) straightly. Then we have

$$\begin{aligned}
 &R(\xi, Y)\tilde{C}(\xi, V)W \\
 &= k[g(Y, \tilde{C}(\xi, V)W)\xi - \eta(\tilde{C}(\xi, V)W))Y] \\
 &\quad + \mu[g(h'Y, \tilde{C}(\xi, V)W)\xi - \eta(\tilde{C}(\xi, V)W))h'Y] \\
 &= k\{A[\eta(Y)g(V, W)\xi - \eta(W)g(Y, V)\xi] \\
 &\quad + E[\eta(Y)g(h'V, W)\xi - \eta(W)g(Y, h'V)\xi]\} \\
 &\quad - k\{A[g(V, W)Y - \eta(W)\eta(V)Y] + Eg(h'V, W)Y\} \\
 &\quad + \mu\{-A\eta(W)g(h'Y, V)\xi - E\eta(W)g(h'Y, h'V)\xi\} \\
 &\quad - \mu\{A[g(V, W)h'Y - \eta(W)\eta(V)h'Y] + Eg(h'V, W)h'Y\}.
 \end{aligned}
 \tag{4.6}$$

$$\begin{aligned}
 &\tilde{C}(R(\xi, Y)\xi, V)W \\
 &= k\eta(Y)\tilde{C}(\xi, V)W - k\tilde{C}(Y, V)W - \mu\tilde{C}(h'Y, V)W \\
 &= k\{A[\eta(Y)g(V, W)\xi - \eta(W)\eta(Y)V] + E[\eta(Y)g(h'V, W)\xi \\
 &\quad - \eta(W)\eta(Y)h'V]\} - k\tilde{C}(Y, V)W - \mu\tilde{C}(h'Y, V)W.
 \end{aligned}
 \tag{4.7}$$

$$\begin{aligned}
& \tilde{C}(\xi, R(\xi, Y)V)W \\
& = kg(Y, V)\tilde{C}(\xi, \xi)W - k\eta(V)\tilde{C}(\xi, Y)W \\
& \quad + \mu g(h'Y, V)\tilde{C}(\xi, \xi)W - \mu\eta(V)\tilde{C}(\xi, h'Y)W \\
(4.8) \quad & = -k\{A[\eta(V)g(Y, W)\xi - \eta(W)\eta(V)Y] \\
& \quad + E[\eta(V)g(h'Y, W)\xi - \eta(W)\eta(V)h'Y]\} \\
& \quad - \mu\{A[\eta(V)g(h'Y, W)\xi - \eta(W)\eta(V)h'Y]\} \\
& \quad + E[\eta(V)g(h'^2Y, W)\xi - \eta(W)\eta(V)h'^2Y]\}.
\end{aligned}$$

$$\begin{aligned}
& \tilde{C}(\xi, V)R(\xi, Y)W \\
& = kg(Y, W)\tilde{C}(\xi, V)\xi - k\eta(W)\tilde{C}(\xi, V)Y \\
& \quad + \mu g(h'Y, W)\tilde{C}(\xi, V)\xi - \mu\eta(W)\tilde{C}(\xi, V)h'Y \\
(4.9) \quad & = k\{A[g(Y, W)\eta(V)\xi - g(Y, W)V] + Dg(Y, W)h'V\} \\
& \quad - k\{A[\eta(W)g(V, Y)\xi - \eta(Y)\eta(W)V] \\
& \quad + E[\eta(W)g(h'V, Y)\xi - \eta(Y)\eta(W)h'V]\} \\
& \quad + \mu\{A[g(h'Y, W)\eta(V)\xi - g(h'Y, W)V] + Dg(h'Y, W)h'V\} \\
& \quad - \mu\{A\eta(W)g(V, h'Y)\xi + E\eta(W)g(h'V, h'Y)\xi\}
\end{aligned}$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Substituting (4.6)-(4.9) into (4.5) and using (3.2) gives

