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Some Results on Linear Models of Nonlinear Systems

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To Ann-Christine

Abstract

Linear time-invariant approximations of nonlinear systems are used in many applications. Such approximations can be obtained in many ways. For example, using system identification and the prediction-error method, it is always possible to estimate a linear model without considering the fact that the input and output measurements in general come from a nonlinear system. The main objective of this thesis is to explain some properties of such estimated models.

More specifically, linear time-invariant models that are optimal approximations in the mean-square error sense are studied. Although this is a classic field of research, relatively few results exist about the properties of such models when they are based on signals from nonlinear systems. In this thesis, some interesting, but in applications usually undesirable, properties of linear approximations of nonlinear systems are pointed out. It is shown that the linear model can be very sensitive to small nonlinearities. Hence, the linear approximation of an almost linear system can be useless for some applications, such as robust control design.

In order to improve the models, conditions are given on the input signal implying various useful properties of the linear approximations. It is shown, for instance, that minimum phase filtered white noise in many senses is a good choice of input signal. Furthermore, some special properties of Gaussian signals are discussed. These signals turn out to be especially useful for approximations of generalized Hammerstein or Wiener systems. Using a Gaussian input, it is possible to estimate the denominator polynomial of the linear part of such a system without compensating for the nonlinearities. In addition, some theoretical results about almost linear systems and about separable input processes are presented. Linear models, both with and without a noise description, are studied.

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Martin Enqvist

Linköping, September 2003

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Notation

Symbols, Operators and Functions

$\mathbb{N}$	the set of natural numbers ($0 \in \mathbb{N}$)
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
q	the shift operator, $qu(t) = u(t+1)$
$\triangleq$	equal by definition
$\in$	belongs to
$A \subset B$	A is a subset of B
$A \setminus B$	set difference, $\{x : x \in A \wedge x \notin B\}$
$\arg\min f(x)$	value of x that minimizes $f(x)$
$\mathrm{E}(x)$	expected value of the random variable x
$\mathrm{E}(x\|y)$	conditional expectation of x given y
θ	parameter vector
$r(t)$	reference signal at time t
$u(t)$	input signal at time t
$y(t)$	output signal at time t
$y_{nf}(t)$	noise-free output signal at time t
$w(t)$	noise signal at time t
$\eta_0(t)$	OE-LTI-SOE residual at time t

$\varepsilon_0(t)$	GE-LTI-SOE residual at time t
$R_u(\tau)$	covariance function of the signal u
$R_{yu}(\tau)$	cross-covariance function of the signals y and u
$\Phi_u(z)$	z-spectrum of the signal u
$\Phi_u(e^{i\omega})$	spectral density function of the signal u
$\Phi_{yu}(z)$	z-cross-spectrum of the signals y and u
$\Phi_{yu}(e^{i\omega})$	cross-spectral density function of the signals y and u
$G_{0,OE}(z)$	Output Error Linear Time-Invariant Second Order Equivalent
$G_{0,GE}(z)$	system model part of the General Error Linear Time-Invariant Second Order Equivalent
$H_{0,GE}(z)$	noise model part of the General Error Linear Time-Invariant Second Order Equivalent
$[G(z)]_{\text{causal}}$	the causal part of the transfer function $G(z)$

Abbreviations and Acronyms

FIR	Finite Impulse Response
GE	General Error
LTI	Linear Time-Invariant
NFIR	Nonlinear Finite Impulse Response
OE	Output Error
SOE	Second Order Equivalent
w.p. 1	with probability one

1

Introduction

Mathematical modeling of real-life systems is a very common methodology in science and engineering. It is used both as a means for achieving deeper knowledge about a system and as a design tool, e.g. as a basis for simulations or for controller design. Sometimes, it is possible to construct a model of a system from physical laws and principles. However, in other cases this is not possible either because of a lack of knowledge of the studied system or because physical modeling is considered too time consuming. In such cases, system identification is an option.

System identification deals with the problem of how to estimate a model of a system from measured input and output signals. Usually, system identification is applied to dynamic systems, i.e., systems with some kind of memory. In practice, linear system models are very common and they are often used also when the true system is nonlinear. It is therefore interesting to understand how an estimated linear model depends on the properties of the true nonlinear system and of the input signal used. The objective of this thesis is to give some answers to this problem.

This section contains a brief discussion about systems and models and some motivating examples. These examples describe some of the phenomena that can occur when a linear model of a nonlinear system is estimated. Furthermore, an outline of the thesis is given and finally, the main contributions are listed.

1.1 Systems and Models

A very important notion in system identification is the difference between a *system* and a *model*. In a wide sense, a system is any kind of physically or conceptually bounded object. Examples of systems are the solar system, a human brain cell and a DC motor. A system is usually affected by external signals. For example, the solar system is affected by the gravity of other stars, a human brain cell is affected by neighboring cells and by the composition of the blood and a DC motor is affected by the voltage over its winding. External signals that can be changed are called *inputs* while other signals that affect the system are called *disturbances*. Measurable signals that describe some property of the system are called *outputs*. Note that also disturbances can be measurable.

From a control engineering perspective, a system is some device whose behavior we want to make more "intelligent" in some way. This can be done by designing a controller that measures the outputs from the system and then alters the input signals in order to achieve the desired behavior. From a control scientist's point of view, the only systems that are of interest are those with both input and output signals. In this thesis, only scalar systems, i.e., systems with one input and one output, are considered.

A mathematical description of a system is called a model. Whenever a system corresponds to a real-life object, it cannot be described exactly and any model of it will thus contain errors. Only in constructed examples, it is possible to give an exact description of the true system. However, any reference to the system always concerns the actual true system. Hence we can talk about model errors but not about system errors.

As already mentioned, it is very common to approximate nonlinear systems using linear models. This approximation can be done in many ways. For example, differentiation can be used to linearize a nonlinear system description locally, or some kind of linear equivalent of a nonlinear system can be derived for a particular input. In this thesis, we will investigate the latter of these two approaches.

More specifically, we will study the behavior of linear model estimates obtained by system identification using input and output data from nonlinear systems. The system identification method that will be used here is the well-known *prediction-error method* (see Section 2.3), and we will only investigate its behavior when the number of measurements tends to infinity.

It can be shown (Ljung, 1978) that the prediction-error model estimate under rather general conditions will converge to the model that minimizes a *mean-square error criterion* $\mathrm{E}((H^{-1}(q)(y(t) - G(q)u(t)))^2)$. Here, $\mathrm{E}(x)$ denotes the expected value of the random value x and q denotes the shift operator, $qu(t) = u(t+1)$. Furthermore, $G(q)$ is the linear model from input to output and $H(q)$ is a description of the noise. It will be assumed that both $H^{-1}(q)G(q)$ and $H^{-1}(q)$ are stable models (see Section 2.1).

In the special case when $H(q) = 1$, the mean-square error optimal model $G(q)$ will here be called the *Output Error Linear Time-Invariant Second Order Equivalent* (OE-LTI-SOE) and it will be denoted $G_{0,OE}(q)$. The corresponding mean-square error optimal model for a general $H(q)$ will be called the *General Error Linear Time-Invariant Second Order Equivalent* (GE-LTI-SOE) and it will be denoted $(G_{0,GE}(q), H_{0,GE}(q))$. In the next section, some motivating examples that illustrate the properties of OE-LTI-SOE:s and GE-LTI-SOE:s will be presented.

1.2 Motivating Examples

Although the use of linear models of nonlinear systems is straightforward in some cases, it can sometimes give rise to rather nonintuitive phenomena. This is shown in the following examples.

Example 1.1
Consider the simple static nonlinear system

$$y(t) = u(t)^3 \tag{1.1}$$

Intuitively, the best linear approximation of this system would be a static linear system $y(t) = c_0 u(t)$*, where* c_0 *is some constant. However, this is not always the case. Let the input to the system (1.1) be*

$$u(t) = e(t) + \frac{1}{2}e(t-1)$$

where $e(t)$ *is a sequence of independent random variables with uniform distribution over the interval* $[-1, 1]$*. In this case, it turns out that the OE-LTI-SOE of the system (1.1) is*

$$G_{0,OE}(q) = \frac{0.85 + 0.575q^{-1}}{1 + 0.5q^{-1}}$$

Hence, a static nonlinear system can have a nonstatic OE-LTI-SOE .

Example 1.2
Let the input signal to (1.1) be generated in a different way according to

$$u(t) = \frac{1}{2}e(t) + e(t-1) \tag{1.2}$$

where $e(t)$ *is the same signal as in Example 1.1. In this way, this input will have the same spectral density* $\Phi_u(e^{i\omega})$ *as the one in the previous example. However, the OE-LTI-SOE of the system (1.1) for the input (1.2) is*

$$G_{0,OE}(q) = \frac{0.925 + 0.425q^{-1}}{1 + 0.5q^{-1}}$$

Hence, a nonlinear system can have different OE-LTI-SOE:s for two input signals with equal spectral densities.

Example 1.3
Consider the static nonlinear system

$$y(t) = u(t)^2 - 3$$

with the input

$$u(t) = e(t) + e(t-1)^2 - 1$$

where $e(t)$ here is a white Gaussian process with zero mean and unit variance. The OE-LTI-SOE of this system is

$$G_{0,OE}(q) = \frac{8}{3} \approx 2.6667$$

while the GE-LTI-SOE is

$$G_{0,GE}(q) = \frac{\sqrt{4161} - 33}{12} \approx 2.6255$$
$$H_{0,GE}(q) = 1 + \frac{65 - \sqrt{4161}}{8} q^{-1}$$

As can be seen from these expressions, $G_{0,OE}(q) \neq G_{0,GE}(q)$ despite the fact that the system operates in open loop.

Hence, the OE-LTI-SOE $G_{0,OE}(q)$ of an open-loop nonlinear system can be different from $G_{0,GE}(q)$ in the corresponding GE-LTI-SOE .

Example 1.4
Consider the nonlinear system

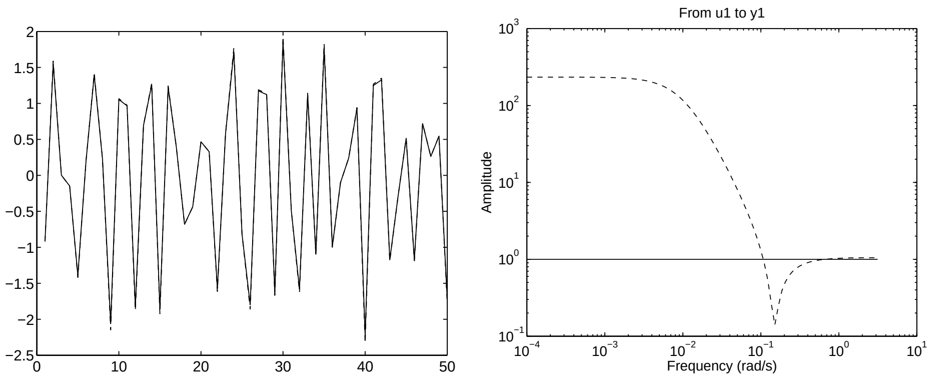
$$y(t) = y_l(t) + 0.01 y_n(t)$$
$$y_l(t) = u(t)$$
$$y_n(t) = u(t)^3$$

The output from this system consists of a linear part, $y_l(t) = G_l(q)u(t)$ with $G_l(q) = 1$, and a nonlinear part, $0.01y_n(t) = 0.01u(t)^3$. Let the input signal be

$$u(t) = (1 - 2cq^{-1} + c^2 q^{-2})e(t)$$

where $c = 0.99$ and where $e(t)$ is a white noise process with uniform distribution over the interval $[-1, 1]$. For this input, it is hard to distinguish the output $y(t)$ of the nonlinear system from the output $y_l(t)$ of the linear part of the system. This can be seen in Figure 1.1a for a particular realization of the input signal. However, the small differences between the output signals $y(t)$ and $y_l(t)$ will make the OE-LTI-SOE very different from the linear part G_l of the system. This difference can be seen in Figure 1.1b.

Hence, the distance between the OE-LTI-SOE and the linear part of the true system can be large also when the nonlinearities are small.



a. The output $y(t)$ (dashed) of the nonlinear system in Example 1.4 and the output $y_l(t) = u(t)$ (solid) of the linear part of that system for a particular realization of the input signal.

b. The gain of the frequency response of the OE-LTI-SOE (dashed) and of the linear part (solid) of the nonlinear system in Example 1.4

Figure 1.1 *The frequency response of the OE-LTI-SOE can be far from the response of the linear part of the system also when the nonlinear contributions to the output are small.*

As can be seen in the previous examples, the OE-LTI-SOE of a nonlinear system is input dependent. Furthermore, there is no guarantee that the OE-LTI-SOE and the GE-LTI-SOE will be equal even for an open-loop nonlinear system. Neither will the OE-LTI-SOE always be close to the linear part of the system.

Especially this last property can in some circumstances be undesirable, e.g. if the OE-LTI-SOE is supposed to be used as a basis for robust control design. Such a design puts restrictions on the control laws in order to guarantee the stability of the resulting true closed-loop system, despite the presence of model errors. A drawback with a model that is far from the linear part of an almost linear system is that the gain of the model errors might be unnecessarily large.

A robust controller design based on a model with large model errors usually implies that the restrictions on the control laws will be rather hard. Hence, the use of an OE-LTI-SOE with large model errors can result in a rather poor control performance for the true system. It is thus interesting to understand under which circumstances the OE-LTI-SOE will be close to the linear part of the system when the nonlinearities are small.

Examples 1.1-1.4 will be discussed in more detail later in this thesis (see Examples 4.2, 4.3, 5.1 and 6.1). Furthermore, some conditions that prevent the behaviors shown in these examples will be presented.

1.3 Outline of the Thesis

The results in this thesis concern systems with stationary stochastic input and output signals. Some background material about such signals and about linear and nonlinear systems can be found in Chapter 2. This chapter contains also a brief description of system identification using the prediction-error method. An overview of some existing linearization approaches can be found in Chapter 3.

The approach used in this thesis is based on mean-square error optimization. The mean-square error optimal LTI approximation of a nonlinear system is here called the LTI Second Order Equivalent and it is described in Chapter 4. Basic properties of LTI-SOE:s are discussed in Chapter 5. There, it is also shown that minimum phase filtered white noise in many cases is a good choice of input signal.