$$\begin{aligned}
& k\tilde{C}(Y, V)W + \mu\tilde{C}(h'Y, V)W - kAg(V, W)Y \\
& \quad - kEg(h'V, W)Y - \mu Ag(V, W)h'Y - \mu Eg(h'V, W)h'Y \\
(4.10) \quad & + kE\eta(V)g(h'Y, W)\xi - kE\eta(W)\eta(V)h'Y - \mu E(k+1)\eta(V)g(Y, W)\xi \\
& \quad + \mu E(k+1)\eta(V)\eta(W)Y + kAg(Y, W)V + kEg(Y, W)h'V \\
& \quad + \mu Ag(h'Y, W)V + \mu Eg(h'Y, W)h'V = 0
\end{aligned}$$

for any $Y, V, W \in \mathfrak{X}(M)$.

Substituting $Y = h'Y$ in (4.10) and using (3.2) we obtain

$$\begin{aligned}
& k\tilde{C}(h'Y, V)W - \mu(k+1)\tilde{C}(Y, V)W - kAg(V, W)h'Y \\
& \quad - kEg(h'V, W)h'Y + \mu A(k+1)g(V, W)Y + \mu E(k+1)g(h'V, W)Y \\
(4.11) \quad & - kE(k+1)\eta(V)g(Y, W)\xi + kE(k+1)\eta(V)\eta(W)Y \\
& \quad - \mu E(k+1)\eta(V)g(h'Y, W)\xi + \mu E(k+1)\eta(V)\eta(W)h'Y + kAg(h'Y, W)V \\
& \quad + kEg(h'Y, W)h'V - \mu A(k+1)g(Y, W)V - \mu E(k+1)g(Y, W)h'V = 0
\end{aligned}$$

for any $Y, V, W \in \mathfrak{X}(M)$. Subtracting μ multiple of (4.11) from k multiple of (4.10) and using $\mu = -2$ implies

$$\begin{aligned}
(4.12) \quad & (k+2)^2\tilde{C}(Y, V)W - (k+2)^2\{Ag(V, W)Y + Eg(h'V, W)Y \\
& \quad - E\eta(V)g(h'Y, W)\xi + E\eta(V)\eta(W)h'Y - Ag(Y, W)V - Eg(Y, W)h'V\} = 0
\end{aligned}$$

for any $Y, V, W \in \mathfrak{X}(M)$. Next, we assume that $Y = V = W \in [-\lambda]'$ in (1.3), where $[-\lambda]'$ is eigenspace of h' corresponding eigenvalue $-\lambda$. Thus, by applying Lemma 3.1 and Lemma 3.2, we get

$$(4.13) \quad \begin{aligned} & \tilde{C}(Y, V)W \\ &= [a(k + 2\lambda) - \frac{r}{2n + 1}(\frac{a}{2n} + 2b) + 4nb(\lambda - 1)][g(V, W)Y - g(Y, W)V] \end{aligned}$$

for any $Y, V, W \in \mathfrak{X}(M)$.

With the help of (4.13) and assuming $Y = V = W \in [-\lambda]'$, from (4.12) we get

$$(4.14) \quad 2nb(k + 2)^2(\lambda - 1 - k)[g(V, W)Y - g(Y, W)V] = 0.$$

Putting (3.4) into (4.14) we have

$$(4.15) \quad \lambda(\lambda - 1)^2(\lambda + 1)^3 = 0.$$

In view of the fact $\lambda > 0$, we obtain $\lambda = 1$ and hence $k = -2$. From [6, Corollary 4.2] and [5, Theorem 6], we know that M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Therefore we have the following:

Theorem 4.1. *If a non-Kenmotsu $(k, \mu)'$ -almost Kenmotsu manifold M^{2n+1} of dimension greater than 3 satisfies $R \cdot \tilde{C} = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Since quasi-conformally symmetric manifold ($\nabla \tilde{C} = 0$) implies $R \cdot \tilde{C} = 0$, therefore from Theorem 4.1 we state the following:

Corollary 4.1. *A quasi-conformally symmetric non-Kenmotsu $(k, \mu)'$ -almost Kenmotsu manifold M^{2n+1} ($n > 1$) is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Since $R \cdot R$ implies $R \cdot \tilde{C} = 0$, we get the following:

Corollary 4.2. *A semisymmetric non-Kenmotsu $(k, \mu)'$ -almost Kenmotsu manifold M^{2n+1} ($n > 1$) is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

The above corollary has been proved by Wang and Liu [15].