Furthermore, LTI approximations of almost linear systems are studied in Chapter 6 while the special case of Gaussian inputs to nonlinear FIR systems is discussed in Chapter 7. This chapter contains also some results about LTI-SOE:s of generalized Hammerstein and Wiener systems. Chapter 8 is focused on the separable class of input signals and, finally, some general conclusions are drawn in Chapter 9.

1.4 Contributions

The main objective of this thesis is to explain some of the behavior of LTI Second Order Equivalents of nonlinear systems. For example, the phenomena shown in Examples 1.1-1.4 are discussed. From a practical point of view, there are two contributions in this thesis that probably are more important than the others. The first one is the observation described in Sections 5.3 and 5.4 that minimum phase filtered white noise in many senses is a good choice of input signal for LTI approximations of nonlinear systems.

The second contribution of practical interest is the result in Corollaries 7.1 and 7.2 about generalized Hammerstein and Wiener systems with Gaussian inputs. With the behavior of LTI Second Order Equivalents shown in Examples 1.1-1.4 in mind, also the continuity result in Theorem 6.1 and the result about separable processes in Theorem 8.2 can be viewed as main contributions.

Some of the material of this thesis has previously been published in conference proceedings. Early versions of the results in Chapter 6 about approximations of almost linear systems can be found in

> M. Enqvist and L. Ljung. Estimating nonlinear systems in a neighborhood of LTI-approximants. In *Proc. of the 41st IEEE Conference on Decision and Control*, pages 1005–1010, Las Vegas, Nevada, Dec. 2002.

The material in Chapter 7 about Gaussian inputs has previously been published in

> M. Enqvist and L. Ljung. Linear models of nonlinear FIR systems with Gaussian inputs. In *Preprints of the 13th IFAC Symposium on System Identification*, pages 1910–1915, Rotterdam, The Netherlands, Aug. 2003.

2

Preliminaries

In this chapter, some background material about linear and nonlinear systems will be presented and the notation that will be used throughout this thesis will be introduced. Furthermore, a brief description of the basic ideas of system identification based on prediction-error methods will be given.

2.1 Linear Systems and Stochastic Processes

Linear time-invariant (LTI) dynamic systems and models are the foundation of control theory and system identification and are described in many textbooks (see, for example, Kailath, 1980; Rugh, 1996). Any discrete-time LTI system can be written as a convolution

$$y(t) = \sum_{k=-\infty}^{\infty} g(k)u(t-k)$$

The sequence $(g(k))_{k=-\infty}^{\infty}$ is called the *impulse response* of the system. An LTI system can also be represented by a *transfer function* $G(z)$, which is obtained by taking the z-transform of the impulse response, i.e.,

$$G(z) = \sum_{k=-\infty}^{\infty} g(k)z^{-k}$$

Similarly, the function $G(q)$, where q is the shift operator $qu(t) = u(t+1)$, will be called the *transfer operator* of the system.

Although the transfer function sometimes can be written more compactly as a rational function of z, it should always be thought of as a certain series expansion in order to avoid any ambiguities. These ambiguities can occur due to the fact that a rational function corresponds to different series expansions in different regions of convergence. However, the series expansion is unique if the region of convergence is specified (Brown and Churchill, 1996). Sometimes, this specification will be done using the following terminology.

Definition 2.1
A sequence $(m(k))_{k=-\infty}^{\infty}$ *is* causal *if* $m(k) = 0$ *for all* $k < 0$ *and* strictly causal *if* $m(k) = 0$ *for all* $k \leq 0$. *The sequence is* anticausal *if* $m(k) = 0$ *for all* $k > 0$ *and* strictly anticausal *if* $m(k) = 0$ *for all* $k \geq 0$.

The notion of causality can be used also for LTI systems.

Definition 2.2
An LTI system is (strictly) causal *if its impulse response is (strictly) causal. Similarly, an LTI system is* (strictly) anticausal *if its impulse response is (strictly) anticausal.*

In some cases, we will need to extract the causal part of a noncausal system. This will be done using the following notation.

$$[G(z)]_{\text{causal}} = \left[\sum_{k=-\infty}^{\infty} g(k)z^{-k}\right]_{\text{causal}} = \sum_{k=0}^{\infty} g(k)z^{-k}$$

Causality of an LTI system implies that the system output only depends on past and present values of the input signal. Hence, all real-life systems are causal. Another important property of LTI systems is stability. In this thesis, we will only use the type of stability called *bounded input bounded output stability*, which is defined as follows.

Definition 2.3
An LTI system with impulse response $g(k)$ *is* stable *if* $\sum_{k=-\infty}^{\infty} |g(k)| < +\infty$.

If a transfer function is said to be stable, it should always be viewed as coming from the series expansion, causal or noncausal, whose region of convergence contains the unit circle. On the other hand if a transfer function is said to be causal it should be viewed as coming from a, possibly unstable, causal series expansion.

Furthermore, an LTI system $G(z)$ is said to be *static* if only $g(0)$ is nonzero and nonstatic if there exists a $k \in \mathbb{Z} \setminus \{0\}$ such that $g(k) \neq 0$. If $g(k)$ is nonzero only for a finite number of k:s, the system is said to be a *finite impulse response* (FIR) system. An LTI system is said to be *monic* if $g(0) = 1$. In some cases, we will use the following notation for the inverse system of a causal LTI system.

$$G^{-1}(z) = \frac{1}{G(z)} = \sum_{k=0}^{\infty} \tilde{g}(k)z^{-k}$$

As indicated above, $G^{-1}(z)$ should always be viewed as a causal series expansion. An important notion for control theory, and also for the discussion later in this thesis, is the concept of *minimum phase* systems.

Definition 2.4
An LTI system is minimum phase *if both $G(z)$ and $G^{-1}(z)$ are stable and causal transfer functions.*

The definitions that have been introduced so far have concerned LTI *systems* but do of course hold for LTI *models* and *filters* as well. The word *filter* will here be used as an alternative name for an LTI system whose main purpose is to change a signal in some way. The signals that will be discussed in this thesis are discrete-time stationary stochastic processes (see, for example, Gardner, 1986; Jazwinski, 1970).

Formally, a discrete-time stochastic process $(u(t))_{t=-\infty}^{\infty}$ is an indexed sequence of random variables where the parameter t corresponds to time. The processes that will be studied in this thesis will be real and *stationary*. Stationarity of a process means that the simultaneous probability density function of any set of variables $\{u(t+\tau), \tau \in D \subset \mathbb{Z}\}$ is independent of t. Furthermore, all processes will have zero mean, i.e., $\mathrm{E}(u(t)) = 0$ for all $t \in \mathbb{Z}$, and well-defined *covariance functions* $R_u(\tau)$. The covariance function of a zero mean process is defined as

$$R_u(\tau) = \mathrm{E}(u(t)u(t-\tau))$$

Furthermore, it will be assumed that the covariance function has a well-defined z-transform $\Phi_u(z)$ whose region of convergence contains the unit circle. The function $\Phi_u(z)$ can be written

$$\Phi_u(z) = \sum_{\tau=-\infty}^{\infty} R_u(\tau) z^{-\tau}$$

and it will, using the terminology in Kailath et al. (2000), be called the *z-spectrum* of the process. Properties like stability and causality that hold for LTI systems can be used also about z-spectra. Note that $\Phi_u(z^{-1}) = \Phi_u(z)$ since $R_u(-\tau) = R_u(\tau)$. The real-valued function $\Phi_u(e^{i\omega})$ of $\omega \in [-\pi, \pi]$ that is obtained when $z = e^{i\omega}$ will be called the *spectral density function* of the process.

If two processes $(u(t))_{t=-\infty}^{\infty}$ and $(y(t))_{t=-\infty}^{\infty}$ are considered, it will be assumed that they are jointly stationary and that the *cross-covariance function* $R_{yu}(\tau)$ between these processes exists. The cross-covariance function is defined as

$$R_{yu}(\tau) = \mathrm{E}(y(t)u(t-\tau))$$

Furthermore, it will be assumed that also this function has a z-transform $\Phi_{yu}(z)$ whose region of convergence contains the unit circle. The function $\Phi_{yu}(z)$ can be written

$$\Phi_{yu}(z) = \sum_{\tau=-\infty}^{\infty} R_{yu}(\tau) z^{-\tau}$$

and will be called the *z-cross-spectrum*. Note that $\Phi_{yu}(z^{-1}) = \Phi_{uy}(z)$ and that all z-spectra and z-cross-spectra should always be interpreted as the series expansion whose region of convergence contains the unit circle.

A very important class of processes is *white noise processes*, which have the property that all $u(t)$, $t \in \mathbb{Z}$, are independent. Hence, for white processes only $R_u(0)$ is nonzero. Using white processes as inputs to LTI filters, it is possible to construct processes with arbitrary z-spectra. This follows from the next lemma about LTI filtering of stationary stochastic processes. This lemma has been taken from Kailath et al. (2000, p. 195).

Lemma 2.1 (Filtering of Stationary Processes)
Let $(y(t))_{t=-\infty}^{\infty}$ be the stationary process that is obtained by passing a zero-mean stationary process $(u(t))_{t=-\infty}^{\infty}$ through a stable LTI system with transfer function $H(z)$. Then the following relations hold.

$$\Phi_y(z) = H(z)\Phi_u(z)H(z^{-1})$$
$$\Phi_{yu}(z) = H(z)\Phi_u(z)$$

Furthermore, if $(x(t))_{t=-\infty}^{\infty}$ is jointly stationary with $(y(t))_{t=-\infty}^{\infty}$ and $(u(t))_{t=-\infty}^{\infty}$ as just defined, then

$$\Phi_{xy}(z) = \Phi_{xu}(z)H(z^{-1})$$

Proof: *See Kailath et al. (2000, pp. 195-197).* □

LTI models and stochastic processes will in this thesis be used to model arbitrary systems. Usually, it will be assumed that these systems contain some noise and hence we need models that include some kind of noise description. One model with this property is the following general LTI model of a system with input $u(t)$ and output $y(t)$,

$$y(t) = G(q)u(t) + H(q)e(t) \tag{2.1}$$

where $H(q)$ is a monic transfer operator that describes how the output depends on the white noise $e(t)$. The structure of the model (2.1) is illustrated in Figure 2.1.

The LTI model (2.1) can be used to define the optimal predictor $\hat{y}(t)$ of $y(t)$ given past output values $(y(t-k))_{k=1}^{\infty}$ and past and present input values $(u(t-k))_{k=0}^{\infty}$ (see, for example, Ljung, 1999, Chap. 3).

$$\hat{y}(t) = H^{-1}(q)G(q)u(t) + (1 - H^{-1}(q))y(t) \tag{2.2}$$

The predictor (2.2) is optimal in the sense that if (2.1) is an accurate description of the true system, it minimizes the *mean-square error* $\mathrm{E}((y(t) - \hat{y}(t))^2)$ and is equal to the conditional expectation of $y(t)$ given past output and past and present input values (Ljung, 1999, Chap. 3). Predictors of this kind are used in the prediction-error method, which will be described in Section 2.3. First, however, we will give a brief overview of some types of nonlinear systems that will be discussed later in this thesis.

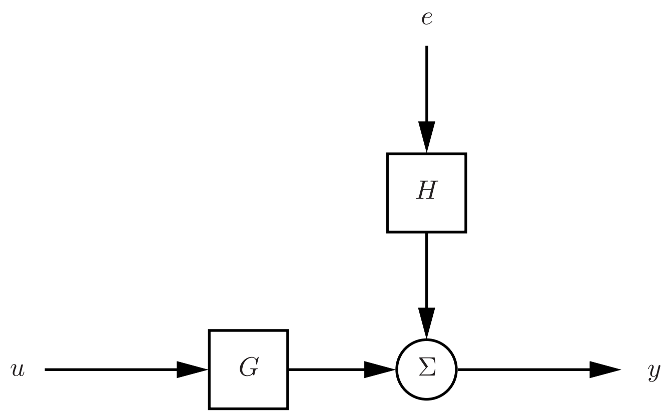


Figure 2.1 *The general LTI model.*

2.2 Nonlinear Systems

In some of the results that will be presented in this thesis, there will be no explicit assumptions on the true nonlinear system that is modeled. Hence, the system can often be viewed as a black box that for a given stationary input signal $(u(t))_{t=-\infty}^{\infty}$ produces the output $(y(t))_{t=-\infty}^{\infty}$. However, in some other results, we will assume that the system belongs to certain classes of nonlinear systems and these classes will be defined here.

Similarly to the LTI case, a nonlinear system will be said to be static if its output $y(t)$ can be written as a function of $u(t)$, i.e., if $y(t) = f(u(t))$, and the system is said to be nonstatic if $y(t)$ depends on any other $u(t-k)$, $k \in \mathbb{Z} \setminus \{0\}$. A class of systems that will be discussed later in this thesis is *nonlinear finite impulse response* (NFIR) systems. An NFIR system can for some $M \in \mathbb{N}$ be written as

$$y(t) = f((u(t-k))_{k=0}^{M})$$

Two other system classes that will be discussed later are *Wiener* and *Hammerstein* systems. A Wiener system consists of an LTI model followed by a static nonlinearity, i.e.,

$$\begin{aligned} y(t) &= f(v(t)) \\ v(t) &= G(q)u(t) \end{aligned}$$

while a Hammerstein system has these linear and the nonlinear subsystems in the opposite order, i.e.,

$$\begin{aligned} y(t) &= G(q)v(t) \\ v(t) &= f(u(t)) \end{aligned}$$

A detailed description and characterization of many other types of nonlinear systems and models can be found in Pearson (1999). In the next section, we will describe some of the basic ideas in system identification.

2.3 System Identification

As was mentioned in the introduction to this thesis, *system identification* can be viewed as a synonym for mathematical modeling of dynamic systems using measurements of the input and output signals. Various identification methods can be found in the literature, but here we will only discuss one of family of methods, namely *prediction-error methods*.

These methods are based on the observation that predictors like (2.2) can be used to compare how well different LTI models can predict the output $y(t)$. The main idea is to use some kind of measure of the distance between the predicted output and the true output and to minimize this distance by adjusting some parameters in the model. Typically, a prediction-error method works with a finite data set $Z^N = (u(t), y(t))_{t=1}^N$ that contains simultaneous measurements of the input and output signals and a parameterized version of the general LTI model (2.1). This parameterized model can be written as

$$y(t, \theta) = G(q, \theta)u(t) + H(q, \theta)e(t) \tag{2.3}$$

where θ is a d-dimensional vector of parameters. For example, θ can be the coefficients of the numerator and denominator polynomials of G and H, provided that these transfer functions are chosen as rational functions.

Different *model structures* can be obtained by imposing some restrictions on the rational functions G and H. For example, the *auto-regressive with external input* (ARX) model structure is obtained by letting $G(q, \theta) = B(q, \theta)/A(q, \theta)$ and $H(q, \theta) = 1/A(q, \theta)$, where A and B are polynomials. Similarly, the *output error* model structure is acquired if $H(q, \theta) = 1$. A family of LTI model structures is described in Ljung (1999, pp. 81-88).

If the model (2.3) would be a perfect description of the true system for some white noise process $e(t)$, the mean-square error optimal predictor $\hat{y}(t, \theta)$ of $y(t)$ would be

$$\hat{y}(t, \theta) = H^{-1}(q, \theta)G(q, \theta)u(t) + (1 - H^{-1}(q, \theta))y(t) \tag{2.4}$$

When a model structure has been selected in the prediction-error method, the corresponding predictor (2.4) is used to compute θ-dependent predictions $\hat{y}(t, \theta)$ based on the data in Z^N. A parameter estimate $\hat{\theta}_N$ can then be computed by minimizing a criterion $V_N(\theta, Z^N)$. For example, this criterion can be chosen to be quadratic such that

$$\hat{\theta}_N = \arg\min_{\theta \in D_M} V_N(\theta, Z^N) = \arg\min_{\theta \in D_M} \frac{1}{N} \sum_{t=1}^{N} (y(t) - \hat{y}(t, \theta))^2 \tag{2.5}$$

Here, θ is bound to be a member some pre-specified set $D_M \subset \mathbb{R}^d$. Usually, D_M is the set of parameters that make the predictor (2.4) stable. In general, the minimization of $V_N(\theta, Z^N)$ has to be performed using some kind of numerical method. A common choice is to use a *Gauss-Newton* or a *damped Gauss-Newton* method. These methods use the gradient and an approximation of the Hessian of $V_N(\theta, Z^N)$ and have good convergence properties, especially in the vicinity of the optimum (see Ljung, 1999).

Detailed studies of the properties of the prediction-error estimate when the number of measurements tends to infinity have been made (Ljung, 1999). In Ljung (1978) it is shown that under rather weak conditions on the true system and the input and output signals, the following convergence result holds with probability one.

$$\hat{\theta}_N \to \theta^* = \arg\min_{\theta \in D_M} \mathrm{E}((y(t) - \hat{y}(t,\theta))^2), \quad \text{w.p. 1 as } N \to \infty \tag{2.6}$$

With some abuse of notation, $y(t)$ and $\hat{y}(t, \theta)$ here denote the stochastic signals while they previously in this section have denoted realizations of these signals. In particular, (2.6) holds also for many nonlinear systems. Since (2.6) shows that the prediction-error estimate usually will converge to the mean-square error optimal estimate θ^*, it is interesting to investigate what can be said about the LTI models that are defined by θ^* when the true system is nonlinear. Actually, this is the main objective of this thesis.

The method described above is not the only prediction-error method, but rather a commonly used member of a family of methods. The main differences between these methods are due to different choices of criterion in (2.5) and to the fact that a prefilter is used in some methods. It should also be noted that prediction-error methods can be used for other model structures than the ones based on (2.1). For example, both linear state-space models and general nonlinear black-box models can be used.

After this brief description of the prediction-error method and the previous discussions about the properties of LTI systems and about some nonlinear systems, we are now ready to move on to the main part of this thesis. First, however, we will in the next chapter give an overview of some of the linearization approaches that can be found in the literature.

3

Different Linearization Frameworks

Since there are many different circumstances when LTI models of nonlinear systems are useful, various linearization approaches have been proposed. These approaches differ in many aspects, for example in the types of signals they are defined for, in their optimality properties and in which mathematical tools that are used. In this chapter, we will give a brief overview of some of the linearization frameworks that can be found in the literature. First, we will in the next section consider linearizations in a deterministic framework while we in the second part of this chapter will discuss LTI approximations for stochastic signals.

3.1 Deterministic Approaches

The most straightforward linearization approach is perhaps to use some kind of local linearization based on a truncated Taylor series expansion. For example, if the true system is an NFIR system (see Section 2.2)

$$y(t) = f((u(t-k))_{k=0}^{M})$$

it can for small inputs be approximated with an LTI system

$$G_0(z) = \sum_{k=0}^{M} g_0(k) z^{-k}$$

where $g_0(k) = f'_{u(t-k)}(0)$. Of course, this approximation is only well-defined if f is differentiable at 0. Similar LTI approximations can be made also for nonlinear state-space systems (see, for example, Ljung and Glad, 1994, pp. 347-349).

The idea of deriving an LTI approximation by differentiation of a nonlinear system is used also by Mäkilä and Partington (2003). They study LTI approximations of Wiener, NFIR and bi-gain systems for l^∞-signals and use the notion of *Fréchet derivatives* to derive some of the approximations. Furthermore, some relations to controller design and identification are studied. Related material can be found also in Partington and Mäkilä (2002) and in Mäkilä (2003a,b).

A different approach has been used by Horowitz (1993). He defines the *LTI equivalent* $P(s)$ of a continuous-time nonlinear system which has the output $y(t)$ for a particular input $u(t)$ as the ratio

$$P(s) = \frac{Y(s)}{U(s)}$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the output and input signal, respectively. Furthermore, it is shown that a set of such LTI equivalents can be used for controller design.

LTI approximations for a class of deterministic signals are also discussed in Sastry (1999). There, the existence of an optimal LTI approximant is mentioned and it is related to the theory of *describing functions*. Describing functions are derived for sinusoidal signals and are parameterized linear approximations of static nonlinearities. They can be used for analysis of closed-loop nonlinear systems (see, for example, Atherton, 1982; Sastry, 1999).

3.2 Stochastic Approaches

Since the work of Wiener (1949), there has been a great activity in estimation and filtering using random signals. We will here present some of the existing approaches to linearization using stochastic signals and we begin with linearizations of static nonlinearities.

3.2.1 Results for Static Nonlinearities

Many problems concerning the interplay between stochastic signals and nonlinear systems are difficult to solve. However, some nonlinear systems can be written as combinations of LTI subsystems and static nonlinear functions. For example, LTI systems with input and/or output saturation turn out to be very common in applications.

It is usually easier to analyze a static nonlinearity than a dynamic one. Hence, it is no surprise that there has been a wide interest in understanding how a static nonlinearity can be linearized for a stochastic input signal. Many results in this area are directly or indirectly related to Bussgang's classic theorem about Gaussian signals (see Bussgang (1952) for the original report and, for example, Papoulis (1984) for a more recent reference).

Bussgang (1952) has shown that the cross-covariance function between the output and the input of a static nonlinear function is a scaled version of the input covariance function if the input is Gaussian. Hence, the following theorem holds.

Theorem 3.1 (Bussgang)
Let $y(t)$ be the output from a static nonlinearity f with a Gaussian input $u(t)$, i.e., $y(t) = f(u(t))$. Assume that the expectations $\mathrm{E}(y(t)) = \mathrm{E}(u(t)) = 0$. *Then*

$$R_{yu}(\tau) = b_0 R_u(\tau)$$

where $b_0 = \mathrm{E}(f'(u(t)))$.

A direct implication of this result is that the mean-square error optimal LTI approximation of a static nonlinearity always will be a constant $b_0 = \mathrm{E}(f'(u(t)))$ when the input to the nonlinearity is Gaussian. It will be shown later in this thesis that this is not always true in the non-Gaussian case. The constant b_0 is called *equivalent gain* by Booton (1954) an can be viewed as a random describing function. Just like ordinary describing functions, it can be used to analyze nonlinear closed-loop systems (Atherton, 1982, Chap. 8). The relation between Bussgang's theorem and some other results about Gaussian processes has been discussed in Gorman and Zaborszky (1968).

Bussgang's theorem has turned out to be very useful for the theory of Hammerstein and Wiener system identification. The reason for this is that Bussgang's theorem explains why it is possible to estimate the linear and nonlinear parts of a Wiener or Hammerstein system separately when the input is Gaussian (Billings and Fakhouri, 1982; Korenberg, 1985; Bendat, 1998). However, this is not a unique property of Gaussian signals but can also be done when, for example, the input is sinusoidal (Bai, 2002, 2003).

Furthermore, Bussgang's theorem has been extended to other classes of signals than Gaussian by Barrett and Lampard (1955), Brown (1957) and Nuttall (1958). Nuttall's extension is especially interesting since it contains the other two, and since it presents a useful condition on the input for the property $R_{yu}(\tau) = b_0 R_u(\tau)$ to hold for an arbitrary static nonlinearity. It turns out that this relation holds for any static nonlinearity in a wide class of functions if and only if the input signal is *separable*. Separability of a process in Nuttall's sense means that

$$\mathrm{E}(u(t-\sigma)|u(t)) = c(\sigma)u(t)$$

where $c(\sigma) = R_u(\sigma)/R_u(0)$. In Nuttall (1958), a number of signals that have this property are listed, e.g. Gaussian processes, sine wave processes and phase modulated processes. In addition, McGraw and Wagner (1968) have shown that signals with *elliptically symmetric distributions* are separable and they have also characterized these signals further.

A result related to separable processes can be found in Balakrishnan (1960). There, it is shown that processes for which conditional expectations like $\mathrm{E}(u(t-\sigma)|u(t))$ are of a specified form have characteristic functions that satisfy certain equations. The use of separable processes in general, and Gaussian processes in particular, for nonlinear system identification is discussed further in Billings and Fakhouri (1978).

3.2.2 Results for General Nonlinear Systems

LTI approximations of nonlinear systems for stochastic signals are often studied in a mean-square error framework. In that case, the optimal LTI model of a nonlinear system with input $u(t)$ and output $y(t)$ can be defined as the stable model $G_0(q)$ that minimizes the mean-square error $\mathrm{E}((y(t) - G(q)u(t))^2)$. It is a well-known result that this optimal approximation can be written

$$G_0(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

and it is sometimes called the *equivalent linear model* (see, for example, Gardner, 1986, p. 382). The model $G_0(z)$ can also be called the *noncausal Wiener filter*, since it in general will be a stable but noncausal LTI system. Later in this thesis, we will also call it the *noncausal LTI Second Order Equivalent* (LTI-SOE).

If the LTI approximation is assumed to be causal, it will in general not be equal to the ratio between $\Phi_{yu}(z)$ and $\Phi_u(z)$. Instead, it can be derived using techniques for causal Wiener filtering (see, for example, Kailath et al., 2000, Chap. 7). The mean-square error optimal stable and *causal* LTI approximation of a nonlinear system has been discussed in Schetzen (1980, p. 330) and Ljung (2001) and is also the main topic of this thesis.

Optimal LTI approximations are also discussed by Pintelon and Schoukens (2001). They use the term *related linear system* for the mean-square error optimal LTI system and view the part of the output signal that this system cannot explain as a *nonlinear distortion*. For some special classes of input signals, including *random multi-sines*, a number of interesting properties of related linear systems can be derived. For example, the asymptotic behavior of the related linear system when the number of harmonics in the input signal tends to infinity has been studied. Furthermore, some results for Wiener and Hammerstein systems and a discussion about how nonlinear distortions can be detected can be found in Pintelon and Schoukens (2001, Chap. 3).

Related material can be found in Pintelon et al. (2001), Pintelon and Schoukens (2002) and Schoukens et al. (2003b). Schoukens et al. (2002) have also discussed benefits and drawbacks of different input signals for LTI approximations. An application of the results about related linear systems is the method for fast approximative identification of nonlinear systems in Schoukens et al. (2003a).

In this chapter, an overview of some existing linearization approaches has been given. These approaches deal either with deterministic or stochastic signals and have rather different properties. In this thesis, we will consider only LTI approximations for stochastic signals. These approximations will be described in the next chapter.

4

The Notion of LTI Second Order Equivalents

The previous chapter contained an overview of the various linearization frameworks that can be found in the literature. All these frameworks have their benefits and drawbacks depending on which type of linearization is desired. In this thesis, our main objective is to understand the behavior of the prediction-error method when the measured input and output signals come from a nonlinear system.

Hence, with the discussion from Section 2.3 about the asymptotic properties of the prediction-error method in mind, it is here natural to study linear approximations of nonlinear systems that are optimal in the mean-square error sense. Such approximations will be called *LTI Second Order Equivalents* (LTI-SOE:s) of the nonlinear systems. This chapter contains both detailed derivations of two types of LTI-SOE:s and some interpretations of these approximations. First, some restrictions on the input and output signals will be imposed in order to make the LTI-SOE:s well-defined.

4.1 Assumptions on the Input and Output Signals

Since the class of nonlinear systems literally contains all kinds of systems, it is too general to be studied as a whole. In many cases, explicit restrictions on the considered types of nonlinear systems are introduced in order to enable further analyses of the properties of these systems. Examples of explicit restrictions are that the nonlinear systems should have finite gain, finite memory or some kind of stability property.

Also in this thesis the class of nonlinear systems has to be restricted. However, we will not impose any explicit restrictions but instead assume that the input and output signals of the nonlinear systems have certain properties. These signal assumptions are listed here and impose implicit restrictions on the class of nonlinear systems that will be studied in the sequel.

Assumption A1: Assume that

(i) The input $u(t)$ and output $y(t)$ are real-valued stationary stochastic processes with $\mathrm{E}(u(t)) = \mathrm{E}(y(t)) = 0$.