5. $(k, \mu)'$ -almost Kenmotsu manifolds satisfying $P(X, Y) \cdot P = 0$

In this section, we consider a non-Kenmotsu $(k, \mu)'$ -almost Kenmotsu manifolds M^{2n+1} satisfying the condition

$$(5.1) \quad P(X, Y) \cdot P = 0,$$

which implies

$$\begin{aligned}
 & (P(X, Y) \cdot P)(U, V)W \\
 (5.2) \quad & = P(X, Y)P(U, V)W - P(P(X, Y)U, V)W \\
 & \quad - P(U, P(X, Y)V)W - P(U, V)P(X, Y)W \\
 & = 0
 \end{aligned}$$

for any $X, Y, U, V, W \in \mathfrak{X}(M)$.

Making use of (1.1), we get

$$\begin{aligned}
 & P(X, Y)P(U, V)W \\
 (5.3) \quad & = R(X, Y)R(U, V)W - \frac{1}{2n}S(V, W)R(X, Y)U + \frac{1}{2n}S(U, W)R(X, Y)V \\
 & - \frac{1}{2n}\{S(Y, R(U, V)W)X - \frac{1}{2n}S(V, W)S(Y, U)X + \frac{1}{2n}S(U, W)S(Y, V)X\} \\
 & + \frac{1}{2n}\{S(X, R(U, V)W)Y - \frac{1}{2n}S(V, W)S(X, U)Y + \frac{1}{2n}S(U, W)S(X, V)Y\},
 \end{aligned}$$

$$\begin{aligned}
 & P(P(X, Y)U, V)W \\
 (5.4) \quad & = R(R(X, Y)U, V)W - \frac{1}{2n}S(Y, U)R(X, V)W + \frac{1}{2n}S(X, U)R(Y, V)W \\
 & - \frac{1}{2n}\{S(V, W)R(X, Y)U - \frac{1}{2n}S(V, W)S(Y, U)X + \frac{1}{2n}S(V, W)S(X, U)Y\} \\
 & + \frac{1}{2n}\{S(R(X, Y)U, W)V - \frac{1}{2n}S(Y, U)S(X, W)V + \frac{1}{2n}S(X, U)S(Y, W)V\},
 \end{aligned}$$

$$\begin{aligned}
 & P(U, P(X, Y)V)W \\
 (5.5) \quad & = R(U, R(X, Y)V)W - \frac{1}{2n}S(Y, V)R(U, X)W + \frac{1}{2n}S(X, V)R(U, Y)W \\
 & - \frac{1}{2n}\{S(R(X, Y)V, W)U - \frac{1}{2n}S(Y, V)S(X, W)U + \frac{1}{2n}S(X, V)S(Y, W)U\} \\
 & + \frac{1}{2n}\{S(U, W)R(X, Y)V - \frac{1}{2n}S(U, W)S(Y, V)X + \frac{1}{2n}S(U, W)S(X, V)Y\},
 \end{aligned}$$

$$\begin{aligned}
 & P(U, V)P(X, Y)W \\
 (5.6) \quad & = R(U, V)R(X, Y)W - \frac{1}{2n}S(Y, W)R(U, V)X + \frac{1}{2n}S(X, W)R(U, V)Y \\
 & - \frac{1}{2n}\{S(V, R(X, Y)W)U - \frac{1}{2n}S(Y, W)S(V, X)U + \frac{1}{2n}S(X, W)S(V, Y)U\} \\
 & + \frac{1}{2n}\{S(U, R(X, Y)W)V - \frac{1}{2n}S(Y, W)S(U, X)V + \frac{1}{2n}S(X, W)S(U, Y)V\}.
 \end{aligned}$$