(ii) There exist $K > 0$ and α, $0 < \alpha < 1$ such that the second order moments $R_u(\tau) = \mathrm{E}(u(t)u(t-\tau))$, $R_{yu}(\tau) = \mathrm{E}(y(t)u(t-\tau))$ and $R_y(\tau) = \mathrm{E}(y(t)y(t-\tau))$ satisfy

$$\begin{aligned} |R_u(\tau)| &< K\alpha^{|\tau|} \quad \forall \tau \in \mathbb{Z} \\ |R_{yu}(\tau)| &< K\alpha^{|\tau|} \quad \forall \tau \in \mathbb{Z} \\ |R_y(\tau)| &< K\alpha^{|\tau|} \quad \forall \tau \in \mathbb{Z} \end{aligned}$$

(iii) The z-spectrum $\Phi_u(z)$ has a canonical spectral factorization

$$\Phi_u(z) = L(z) r_u L(z^{-1}) \tag{4.1}$$

where $L(z)$ and $L^{-1}(z) = 1/L(z)$ are causal transfer functions that are analytic in $\{z \in \mathbb{C} : |z| \geq 1\}$, $L(+\infty) = 1$ and r_u is a positive constant.

The condition (ii) above implies that the second order moments of $u(t)$ and $y(t)$ are bounded and that the z-spectra $\Phi_u(z)$ and $\Phi_y(z)$ and the the z-cross-spectrum $\Phi_{yu}(z)$ converge absolutely and are analytic in the annulus $\{z \in \mathbb{C} : \alpha < |z| < \frac{1}{\alpha}\}$. It should be mentioned that the canonical spectral factor $L(z)$ in Assumption A1-(iii) is unique and that the factorization exists for all rational $\Phi_u(z)$ without zeros on the unit circle (Kailath et al., 2000, pp. 198-199). However, we will not restrict ourselves only to rational spectra here. The following example shows that also a nonrational z-spectrum can have a canonical spectral factorization.

Example 4.1
The z-spectrum

$$\Phi_u(z) = e^{1/z+z}$$

can be factorized as $\Phi_u(z) = L(z) r_u L(z^{-1})$ *with* $L(z) = e^{1/z}$ *and* $r_u = 1$*. The causal series expansions*

$$\begin{aligned} L(z) &= e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k} \\ L^{-1}(z) &= e^{-1/z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^{-k} \end{aligned}$$

converge absolutely and are analytic for $\{z \in \mathbb{C} : |z| > 0\}$ *and* $L(+\infty) = 1$*. Hence,* $L(z)r_u L(z^{-1})$ *is the unique canonical spectral factorization of the nonrational z-spectrum* $\Phi_u(z)$*.*

Assumption A1 gives sufficient conditions for the type of LTI-SOE that will be defined in Section 4.2 to be well-defined. However, for the second type of LTI-SOE that will be introduced in Section 4.3 we will need another assumption. Let $\zeta(t) = (u(t)\ y(t-1))^T$. Then

$$R_\zeta(\tau) = \begin{pmatrix} R_u(\tau) & R_{uy}(\tau+1) \\ R_{yu}(\tau-1) & R_y(\tau) \end{pmatrix} \tag{4.2a}$$

$$\Phi_\zeta(z) = \begin{pmatrix} \Phi_u(z) & z\Phi_{uy}(z) \\ z^{-1}\Phi_{yu}(z) & \Phi_y(z) \end{pmatrix} \tag{4.2b}$$

Assumption A2: Assume that the signals $u(t)$ and $y(t)$ fulfill Assumption A1 and that they also are such that $\Phi_\zeta(z)$ in (4.2b) has a canonical spectral factorization

$$\Phi_\zeta(z) = T(z)Q_\zeta T^T(z^{-1})$$

where $T(z)$ and $T^{-1}(z)$ are analytic in $\{z \in \mathbb{C} : |z| \geq 1\}$, $T(+\infty) = I$ and Q_ζ is a positive definite matrix, i.e., $Q_\zeta > 0$. (Here, T^{-1} denotes the *matrix inverse*.)

Also the canonical factorization of a matrix-valued z-spectrum is unique and its existence is guaranteed if, for example, $\Phi_\zeta(z)$ is a rational matrix without unit circle zeros (Kailath et al., 2000, p. 205). Assumptions A1 and A2 will be used in the derivations of LTI-SOE:s in the next two sections and will be the standard assumptions throughout this thesis.

4.2 The Output Error Model Type

In Section 2.1 it was mentioned that a general LTI model can be written as in (2.1). In general, both G and H in this model can be transfer functions of any order. Here, however, at first we will only consider models where H is fixed to 1, i.e., *output error* models (Ljung, 1999). If G is causal this implies that only $u(t)$ and past inputs $u(t-1)$, $u(t-2)$, ... are used to predict the output $y(t)$ according to (2.2). The structure of an output error model is shown in Figure 4.1.

Using only output error models, the mean-square error optimal LTI approximation of a certain nonlinear system is simply the causal and stable LTI model $G_{0,OE}$ that minimizes $\mathrm{E}((y(t) - G(q)u(t))^2)$. This model is often called the Wiener filter for prediction of $y(t)$ from $(u(t-k))_{k=0}^{\infty}$ (Wiener, 1949). However, we will not use the term Wiener filter here, but instead call $G_{0,OE}$ the *Output Error LTI Second Order Equivalent* (OE-LTI-SOE) of the nonlinear system. Hence, we have the following definition.

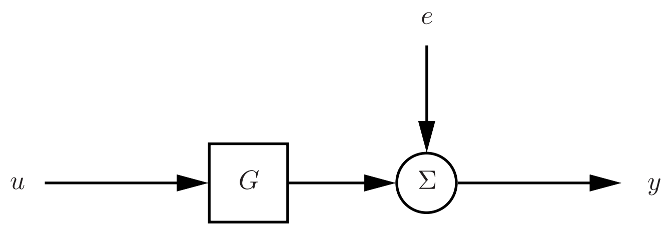


Figure 4.1 *The output error model.*

Definition 4.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. The Output Error LTI Second Order Equivalent *(OE-LTI-SOE) of this system is the stable and causal LTI model $G_{0,OE}(q)$ that minimizes the mean-square error $\mathrm{E}((y(t) - G(q)u(t))^2)$, i.e.,*

$$G_{0,OE}(q) = \arg\min_{G \in \mathcal{G}} \mathrm{E}((y(t) - G(q)u(t))^2)$$

where $\mathcal{G}$ denotes the set of all stable and causal LTI models.

There are two main reasons for the change of name from the commonly used Wiener filter to OE-LTI-SOE. First, we want to avoid any ambiguities. Many different Wiener filters can be constructed for a given pair of input and output signals. Here, however, we are only interested in the Wiener filter that predicts $y(t)$ from $(u(t-k))_{k=0}^{\infty}$.

The second reason for the change of name is that we want to emphasize the connection between the OE-LTI-SOE and the *General Error LTI Second Order Equivalent* (GE-LTI-SOE), with $H \neq 1$, that will be defined in Section 4.3. There it is shown that although the GE-LTI-SOE is derived from a certain Wiener filter, it has a different interpretation.

It should be noted that we are not only interested in the filtering and prediction capabilities of the OE-LTI-SOE/Wiener filter (and the GE-LTI-SOE), but also in the model itself. For example, we are not only interested in how good estimate of $y(t)$ the model can produce, but also in issues like how the model order and model coefficients depend on the nonlinear system and on the input signal.

The concept of LTI Second Order Equivalents has, for example, been discussed by Ljung (2001). As the name indicates, the LTI-SOE:s are equivalent to the true nonlinear system in a certain sense. More specifically, the GE-LTI-SOE has the property that it is impossible to distinguish it from the true system if only second order properties of the input, output and model residuals are considered. This will be discussed in more detail in Section 4.3 (see the comments to Corollary 4.3 on page 34).

It should immediately be pointed out that the OE-LTI-SOE of a nonlinear system depends on which input signal that is used. Hence, we can only talk about the OE-LTI-SOE of a nonlinear system with respect to a particular input signal. The fact that the OE-LTI-SOE is input-dependent is natural if we view it as an example of undermodeling. Undermodeling occurs whenever a system is approximated by a model of lower complexity. An often studied example of undermodeling is when an LTI system is approximated with an LTI model of lower order than the true system. Actually, also in this case of linear undermodeling the mean-square error optimal approximation is input-dependent (Ljung, 1999, Sec. 8.5).

Another property of OE-LTI-SOE:s that should be pointed out, besides their input-dependency, is that they are assumed to be initialized at $t = -\infty$ such that all transients have died out. In Definition 4.1 it is assumed that the complete infinite sequence $(u(t-k))_{k=0}^{\infty}$ of random variables is available for the computation of an estimate of $y(t)$.

The following theorem is a direct consequence of classic Wiener filter theory, and the proof of the theorem is almost identical to the derivation of the scalar Wiener filter in Kailath et al. (2000, pp. 231-234).

Theorem 4.1 (Output Error LTI Second Order Equivalents)
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. Then the Output Error LTI Second Order Equivalent (OE-LTI-SOE) $G_{0,OE}$ of this system is

$$G_{0,OE}(z) = \frac{1}{r_u L(z)} \left[\frac{\Phi_{yu}(z)}{L(z^{-1})} \right]_{\text{causal}} \tag{4.3}$$

where $[\dots]_{\text{causal}}$ denotes taking the causal part (see Section 2.1), and where $L(z)$ is the canonical spectral factor of $\Phi_u(z)$ from (4.1).

Proof: *The criterion* $\mathrm{E}((y(t) - G(q)u(t))^2)$ *that $G_{0,OE}$ should minimize is equivalent to the* Wiener-Hopf condition

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} g_{0,OE}(k) R_u(\tau - k) = 0, \quad \tau \geq 0 \tag{4.4}$$

or, alternatively, to

$$\Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z) = K_{OE}(z) \tag{4.5}$$

where $K_{OE}(z)$ is a stable and strictly anticausal transfer function. The Wiener-Hopf criterion follows from the fact that for the optimal model, the error

$$y(t) - G_{0,OE}(q)u(t)$$

should be orthogonal to $\{u(t-k)\}_{k=0}^{\infty}$ (using the inner-product $< u, v >= \mathrm{E}(uv)$). Using the spectral factorization according to (4.1) and multiplying by $L^{-1}(z^{-1})$ now gives

$$\tilde{K}_{OE}(z) \triangleq K_{OE}(z)L^{-1}(z^{-1}) = \Phi_{yu}(z)L^{-1}(z^{-1}) - G_{0,OE}(z)L(z)r_u \tag{4.6}$$

where $\tilde{K}_{OE}(z)$ is a stable and strictly anticausal transfer function due to the fact that it is a product of the stable and strictly anticausal transfer function $K_{OE}(z)$ and the stable and anticausal transfer function $L^{-1}(z^{-1})$. For (4.6) to hold it is necessary that the second right hand term, which is a stable and causal transfer function, is equal to the causal part of the first, i.e.,

$$G_{0,OE}(z)L(z)r_u = \left[\frac{\Phi_{yu}(z)}{L(z^{-1})}\right]_{\text{causal}} \tag{4.7}$$

and (4.3) follows. □

In general, the OE-LTI-SOE has to be calculated as in (4.3), which means that the canonical spectral factor $L(z)$ of the input z-spectrum has to be obtained. However, in some cases this is not necessary and the OE-LTI-SOE can be calculated using a simplified expression. This is shown in the following corollary.

Corollary 4.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled, and assume that the ratio $\Phi_{yu}(z)/\Phi_u(z)$ defines a stable and causal LTI system. Then

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

Proof: *Assume that*

$$C(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

is a stable and causal transfer function. Then

$$\Phi_{yu}(z) = C(z)\Phi_u(z) = C(z)L(z)r_uL(z^{-1})$$

and (4.3) gives

$$G_{0,OE}(z) = \frac{1}{r_uL(z)}\left[\frac{C(z)L(z)r_uL(z^{-1})}{L(z^{-1})}\right]_{\text{causal}} = C(z)$$

since $C(z)L(z)r_u$ is a stable and causal transfer function. □

The Wiener-Hopf condition (4.4) implies that the model residuals for the OE-LTI-SOE are uncorrelated with past and current inputs. This is stated more clearly in the following corollary.

Corollary 4.2
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. Let the residuals be defined by

$$\eta_0(t) = y(t) - G_{0,OE}(q)u(t) \tag{4.8}$$

Then

$$\Phi_{\eta_0 u}(z) = \Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z) \quad \textit{is strictly anticausal} \tag{4.9}$$

and

$$R_{\eta_0}(0) = R_y(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega})|^2 \Phi_u(e^{i\omega}) d\omega \tag{4.10}$$

Proof: *The result (4.9) follows directly from (4.4). Furthermore, we can write*

$$\begin{aligned}\Phi_{\eta_0}(z) &= \Phi_y(z) - \Phi_{yu}(z)G_{0,OE}(z^{-1}) - G_{0,OE}(z)\Phi_{uy}(z) \\ &\quad + G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}) \\ &= \Phi_y(z) - \left(\Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z)\right) G_{0,OE}(z^{-1}) \\ &\quad - G_{0,OE}(z)\left(\Phi_{uy}(z) - \Phi_u(z)G_{0,OE}(z^{-1})\right) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1})\end{aligned}$$

Since $G_{0,OE}(z^{-1})$ is anticausal and since $\Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z)$ by (4.9) is strictly anticausal it follows that

$$\int_{-\pi}^{\pi} \left(\Phi_{yu}(e^{i\omega}) - G_{0,OE}(e^{i\omega})\Phi_u(e^{i\omega})\right) G_{0,OE}(e^{-i\omega}) d\omega = 0$$

and that

$$\int_{-\pi}^{\pi} G_{0,OE}(e^{i\omega}) \left(\Phi_{uy}(e^{i\omega}) - \Phi_u(e^{i\omega})G_{0,OE}(e^{-i\omega})\right) d\omega = 0$$

Because

$$R_{\eta_0}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\eta_0}(e^{i\omega}) d\omega$$

we get

$$R_{\eta_0}(0) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_y(e^{i\omega}) d\omega}_{=R_y(0)} - \frac{1}{2\pi}\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega})|^2 \Phi_u(e^{i\omega}) d\omega$$

□

Corollary 4.2 shows that the OE-LTI-SOE really is the “best” output error model of the true system as the remaining residuals $\eta_0(t)$ are uncorrelated with past and present input signal components.

An example of the OE-LTI-SOE of a simple nonlinear system can be found below. Note that the OE-LTI-SOE is nonstatic in this example although the nonlinear system is static.

Example 4.2
Consider the static nonlinear system

$$y(t) = u(t)^3 \tag{4.11}$$

with the input

$$u(t) = e(t) + \frac{1}{2}e(t-1)$$

where $e(t)$ is a sequence of independent random variables with uniform distribution over the interval $[-1, 1]$. Hence, $\mathrm{E}(e(t)^2) = \frac{1}{3}$ and $\mathrm{E}(e(t)^4) = \frac{1}{5}$ and, using the fact that $e(t)$ and $e(t-1)$ are independent, we get

$$\begin{aligned}
R_{yu}(0) &= \mathrm{E}(u(t)^4) = \mathrm{E}(e(t)^4) + \frac{6}{4}\mathrm{E}(e(t)^2 e(t-1)^2) + \frac{1}{16}\mathrm{E}(e(t-1)^4) \\
&= \frac{1}{5} + \frac{6}{4}\cdot\frac{1}{9} + \frac{1}{16}\cdot\frac{1}{5} = \frac{91}{240} \\
R_{yu}(1) &= \mathrm{E}(u(t)^3 u(t-1)) \\
&= \mathrm{E}(\left(e(t)^3 + \frac{3}{2}e(t)^2 e(t-1) + \frac{3}{4}e(t)e(t-1)^2 + \frac{1}{8}e(t-1)^3\right)\cdot \\
&\quad \cdot\left(e(t-1) + \frac{1}{2}e(t-2)\right)) = \frac{3}{2}\mathrm{E}(e(t)^2 e(t-1)^2) + \frac{1}{8}\mathrm{E}(e(t-1)^4) \\
&= \frac{3}{2}\cdot\frac{1}{9} + \frac{1}{8}\cdot\frac{1}{5} = \frac{46}{240} \\
R_{yu}(-1) &= \mathrm{E}(u(t)^3 u(t+1)) \\
&= \mathrm{E}(\left(e(t)^3 + \frac{3}{2}e(t)^2 e(t-1) + \frac{3}{4}e(t)e(t-1)^2 + \frac{1}{8}e(t-1)^3\right)\cdot \\
&\quad \cdot\left(e(t+1) + \frac{1}{2}e(t)\right)) = \frac{1}{2}\mathrm{E}(e(t)^4) + \frac{3}{8}\mathrm{E}(e(t)^2 e(t-1)^2) \\
&= \frac{1}{2}\cdot\frac{1}{5} + \frac{3}{8}\cdot\frac{1}{9} = \frac{34}{240} \\
R_{yu}(\tau) &= 0 \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}
\end{aligned}$$

This gives

$$\Phi_{yu}(z) = \frac{1}{240}(34z + 91 + 46z^{-1}) = \frac{1}{240}\left(1 + \frac{1}{2}z\right)\left(68 + 46z^{-1}\right)$$

Furthermore, Lemma 2.1 gives

$$\Phi_u(z) = \left(1 + \frac{1}{2}z^{-1}\right)\cdot\frac{1}{3}\cdot\left(1 + \frac{1}{2}z\right) = \frac{1}{12}\left(2z + 5 + 2z^{-1}\right)$$

and hence

$$\frac{\Phi_{yu}(z)}{\Phi_u(z)} = \frac{1}{40} \cdot \frac{34 + 23z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

Since the ratio $\Phi_{yu}(z)/\Phi_u(z)$ here is stable and causal, the OE-LTI-SOE of the system (4.11) for this input is

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \frac{1}{40} \cdot \frac{34 + 23z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{0.85 + 0.575z^{-1}}{1 + 0.5z^{-1}}$$

Note that although the nonlinear system is static the OE-LTI-SOE is not.

The fact that a static nonlinear system can have a nonstatic OE-LTI-SOE is a first indication that LTI approximations of nonlinear systems in a mean-square error framework are not as straightforward as one might expect. More examples of this will be presented later in this thesis. In Chapters 7 and 8 we will also discuss classes of input signals which guarantee that the output of the OE-LTI-SOE will depend on the same number of input signal components as the nonlinear system.

For most systems, the order of the OE-LTI-SOE is unknown. In practice, this implies that several output error models have to be estimated and that a validation procedure has to be used in order to find the best model. Naturally, there is no guarantee that the correct order of the OE-LTI-SOE will be found. As a matter of fact, the OE-LTI-SOE can sometimes be an infinite order model. Hence, it is interesting to characterize in what sense an output error model with lower order than the OE-LTI-SOE approximates the OE-LTI-SOE.

This is interesting also when the true system is an LTI system. In that case, it can be shown that a low order model will approximate the true system mainly for frequencies where $\Phi_u(e^{i\omega})$ is large (Ljung, 1999, p. 266). As a matter of fact, this result holds also when the true system is nonlinear. In this case, a low order output error model will approximate the OE-LTI-SOE instead of the true system as well as possible for frequencies where $\Phi_u(e^{i\omega})$ is large according to the following theorem. This theorem is basically a special case of Theorem 4.1 in Ljung (2001) and the proof is very similar to the outlined proof in Problem 8G.5 in Ljung (1999).

Theorem 4.2
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. Let $G_{0,OE}$ be the corresponding OE-LTI-SOE according to Theorem 4.1. Suppose that a parameterized stable and causal output error model $G(q, \theta)$ is fitted to the signals u and y according to

$$\hat{\theta} = \arg\min_{\theta} \mathrm{E}(\eta(t,\theta)^2) \tag{4.12}$$

where

$$\eta(t,\theta) = y(t) - G(q,\theta)u(t) \tag{4.13}$$

Then it follows that

$$\hat{\theta} = \arg\min_{\theta} \int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G(e^{i\omega}, \theta)|^2 \Phi_u(e^{i\omega}) d\omega \tag{4.14}$$

Proof: *The z-spectrum of $\eta(t,\theta)$ is*

$$\begin{aligned}
\Phi_\eta(z,\theta) &= \begin{pmatrix} -G(z,\theta) & 1 \end{pmatrix} \begin{pmatrix} \Phi_u(z) & \Phi_{uy}(z) \\ \Phi_{yu}(z) & \Phi_y(z) \end{pmatrix} \begin{pmatrix} -G(z^{-1},\theta) \\ 1 \end{pmatrix} \\
&= \Phi_y(z) - G(z,\theta)\Phi_{uy}(z) - G(z^{-1},\theta)\Phi_{yu}(z) + G(z,\theta)\Phi_u(z)G(z^{-1},\theta) \\
&= \left(G(z,\theta) - \frac{\Phi_{yu}(z)}{\Phi_u(z)} \right) \Phi_u(z) \left(G(z^{-1},\theta) - \frac{\Phi_{yu}(z^{-1})}{\Phi_u(z^{-1})} \right) \\
&\quad - \frac{\Phi_{yu}(z)\Phi_{yu}(z^{-1})}{\Phi_u(z)} + \Phi_y(z)
\end{aligned}$$

Let

$$\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\Phi_y(e^{i\omega}) - \frac{|\Phi_{yu}(e^{i\omega})|^2}{\Phi_u(e^{i\omega})} \right) d\omega \\
B_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\Phi_{\eta_0 u}(e^{i\omega})|^2}{\Phi_u(e^{i\omega})} d\omega
\end{aligned}$$

Parseval's relation gives

$$\begin{aligned}
\mathrm{E}(\eta(t,\theta)^2) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_\eta(e^{i\omega},\theta) d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\Phi_{yu}(e^{i\omega})}{\Phi_u(e^{i\omega})} - G(e^{i\omega},\theta) \right|^2 \Phi_u(e^{i\omega}) d\omega + A_0 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G_{0,OE}(e^{i\omega}) + \frac{\Phi_{\eta_0 u}(e^{i\omega})}{\Phi_u(e^{i\omega})} - G(e^{i\omega},\theta) \right|^2 \Phi_u(e^{i\omega}) d\omega + A_0 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta) \right|^2 \Phi_u(e^{i\omega}) d\omega \\
&+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{\eta_0 u}(e^{i\omega}) (G_{0,OE}(e^{-i\omega}) - G(e^{-i\omega},\theta)) d\omega \\
&+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{\eta_0 u}(e^{-i\omega}) (G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta)) d\omega + A_0 + B_0
\end{aligned}$$

where we have used (4.9) in the third equality. Since $\Phi_{\eta_0 u}(z)$ by (4.9) is strictly anticausal and since $G_{0,OE}(z)$ and $G(z,\theta)$ both are causal, a term-by-term integration shows that

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{\eta_0 u}(e^{i\omega}) (G_{0,OE}(e^{-i\omega}) - G(e^{-i\omega},\theta)) d\omega = 0 \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{\eta_0 u}(e^{-i\omega}) (G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta)) d\omega = 0
\end{aligned}$$

Thus

$$\mathrm{E}(\eta(t,\theta)^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta)\right|^2 \Phi_u(e^{i\omega})d\omega + A_0 + B_0$$

and (4.14) follows as A_0 and B_0 are independent of θ. □

Theorem 4.2 shows that a low order output error model approximation of an OE-LTI-SOE results in the same kind of approximation as a low order approximation of an LTI system. More specifically, (4.14) shows that if $\Phi_u(e^{i\omega})$ is large in a certain frequency region, the parameter vector θ will be chosen such that

$$|G_{0,OE}(e^{i\omega}) - G(e^{i\omega},\theta)|$$

is small in that frequency region.

It is however important to remember that there is a major difference between the linear and the nonlinear cases. If the true system is an LTI system, it is always desirable to approximate it as well as possible, at least for some frequencies. On the other hand, if the system is nonlinear, there is no guarantee that the OE-LTI-SOE is a good model of the system for any other input signals than the one that it was defined for. Actually, it might be a bad model also for this signal.

If, for example, a second order output error model is estimated and the input power is focused in a certain frequency region the model will in general approximate a *different* OE-LTI-SOE than if, for example, a white input signal had been used. Actually, input signals with equal $\Phi_u(z)$ but different amplitude distributions will in general give rise to different OE-LTI-SOE:s of a nonlinear system. A simple example of this will be shown later in Section 5.3. These observations make it much harder to design the input such that it is suitable for low order LTI approximations when the system is nonlinear. Some examples of input signals that might be suitable for this purpose will be given later in this thesis. First, we will in the next section turn our attention to LTI-SOE:s that contain a noise model.

4.3 The General Error Model Type

Consider once again the general LTI model (2.1) from Section 2.1. For this model, the optimal predictor can be written as in (2.2). Predictors with this structure are used in the prediction-error method to compute parameter dependent predictions of the output signal of the system (cf. Section 2.3). This predicted output is compared with the measured output and the parameters are selected such that a least squares criterion is minimized.

With this in mind, it is natural to define the best general LTI model in the mean-square error sense as the LTI model whose predictor (2.2) minimizes the mean-square error. The optimal LTI model according to this definition will here be called the *General Error LTI Second Order Equivalent* (GE-LTI-SOE). It is described more clearly in the following definition.

Definition 4.2
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. The General Error LTI Second Order Equivalent *(GE-LTI-SOE) of this system is the pair of transfer operators $(G_{0,GE}(q),\ H_{0,GE}(q))$ that are formed as*

$$G_{0,GE}(q) = (1 - q^{-1}\tilde{W}_y(q))^{-1}\tilde{W}_u(q)$$
$$H_{0,GE}(q) = (1 - q^{-1}\tilde{W}_y(q))^{-1}$$

Here $\tilde{W}_u(q)$ and $\tilde{W}_y(q)$ are the stable and causal LTI filters that minimize the mean-square error $\mathrm{E}((y(t) - W_u(q)u(t) - W_y(q)y(t-1))^2)$, i.e.,

$$(\tilde{W}_u(q),\ \tilde{W}_y(q)) = \operatorname*{arg\,min}_{W_u, W_y \in \mathcal{G}} \mathrm{E}((y(t) - W_u(q)u(t) - W_y(q)y(t-1))^2)$$

where $\mathcal{G}$ denotes the set of all stable and causal LTI models.

GE-LTI-SOE:s are, just like OE-LTI-SOE:s, input-dependent and hence it is only possible to talk about the GE-LTI-SOE of a nonlinear system with respect to a particular input signal. Another similarity between GE-LTI-SOE:s and OE-LTI-SOE:s is that $\tilde{W}_u(q)$ and $\tilde{W}_y(q)$ are assumed to be initialized at $t = -\infty$.

In the next theorem, expressions for the GE-LTI-SOE similar to the ones in Ljung (2001) will be derived. The only differences between these two versions of GE-LTI-SOE expressions are that the GE-LTI-SOE here is allowed to contain a direct term from the input and that it is expressed explicitly using components of the canonical spectral factor $T(z)$ from Assumption A2. The proof of the theorem is very similar to the proof of Theorem 4.1 and hence to the proof of the scalar Wiener filter in Kailath et al. (2000).