Substituting (5.3)-(5.6) into (5.2), we have

$$\begin{aligned}
 & (R(X, Y) \cdot R)(U, V)W - \frac{1}{2n}\{S(Y, R(U, V)W)X - S(X, R(U, V)W)Y\} \\
 & + \frac{1}{2n}\{S(Y, U)R(X, V)W - S(X, U)R(Y, V)W - S(R(X, Y)U, W)V\} \\
 (5.7) \quad & + \frac{1}{2n}\{S(Y, V)R(U, X)W - S(X, V)R(U, Y)W + S(R(X, Y)V, W)U\} \\
 & + \frac{1}{2n}\{S(Y, W)R(U, V)X - S(X, W)R(U, V)Y + S(V, R(X, Y)W)U \\
 & - S(U, R(X, Y)W)V\} = 0
 \end{aligned}$$

for any vector fields $X, Y, U, V, W \in \mathfrak{X}(M)$. If (5.1) holds, putting $Y = U = \xi$ into (5.7), we obtain

$$\begin{aligned}
 & (R(X, \xi) \cdot R)(\xi, V)W - \frac{1}{2n}\{S(\xi, R(\xi, V)W)X - S(X, R(\xi, V)W)\xi\} \\
 & + \frac{1}{2n}\{S(\xi, \xi)R(X, V)W - S(X, \xi)R(\xi, V)W - S(R(X, \xi)\xi, W)V\} \\
 (5.8) \quad & + \frac{1}{2n}\{S(\xi, V)R(\xi, X)W - S(X, V)R(\xi, \xi)W + S(R(X, \xi)V, W)\xi\} \\
 & + \frac{1}{2n}\{S(\xi, W)R(\xi, V)X - S(X, W)R(\xi, V)\xi + S(V, R(X, \xi)W)\xi \\
 & - S(\xi, R(X, \xi)W)V\} = 0
 \end{aligned}$$

for any vector fields $X, V, W \in \mathfrak{X}(M)$. In Section 4, we know that $S(\xi, V) = 2nk\eta(V)$, using the equation and (3.1), we have

$$\begin{aligned}
 & S(R(\xi, X)Y, Z) \\
 (5.9) \quad & = 2n\{\eta(Z)[k^2g(X, Y) - 2kg(h'X, Y)] \\
 & + \eta(Y)[kg(X, Z) - k(k+1)\eta(Z)\eta(X) \\
 & + kg(X, h'Z) - 2g(h'X, Z) - 2g(h'X, h'Z)]\}
 \end{aligned}$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$. Combining (5.9) with (5.8) and assuming that $X \in [\lambda]$ and $V = W \in [-\lambda]$ in (5.8) are eigenvector fields of h' corresponding two eigenvalues λ and $-\lambda$, respectively. Thus, by applying Lemma 3.1, we obtain

$$(5.10) \quad (R(X, \xi) \cdot R)(\xi, V)W = [k^2 + 2k\lambda + k(k+2)]g(V, W)X.$$

On the other hand, by a straightforward computation and applying Lemma 3.1, Wang and Liu [15, Theorem 1.1] obtained the following relation (one can check it by a direct calculation).

$$\begin{aligned}
 & (R(X, \xi) \cdot R)(\xi, V)W \\
 (5.11) \quad & = R(X, \xi)R(\xi, V)W - R(R(X, \xi)\xi, V)W \\
 & - R(\xi, R(X, \xi)V)W - R(\xi, V)R(X, \xi)W \\
 & = [(k - 2\lambda)(k + 2) - k^2 + 4\lambda^2]g(V, W)X.
 \end{aligned}$$

From (5.10) and (5.11), we get $\lambda^2(\lambda - 1) = 0$. In view of the fact $\lambda > 0$, we obtain $\lambda = 1$ and hence $k = -2$. From [6, Corollary 4.2] and [5, Theorem 6] we can know that M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Consequently, we have the following theorem:

Theorem 5.1. *If a non-Kenmotsu $(k, \mu)^l$ -almost Kenmotsu manifold M^{2n+1} satisfies $P \cdot P = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

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