Theorem 4.3 (General Error LTI Second Order Equivalents)
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Then the General Error LTI Second Order Equivalent (GE-LTI-SOE) $(G_{0,GE}(z),\ H_{0,GE}(z))$ of this system is

$$G_{0,GE}(z) = \frac{zT_{21}(z)}{T_{11}(z)} \tag{4.15a}$$

$$H_{0,GE}(z) = \frac{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}{T_{11}(z)} \tag{4.15b}$$

where $T_{11}(z)$, $T_{12}(z)$, $T_{21}(z)$ and $T_{22}(z)$ are elements of the canonical spectral factor of the z-spectrum for $\zeta(t) = (u(t),\ y(t-1))^T$, i.e.,

$$\Phi_\zeta(z) = \begin{pmatrix} \Phi_u(z) & z\Phi_{uy}(z) \\ z^{-1}\Phi_{yu}(z) & \Phi_y(z) \end{pmatrix} = T(z)Q_\zeta T^T(z^{-1}) \tag{4.16a}$$

$$T(z) = \begin{pmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{pmatrix} \tag{4.16b}$$

Proof: *The GE-LTI-SOE is defined by means of the Wiener filter* $(\tilde{W}_u(q),\ \tilde{W}_y(q))$ *that predicts* $y(t)$ *from* $(y(t-k))_{k=1}^{\infty}$ *and* $(u(t-k))_{k=0}^{\infty}$.

$$\hat{y}(t) = \tilde{W}_u(q)u(t) + \tilde{W}_y(q)y(t-1) \tag{4.17}$$

The filter components $\tilde{W}_u(q)$ *and* $\tilde{W}_y(q)$ *are defined as the stable and causal LTI filters that minimize* $\mathrm{E}((y(t) - W_u(q)u(t) - W_y(q)y(t-1))^2)$ *or, equivalently, the filters that satisfy the* Wiener-Hopf conditions

$$R_{yu}(\tau) - \sum_{k=0}^{\infty} \tilde{w}_y(k) R_{yu}(\tau - k - 1) - \sum_{k=0}^{\infty} \tilde{w}_u(k) R_u(\tau - k) = 0, \quad \tau \geq 0 \tag{4.18a}$$

$$R_y(\tau + 1) - \sum_{k=0}^{\infty} \tilde{w}_y(k) R_y(\tau - k) - \sum_{k=0}^{\infty} \tilde{w}_u(k) R_{uy}(\tau + 1 - k) = 0, \quad \tau \geq 0 \tag{4.18b}$$

Using the z-transform, these conditions can be rewritten as

$$\Phi_{yu}(z) - \tilde{W}_y(z) z^{-1} \Phi_{yu}(z) - \tilde{W}_u(z)\Phi_u(z) = K_1(z) \tag{4.19a}$$

$$z\Phi_y(z) - \tilde{W}_y(z)\Phi_y(z) - \tilde{W}_u(z) z \Phi_{uy}(z) = K_2(z) \tag{4.19b}$$

where $K_1(z)$ *and* $K_2(z)$ *are stable and strictly anticausal transfer functions. Equations (4.19) and (4.2b) give*

$$\begin{pmatrix} K_1(z) & K_2(z) \end{pmatrix} = \underbrace{\begin{pmatrix} \Phi_{yu}(z) & z\Phi_y(z) \end{pmatrix}}_{=(0\ z)\Phi_\zeta(z)} - \begin{pmatrix} \tilde{W}_u(z) & \tilde{W}_y(z) \end{pmatrix} \Phi_\zeta(z) \tag{4.20}$$

Using the spectral factorization of $\Phi_\zeta(z)$ *and multiplying by* $T^{-T}(z^{-1})$ *now gives*

$$\tilde{K}(z) \triangleq \begin{pmatrix} K_1(z) & K_2(z) \end{pmatrix} T^{-T}(z^{-1}) = \begin{pmatrix} 0 & z \end{pmatrix} T(z) Q_\zeta - \begin{pmatrix} \tilde{W}_u(z) & \tilde{W}_y(z) \end{pmatrix} T(z) Q_\zeta \tag{4.21}$$

where $\tilde{K}(z)$ *is a stable and strictly anticausal transfer matrix, due to the fact that it is a product of the stable and strictly anticausal transfer matrix* $(K_1(z)\ K_2(z))$ *and the stable and anticausal transfer matrix* $T^{-T}(z^{-1})$. *For (4.21) to hold it is necessary that the second right hand term, which is a stable and causal transfer matrix, is equal to the causal part of the first, i.e.,*

$$\begin{aligned}
&\begin{pmatrix} \tilde{W}_u(z) & \tilde{W}_y(z) \end{pmatrix} T(z) Q_\zeta = \left[\begin{pmatrix} 0 & z \end{pmatrix} T(z) Q_\zeta\right]_{\text{causal}} \\
&= \left[\begin{pmatrix} zT_{21}(z)Q_{\zeta 11} + zT_{22}(z)Q_{\zeta 21} & zT_{21}(z)Q_{\zeta 12} + zT_{22}(z)Q_{\zeta 22} \end{pmatrix}\right]_{\text{causal}} \\
&= \begin{pmatrix} zT_{21}(z)Q_{\zeta 11} + z(T_{22}(z) - 1)Q_{\zeta 21} & zT_{21}(z)Q_{\zeta 12} + z(T_{22}(z) - 1)Q_{\zeta 22} \end{pmatrix} \\
&= \left(\begin{pmatrix} 0 & z \end{pmatrix} T(z) - \begin{pmatrix} 0 & z \end{pmatrix}\right) Q_\zeta
\end{aligned}$$

where the third equality follows since $T_{21}(z)$ *is a stable and strictly causal transfer function while* $T_{22}(z)$ *is a monic stable and causal transfer function. This gives*

$$\begin{aligned}
\begin{pmatrix} \tilde{W}_u(z) & \tilde{W}_y(z) \end{pmatrix} &= \left(\begin{pmatrix} 0 & z \end{pmatrix} T(z) - \begin{pmatrix} 0 & z \end{pmatrix}\right) T^{-1}(z) \\
&= \begin{pmatrix} \frac{zT_{21}(z)}{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)} & z\left(1 - \frac{T_{11}(z)}{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}\right) \end{pmatrix}
\end{aligned} \tag{4.22}$$

Let

$$\varepsilon_0(t) = y(t) - \hat{y}(t) = (1 - q^{-1}\tilde{W}_y(q))y(t) - \tilde{W}_u(q)u(t) \qquad (4.23)$$

and rewrite this expression in analogy with (2.1) and (2.2) as

$$y(t) = (1 - q^{-1}\tilde{W}_y(q))^{-1}\tilde{W}_u(q)u(t) + (1 - q^{-1}\tilde{W}_y(q))^{-1}\varepsilon_0(t) \qquad (4.24)$$

With

$$\begin{aligned} G_{0,GE}(q) &= (1 - q^{-1}\tilde{W}_y(q))^{-1}\tilde{W}_u(q) \\ H_{0,GE}(q) &= (1 - q^{-1}\tilde{W}_y(q))^{-1} \end{aligned}$$

(4.24) can be written as

$$y(t) = G_{0,GE}(q)u(t) + H_{0,GE}(q)\varepsilon_0(t) \qquad (4.25)$$

Hence, using (4.22) the General Error LTI Second Order Equivalent of the nonlinear system turns out to be

$$\begin{aligned} G_{0,GE}(z) &= \frac{zT_{21}(z)}{T_{11}(z)} \\ H_{0,GE}(z) &= \frac{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}{T_{11}(z)} \end{aligned}$$

□

From (4.15) we see that the calculation of a GE-LTI-SOE requires knowledge of the canonical spectral factorization of the matrix $\Phi_\zeta(z)$. Although there exists a number of methods for making this factorization, it is definitely more complicated to perform than the canonical factorization of the input z-spectrum that is required for the calculation of the OE-LTI-SOE in Theorem 4.1. In Section 5.5, we will describe some classes of input signals that simplify the calculation of GE-LTI-SOE:s significantly.

It should be noted that the factor $\frac{1}{T_{11}(z)}$ in (4.15) should as usual be interpreted as a causal series expansion. This is always possible because $T_{11}(z)$ is analytic on and outside the unit circle and $T_{11}(\infty) = 1$. Let $H_{0,GE}^{-1}$ denote the transfer function of the inverse model of $H_{0,GE}$, i.e.,

$$H_{0,GE}^{-1}(z) = \frac{1}{H_{0,GE}(z)} \qquad (4.27)$$

By the construction of the GE-LTI-SOE, $H_{0,GE}^{-1}(z)G_{0,GE}(z)$ and $H_{0,GE}^{-1}(z)$ will be stable transfer functions. However, neither $G_{0,GE}(z)$ nor $H_{0,GE}(z)$ needs to be stable since $T_{11}(z)$ might have zeros outside the unit circle. The only thing that is guaranteed is that all unstable poles to $G_{0,GE}(z)$ are also poles to $H_{0,GE}(z)$ and vice versa.

In the previous section, it was shown that the OE-LTI-SOE of a nonlinear system can explain all correlations between the output and past and present input signal components (see Corollary 4.2). As the inclusion of a noise model in the GE-LTI-SOE makes this model more flexible than the OE-LTI-SOE, one might expect that the GE-LTI-SOE should be able to explain more correlations than the OE-LTI-SOE. In the following theorem it will be shown that this is also the case.

Corollary 4.3

Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Let $(G_{0,GE}, H_{0,GE})$ be the corresponding GE-LTI-SOE according to Theorem 4.3 and let $\varepsilon_0(t)$ be defined by (4.23). Then the following holds

$$\begin{aligned}\Phi_{\varepsilon_0 u}(z) &= H_{0,GE}^{-1}(z)(\Phi_{yu}(z) - G_{0,GE}(z)\Phi_u(z)) \\ &= zQ_{\zeta 21}T_{11}(z^{-1}) + zQ_{\zeta 22}T_{12}(z^{-1}) \qquad (4.28a)\\ \Phi_{\varepsilon_0 y}(z) &= H_{0,GE}^{-1}(z)(\Phi_y(z) - G_{0,GE}(z)\Phi_{uy}(z)) \\ &= Q_{\zeta 21}T_{21}(z^{-1}) + Q_{\zeta 22}T_{22}(z^{-1}) \qquad (4.28b)\\ \Phi_{\varepsilon_0}(z) &= H_{0,GE}^{-1}(z)(\Phi_{y\varepsilon_0}(z) - G_{0,GE}(z)\Phi_{u\varepsilon_0}(z)) = Q_{\zeta 22} \triangleq \lambda_0 \qquad (4.28c)\end{aligned}$$

Furthermore, an alternative way to describe the relations between the z-spectra of u, y and ε_0 by the GE-LTI-SOE is

$$\Phi_{yu}(z) = G_{0,GE}(z)\Phi_u(z) + H_{0,GE}(z)\Phi_{\varepsilon_0 u}(z) \qquad (4.29a)$$

$$\Phi_{y\varepsilon_0}(z) = G_{0,GE}(z)\Phi_{u\varepsilon_0}(z) + H_{0,GE}(z)\lambda_0 \qquad (4.29b)$$

$$\Phi_y(z) = \begin{pmatrix} G_{0,GE}(z) & H_{0,GE}(z)\end{pmatrix} \begin{pmatrix} \Phi_u(z) & \Phi_{u\varepsilon_0}(z) \\ \Phi_{\varepsilon_0 u}(z) & \lambda_0 \end{pmatrix} \begin{pmatrix} G_{0,GE}(z^{-1}) \\ H_{0,GE}(z^{-1})\end{pmatrix} \qquad (4.29c)$$

Proof: *Note that (4.23) can be written*

$$\varepsilon_0(t) = H_{0,GE}^{-1}(q)(y(t) - G_{0,GE}(q)u(t)) \qquad (4.30)$$

This gives

$$\begin{aligned}\Phi_{\varepsilon_0 u}(z) &= H_{0,GE}^{-1}(z)(\Phi_{yu}(z) - G_{0,GE}(z)\Phi_u(z)) \\ \Phi_{\varepsilon_0 y}(z) &= H_{0,GE}^{-1}(z)(\Phi_y(z) - G_{0,GE}(z)\Phi_{uy}(z))\end{aligned}$$

Together with (4.15) and (4.16a) these expressions can be rewritten as

$$\begin{aligned}&\begin{pmatrix} z^{-1}\Phi_{\varepsilon_0 u}(z) & \Phi_{\varepsilon_0 y}(z)\end{pmatrix} \\ &= H_{0,GE}^{-1}(z)\begin{pmatrix} z^{-1}\Phi_{yu}(z) - z^{-1}G_{0,GE}(z)\Phi_u(z) & \Phi_y(z) - G_{0,GE}(z)\Phi_{uy}(z)\end{pmatrix} \\ &= \frac{T_{11}(z)}{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}\begin{pmatrix} -\frac{T_{21}(z)}{T_{11}(z)} & 1\end{pmatrix}\Phi_\zeta(z) = \begin{pmatrix} 0 & 1\end{pmatrix} T^{-1}(z)\Phi_\zeta(z) \\ &= \begin{pmatrix} 0 & 1\end{pmatrix} Q_\zeta T^T(z^{-1}) = \begin{pmatrix} Q_{\zeta 21} & Q_{\zeta 22}\end{pmatrix} T^T(z^{-1}) \\ &= \begin{pmatrix} Q_{\zeta 21}T_{11}(z^{-1}) + Q_{\zeta 22}T_{12}(z^{-1}) & Q_{\zeta 21}T_{21}(z^{-1}) + Q_{\zeta 22}T_{22}(z^{-1})\end{pmatrix}\end{aligned}$$

Hence, (4.28a) and (4.28b) have been shown. Furthermore, (4.30) gives, using (4.15), (4.28a) and (4.28b), the following expression

$$\begin{aligned}\Phi_{\varepsilon_0}(z) &= H_{0,GE}^{-1}(z)(\Phi_{y\varepsilon_0}(z) - G_{0,GE}(z)\Phi_{u\varepsilon_0}(z)) \\ &= \frac{T_{11}(z)}{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}(Q_{\zeta 21}T_{21}(z) + Q_{\zeta 22}T_{22}(z) \\ &\quad - \frac{zT_{21}(z)}{T_{11}(z)}z^{-1}\left(Q_{\zeta 21}T_{11}(z) + Q_{\zeta 22}T_{12}(z)\right)) \\ &= \frac{T_{11}(z)}{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}\left(Q_{\zeta 22}T_{22}(z) - Q_{\zeta 22}\frac{T_{12}(z)T_{21}(z)}{T_{11}(z)}\right) \\ &= Q_{\zeta 22}\end{aligned}$$

and (4.28c) follows.

The expressions (4.28a) and (4.28c) can be rewritten as (4.29a) and (4.29b), respectively. Finally, (4.29c) follows if (4.29a) and (4.29b) are inserted in (4.28b).

□

The main result of Corollary 4.3 is that $\Phi_{\varepsilon_0 u}(z)$ and $\Phi_{\varepsilon_0 y}(z)$ are strictly anticausal and anticausal, respectively, and that $\Phi_{\varepsilon_0}(z)$ is a constant. Hence, (4.28) illustrates that the GE-LTI-SOE really is the "best" LTI model of the nonlinear system as the remaining residuals $\varepsilon_0(t)$ are uncorrelated with past outputs, past and present inputs and with residuals at all other time instants.

In addition, Corollary 4.3 explains why the name *LTI Second Order Equivalent* is natural. The alternative version of (4.28) in (4.29) emphasizes the filtering capabilities of $G_{0,GE}$ and $H_{0,GE}$. As a matter of fact, (4.29) shows that the GE-LTI-SOE is impossible to distinguish from the true nonlinear system only by looking at second order properties of y, u and ε_0. The LTI system $(G_{0,GE}(q),\ H_{0,GE}(q))$ is thus equivalent to the nonlinear system for the input in question if only second order properties are considered, hence the name GE-LTI-SOE. The additional *General Error* in the name GE-LTI-SOE is added in order to distinguish this type of LTI model from the previously described output error model type, which does not include a noise model.

A fundamental observation about LTI-SOE:s is that the OE-LTI-SOE $G_{0,OE}$ and the GE-LTI-SOE $G_{0,GE}$ are not always equal. This is shown in Example 4.3. The particular system and input signal used in this example have been taken from Forssell and Ljung (2000), where it is used to show that $\Phi_{yu}(z)/\Phi_u(z)$ can be noncausal. Here, however, we will also derive the GE-LTI-SOE of this system.

Example 4.3
Consider the static nonlinear system

$$y(t) = u(t)^2 - 3 \tag{4.31}$$

with the input

$$u(t) = e(t) + e(t-1)^2 - 1$$

where $e(t)$ here is a white Gaussian process with zero mean and unit variance. Straightforward calculations (see Appendix A.1), which are similar to the ones in Example 4.2, give

$$\begin{aligned} \Phi_u(z) &= 3 \\ \Phi_{yu}(z) &= 2z + 8 \\ \Phi_y(z) &= 8z + 66 + 8z^{-1} \end{aligned}$$

This gives

$$\begin{aligned} \Phi_\zeta(z) &= \begin{pmatrix} 3 & 2+8z \\ 2+8z^{-1} & 8z+66+8z^{-1} \end{pmatrix} = T(z)Q_\zeta T^T(z^{-1}) \\ T(z) &= \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{4161}-33}{12} z^{-1} & 1 + \frac{65-\sqrt{4161}}{8} z^{-1} \end{pmatrix} \\ Q_\zeta &= \begin{pmatrix} 3 & 2 \\ 2 & 23 + \frac{\sqrt{4161}}{3} \end{pmatrix} \end{aligned}$$

The spectral factor $T(z)$ has been computed by a diagonalization of $\Phi_\zeta(z)$ followed by a factorization of the derived diagonal matrix and an adjustment in order to achieve $T(+\infty) = I$. The complete derivation of $T(z)$ can be found in Appendix A.1. From the derived transform expressions, the OE-LTI-SOE of the system (4.31) for this input is found to be

$$G_{0,OE}(z) = \frac{8}{3} \approx 2.6667$$

while the GE-LTI-SOE is

$$\begin{aligned} G_{0,GE}(z) &= \frac{\sqrt{4161}-33}{12} \approx 2.6255 \\ H_{0,GE}(z) &= 1 + \frac{65-\sqrt{4161}}{8} z^{-1} \end{aligned}$$

Note that $G_{0,OE}(z) \neq G_{0,GE}(z)$ despite the fact that the system operates in open loop.

The fact that $G_{0,OE}(z) \neq G_{0,GE}(z)$ for some open-loop nonlinear systems and inputs is the reason why a matrix-valued spectral factorization in general has to be performed when the GE-LTI-SOE is calculated. If $G_{0,OE}(z)$ and $G_{0,GE}(z)$ always would have been equal, the GE-LTI-SOE could have been calculated using only scalar spectral factorizations. In Section 5.5 it will be shown that such a simplified calculation of the GE-LTI-SOE is possible for some classes of input signals.

Just like in the case of OE-LTI-SOE:s, it is in practice often hard to know the correct order of the GE-LTI-SOE. As a matter of fact, it might actually be infinite dimensional. Hence, it is also here interesting to understand in what sense a low order model can approximate the GE-LTI-SOE of a nonlinear system. Also for GE-LTI-SOE:s, this approximation is similar to a low order approximation of an LTI system, something which is shown in the next theorem. Apart from the fact that a GE-LTI-SOE is here allowed to contain a direct term from the input, this theorem is identical to Theorem 4.1 in Ljung (2001). The proof is, just like for Theorem 4.2, similar to the outlined proof in Problem 8G.5 in Ljung (1999).

Theorem 4.4
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Let $(G_{0,GE}, H_{0,GE})$ be the corresponding GE-LTI-SOE according to Theorem 4.3, let $\varepsilon_0(t)$ be defined by (4.23) and assume that both $G_{0,GE}$ and $H_{0,GE}$ are stable. Consider a parameterized stable and causal model $(G(q,\theta),\ H(q,\theta))$, where $H(q,\theta)$ is monic and where $H^{-1}(q,\theta)$ is stable and causal. Suppose that this model is fit to the signals u and y according to

$$\hat{\theta} = \arg\min_{\theta} \mathrm{E}(\varepsilon(t,\theta)^2) \tag{4.32}$$

where

$$\begin{aligned}\varepsilon(t,\theta) &= y(t) - H^{-1}(q,\theta)G(q,\theta)u(t) - (1 - H^{-1}(q,\theta))y(t)\\ &= H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t))\end{aligned} \tag{4.33}$$

Then it follows that

$$\hat{\theta} = \arg\min_{\theta} \int_{-\pi}^{\pi} \Delta_{GH}(e^{i\omega},\theta)^T \begin{pmatrix}\Phi_u(e^{i\omega}) & \Phi_{u\varepsilon_0}(e^{i\omega})\\ \Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0\end{pmatrix} \Delta_{GH}(e^{-i\omega},\theta)d\omega \tag{4.34}$$

where

$$\Delta_{GH}(z,\theta) = \frac{1}{H(z,\theta)}\begin{pmatrix}G_{0,GE}(z) - G(z,\theta)\\ H_{0,GE}(z) - H(z,\theta)\end{pmatrix} \tag{4.35}$$

Proof: *Equation (4.25) can be used to rewrite (4.33) as*

$$\begin{aligned}\varepsilon(t,\theta) &= H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t))\\ &= H^{-1}(q,\theta)(G_{0,GE}(q)u(t) + H_{0,GE}(q)\varepsilon_0(t) - G(q,\theta)u(t))\\ &= H^{-1}(q,\theta)((G_{0,GE}(q) - G(q,\theta))u(t) + (H_{0,GE}(q) - H(q,\theta))\varepsilon_0(t))\\ &\quad + \varepsilon_0(t) = \left(\Delta_{GH}(q,\theta)^T + \begin{pmatrix}0 & 1\end{pmatrix}\right)\begin{pmatrix}u(t)\\ \varepsilon_0(t)\end{pmatrix}\end{aligned}$$

This gives

$$\begin{aligned}
\Phi_\varepsilon(z,\theta) &= \big(\Delta_{GH}(z,\theta)^T + \begin{pmatrix}0 & 1\end{pmatrix}\big)\begin{pmatrix}\Phi_u(z) & \Phi_{u\varepsilon_0}(z)\\ \Phi_{\varepsilon_0 u}(z) & \lambda_0\end{pmatrix}\left(\Delta_{GH}(z^{-1},\theta) + \begin{pmatrix}0\\1\end{pmatrix}\right)\\
&= \Delta_{GH}(z,\theta)^T\begin{pmatrix}\Phi_u(z) & \Phi_{u\varepsilon_0}(z)\\ \Phi_{\varepsilon_0 u}(z) & \lambda_0\end{pmatrix}\Delta_{GH}(z^{-1},\theta)\\
&\quad + \Delta_{GH}(z,\theta)^T\begin{pmatrix}\Phi_{u\varepsilon_0}(z)\\ \lambda_0\end{pmatrix}\\
&\quad + \begin{pmatrix}\Phi_{\varepsilon_0 u}(z) & \lambda_0\end{pmatrix}\Delta_{GH}(z^{-1},\theta) + \lambda_0
\end{aligned}$$

Parseval's relation gives

$$\begin{aligned}
\mathrm{E}(\varepsilon(t,\theta)^2) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_\varepsilon(e^{i\omega},\theta)d\omega\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{GH}(e^{i\omega},\theta)^T\begin{pmatrix}\Phi_u(e^{i\omega}) & \Phi_{u\varepsilon_0}(e^{i\omega})\\ \Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0\end{pmatrix}\Delta_{GH}(e^{-i\omega},\theta)d\omega\\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{GH}(e^{i\omega},\theta)^T\begin{pmatrix}\Phi_{u\varepsilon_0}(e^{i\omega})\\ \lambda_0\end{pmatrix}d\omega\\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}\begin{pmatrix}\Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0\end{pmatrix}\Delta_{GH}(e^{-i\omega},\theta)d\omega + \lambda_0
\end{aligned}$$

The transfer function $H^{-1}(z,\theta)(G_{0,GE}(z) - G(z,\theta))$ is causal and, since both $H_{0,GE}(z)$ and $H(z,\theta)$ are monic, the transfer function

$$H^{-1}(z,\theta)(H_{0,GE}(z) - H(z,\theta))$$

is strictly causal. Since $\Phi_{u\varepsilon_0}(z)$ is strictly causal (cf. (4.28a)), this implies that a term-by-term integration gives

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{GH}(e^{i\omega},\theta)^T\begin{pmatrix}\Phi_{u\varepsilon_0}(e^{i\omega})\\ \lambda_0\end{pmatrix}d\omega = 0\\
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\begin{pmatrix}\Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0\end{pmatrix}\Delta_{GH}(e^{-i\omega},\theta)d\omega = 0
\end{aligned}$$

Hence,

$$\mathrm{E}(\varepsilon(t,\theta)^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{GH}(e^{i\omega},\theta)^T\begin{pmatrix}\Phi_u(e^{i\omega}) & \Phi_{u\varepsilon_0}(e^{i\omega})\\ \Phi_{\varepsilon_0 u}(e^{i\omega}) & \lambda_0\end{pmatrix}\Delta_{GH}(e^{-i\omega},\theta)d\omega + \lambda_0$$

and (4.34) follows since λ_0 is independent of θ. □

The previous theorem shows that the GE-LTI-SOE can be approximated by an LTI model of lower order than the GE-LTI-SOE according to (4.34). Just like for OE-LTI-SOE:s, it is here important to remember that the GE-LTI-SOE is in general input-dependent. Hence, different input signals will make a parameterized LTI model close to *different* GE-LTI-SOE:s.

Theorem 4.4 can be simplified if the true system actually is an open-loop LTI system. In this case, there is no correlation between the residuals and the input signal, i.e., $\Phi_{u\varepsilon_0}(z)$ is identically equal to zero (see Ljung, 1999, p. 265). However, for a open-loop nonlinear system this is in general not true.

Consider once again the system and input signal in Example 4.3. Using (4.28a) and the expressions for $T(z)$ and Q_ζ from Appendix A.1 it is easy to verify that $\Phi_{u\varepsilon_0}(z) = 2z^{-1}$ in this case, despite the fact that the system in Example 4.3 is an open-loop system.

Without any prior knowledge about the structure of this system and using an intuitive reasoning based on the linear case, a nonzero strictly causal $\Phi_{u\varepsilon_0}(z)$ might have been taken as an indication that the system actually was a closed-loop system. Hence, it is a justified question whether such a closed-loop interpretation of a GE-LTI-SOE can always be made. This is the topic of the next section.

4.4 Interpretations of the GE-LTI-SOE

The fact that $\Phi_{u\varepsilon_0}(z)$ in general is nonzero in (4.29) has some implications on the LTI interpretation. If we would like to interpret all second order correlations between u, y and ε_0 as results of linear filter connections we have to allow the overall linear system to include some kind of feedback or feedforward connections. This is natural as we have not imposed any restrictions on the true system that exclude the possibility that it in fact is a linear closed-loop system.

It turns out that when the GE-LTI-SOE is well-defined according to Theorem 4.3 it is always possible to compose a stable linear closed-loop system which fulfills (4.29) and which also explains the correlation between $u(t)$ and past $\varepsilon_0(t-k)$, i.e., it explains why $\Phi_{u\varepsilon_0}(z) \neq 0$. Consider the closed-loop system in Figure 4.2. This system is described by

$$u(t) = r_0(t) - F_0(q)N_0(q)\tilde{H}_0(q)\varepsilon_0(t) - F_0(q)N_0(q)\tilde{G}_0(q)u(t)$$

$$\Rightarrow u(t) = \frac{1}{1 + F_0(q)N_0(q)\tilde{G}_0(q)} r_0(t) + \frac{-F_0(q)N_0(q)\tilde{H}_0(q)}{1 + F_0(q)N_0(q)\tilde{G}_0(q)} \varepsilon_0(t) \tag{4.36a}$$

$$y(t) = N_0(q)\tilde{H}_0(q)\varepsilon_0(t) + N_0(q)\tilde{G}_0(q)r_0(t) - N_0(q)\tilde{G}_0(q)F_0(q)y(t)$$

$$\Rightarrow y(t) = \frac{N_0(q)\tilde{G}_0(q)}{1 + F_0(q)N_0(q)\tilde{G}_0(q)} r_0(t) + \frac{N_0(q)\tilde{H}_0(q)}{1 + F_0(q)N_0(q)\tilde{G}_0(q)} \varepsilon_0(t) \tag{4.36b}$$

Let $N_0(z)$ be a causal, possibly unstable, LTI system such that $G_{0,GE}(z)$ and $H_{0,GE}(z)$ can be factorized as

$$G_{0,GE}(z) = \tilde{G}_0(z)N_0(z) \tag{4.37a}$$

$$H_{0,GE}(z) = \tilde{H}_0(z)N_0(z) \tag{4.37b}$$

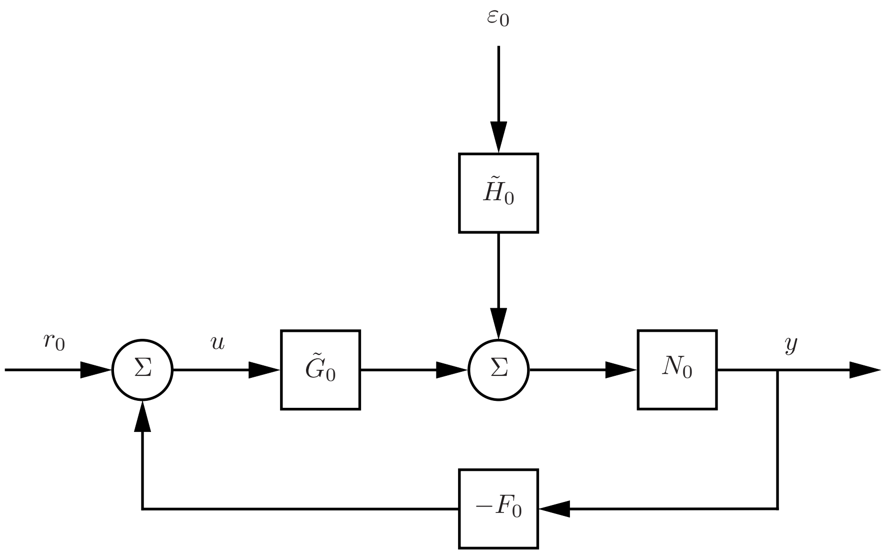


Figure 4.2 *The GE-LTI-SOE can be interpreted as being part of a linear closed-loop system.*

where $\tilde{G}_0$ and $\tilde{H}_0$ are causal and stable LTI systems. Furthermore, let

$$F_0(z) = -\frac{\Phi_{u\varepsilon_0}(z)}{\Phi_{y\varepsilon_0}(z)} \tag{4.38}$$

and, as usual, interpret the factor $\frac{1}{\Phi_{y\varepsilon_0}(z)}$ in this expression as a causal series expansion. This is always possible because $\Phi_{y\varepsilon_0}(z) = Q_{\zeta 21}T_{21}(z) + Q_{\zeta 22}T_{22}(z)$ is analytic on and outside the unit circle and $\Phi_{y\varepsilon_0}(\infty) = Q_{\zeta 22} > 0$ (see (4.28b)). Since $\Phi_{u\varepsilon_0}(z)$ is strictly causal (see (4.28a)), also $F_0(z)$ will be strictly causal. Note, however, that $F_0(z)$ might be unstable.

The interpretation of the GE-LTI-SOE as a part of a linear closed-loop system is meaningless if this system is unstable. Hence, it is crucial to check whether the transfer functions in (4.36) are stable when the definitions (4.37) and (4.38) are used. First, (4.29b), (4.37a) and (4.38) give

$$1 + F_0(z)\underbrace{N_0(z)\tilde{G}_0(z)}_{=G_{0,GE}(z)} = \frac{\Phi_{y\varepsilon_0}(z) - G_{0,GE}(z)\Phi_{u\varepsilon_0}(z)}{\Phi_{y\varepsilon_0}(z)} = \frac{H_{0,GE}(z)\lambda_0}{\Phi_{y\varepsilon_0}(z)}$$

Thus

$$\frac{1}{1 + F_0(z)N_0(z)\tilde{G}_0(z)} = \frac{\Phi_{y\varepsilon_0}(z)}{H_{0,GE}(z)\lambda_0} \tag{4.39}$$

is stable since $H_{0,GE}^{-1}(z)$ is stable and $\Phi_{y\varepsilon_0}(z)$ by its construction (4.28b) is analytic and hence absolutely convergent on the unit circle. Since $H_{0,GE}^{-1}(z)G_{0,GE}(z)$ is stable as well, the stability property of $\Phi_{y\varepsilon_0}(z)$ also imply that

$$\begin{aligned}\frac{N_0(z)\tilde{G}_0(z)}{1+F_0(z)N_0(z)\tilde{G}_0(z)} &= \frac{G_{0,GE}(z)\Phi_{y\varepsilon_0}(z)}{H_{0,GE}(z)\lambda_0} \\ \frac{N_0(z)\tilde{H}_0(z)}{1+F_0(z)N_0(z)\tilde{G}_0(z)} &= \frac{\Phi_{y\varepsilon_0}(z)}{\lambda_0}\end{aligned}$$

are stable. Finally,

$$\frac{-F_0(z)N_0(z)\tilde{H}_0(z)}{1+F_0(z)N_0(z)\tilde{G}_0(z)} = \frac{\Phi_{u\varepsilon_0}(z)}{\lambda_0} \tag{4.40}$$

is stable since $\Phi_{u\varepsilon_0}(z)$ by (4.28a) is analytic and hence absolutely convergent on the unit circle. Hence, all transfer functions in (4.36) are stable, and since

$$\frac{F_0(z)}{1+F_0(z)N_0(z)\tilde{G}_0(z)} = -\frac{\Phi_{u\varepsilon_0}(z)}{H_{0,GE}(z)\lambda_0}$$

is also stable, the closed-loop system is internally stable (Glad and Ljung, 2000, Def. 6.1).

As indicated in Figure 4.2, a reference signal r_0 might have to be included as an external input signal to the closed-loop model in order to explain the part of u that does not originate from ε_0. Unlike $G_{0,GE}$ and $H_{0,GE}$, which are defined such that (4.25) holds, F_0 can only explain the correlation between u and ε_0 and not the complete signal u. If (4.37) and (4.38) are inserted in (4.36a) and the result is used to express $\Phi_{u\varepsilon_0}(z)$ in $\Phi_{r_0\varepsilon_0}(z)$ and λ_0 we get

$$\begin{aligned}\Phi_{u\varepsilon_0}(z) &= \frac{1}{1+F_0(z)N_0(z)\tilde{G}_0(z)}\Phi_{r_0\varepsilon_0}(z) + \frac{-F_0(z)N_0(z)\tilde{H}_0(z)}{1+F_0(z)N_0(z)\tilde{G}_0(z)}\lambda_0 \\ \Rightarrow \Phi_{u\varepsilon_0}(z) &= \frac{\Phi_{y\varepsilon_0}(z)}{H_{0,GE}(z)\lambda_0}\Phi_{r_0\varepsilon_0}(z) + \frac{\Phi_{u\varepsilon_0}(z)}{\lambda_0}\lambda_0 \\ \Rightarrow \Phi_{r_0\varepsilon_0}(z) &= 0\end{aligned} \tag{4.41}$$

where (4.39) and (4.40) have been used to rewrite the closed-loop transfer functions. Equation (4.41) shows that r_0 is uncorrelated with ε_0, which is natural if we want a closed-loop interpretation.

Since the linear closed-loop system in Figure 4.2 is internally stable when F_0 is defined as in (4.38), it can always be used to show how the signals u and y could have been generated from an LTI model. More specifically, it is impossible to disprove that the linear closed-loop system has not generated the signals merely by looking at second order properties of u, y and ε_0. Hence, the closed-loop model in Figure 4.2 with the definitions in (4.37) and (4.38) might be called the complete

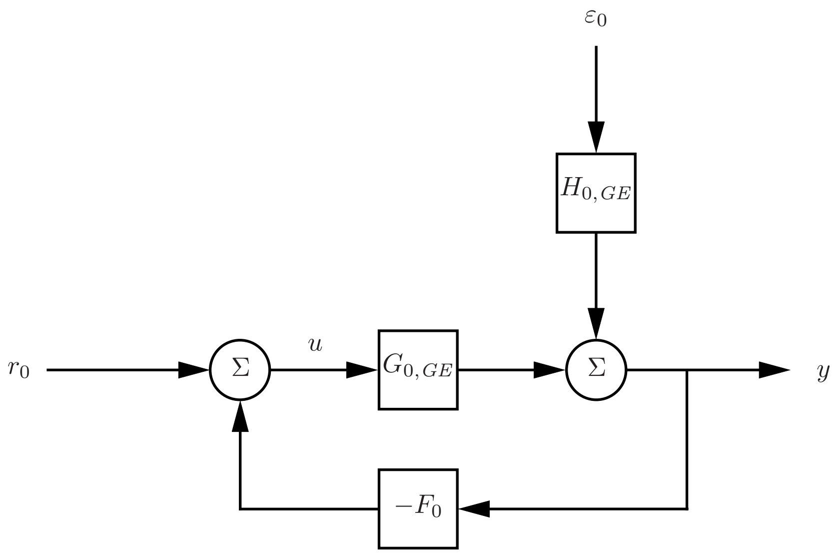


Figure 4.3 *The GE-LTI-SOE can be interpreted as being part of a simplified linear closed-loop system if both $G_{0,GE}$ and $H_{0,GE}$ are stable.*

second order equivalent LTI description of the true system. Of course, since the GE-LTI-SOE is input-dependent, this description is too.

Actually, it is sometimes possible to directly judge the quality of the GE-LTI-SOE if some additional information about the system is available. For example, assume that the input signal u has been generated in such a way that there can be no correlation between $u(t)$ and any system disturbances. If the GE-LTI-SOE of the true system for this u indeed results in a nonzero $\Phi_{u\varepsilon_0}(z)$, then the GE-LTI-SOE cannot be a correct description of the true system.

The interpretation of the GE-LTI-SOE can be somewhat simplified if both $G_{0,GE}$ and $H_{0,GE}$ turn out to be stable. If this is the case, N_0 can be set to 1 in (4.37). This makes it possible to draw a simplified version of the closed-loop model from Figure 4.2. This simplified closed-loop model is shown in Figure 4.3.

However, the closed-loop model in Figure 4.3 is not the only interpretation of a stable GE-LTI-SOE. An alternative explanation of the cross-correlation between u and ε_0 can be given by the feedforward model in Figure 4.4. In this model, let

$$F_{f0}(z) = \frac{\Phi_{u\varepsilon_0}(z)}{\lambda_0} \tag{4.42a}$$

$$r_{f0}(t) = u(t) - F_{f0}(q)\varepsilon_0(t) \tag{4.42b}$$

With this definition, the transfer function $F_{f0}(z)$ is stable since $\Phi_{u\varepsilon_0}(z)$ by

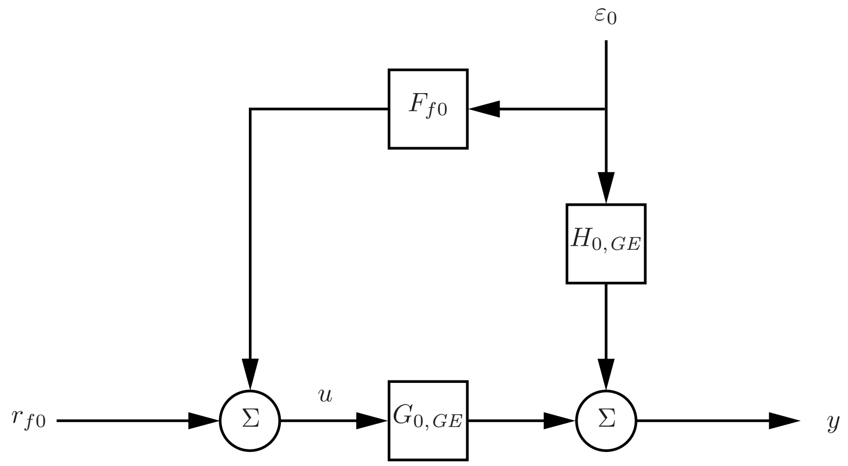


Figure 4.4 *The interpretation of the GE-LTI-SOE as a linear feedforward structure when $G_{0,GE}$ and $H_{0,GE}$ are stable.*

(4.28a) is absolutely convergent on the unit circle. Furthermore, it follows that

$$\Phi_{r_{f0}\varepsilon_0}(z) = \Phi_{u\varepsilon_0}(z) - F_{f0}(z)\lambda_0 = 0 \tag{4.43}$$

and r_{f0} and ε_0 are thus uncorrelated.

The conclusion that can be drawn from the discussion in this section is that a GE-LTI-SOE always can be interpreted as being a part of an internally stable feedback or, in the case of a stable GE-LTI-SOE, feedforward system. Hence, by looking only at second order properties, it is impossible to disprove that any data set, with input and output measurements that fulfill Assumptions A1 and A2, might have been generated by this closed-loop system. However, in some cases additional prior knowledge about the structure of the nonlinear system is available, and this knowledge can influence the interpretation of the LTI-SOE. In the next section, additional knowledge about the noise in the system will be discussed.

4.5 Assumptions on the Noise

In the derivation of the OE-LTI-SOE and GE-LTI-SOE in the previous sections, no assumptions were made about the structure of the nonlinear system. Although structural assumptions are not necessary for the existence of the OE-LTI-SOE and the GE-LTI-SOE, it is hard to draw any conclusions about the properties and usefulness of these second order equivalents without any further information about the nonlinear system.

One important structural property is how the noise enters the system. For most of the results in this thesis we will need the following assumption that says that the noise is additive and uncorrelated with the input and the noise-free output.

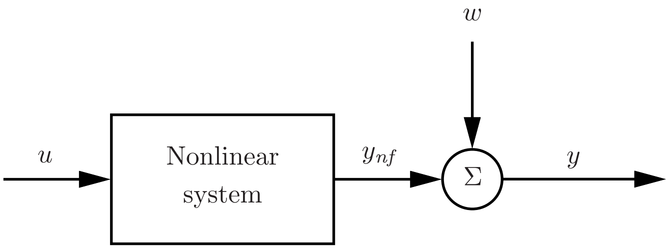


Figure 4.5 *In some cases, the noise is assumed to be output additive.*

Assumption A3: Assume that the output $y(t)$ can be written

$$y(t) = y_{nf}(t) + w(t) \tag{4.44}$$

where y_{nf} is the noise-free response of the nonlinear system and not dependent of other external signals than u, and where w is a noise term which is uncorrelated with u and y_{nf}.

In most cases, this assumption does not hold for a closed-loop system. This is due to the fact that the measured output that is fed back to the controller in such a system usually contains some noise. The input is thus in general correlated with the noise when it comes from a closed-loop system. Hence, Assumption A3 can be viewed as an assumption that the system has the structure shown in Figure 4.5.

However, the fact that Assumption A3 is needed for most of the results in this thesis is not the only reason why LTI approximations for systems in closed loop are not studied much here. Most of the results later in this thesis hold only for input signals with special distributions, e.g. Gaussian inputs or LTI filtered white noise inputs. For an open-loop system, this limitation is often not a problem since the input signal in many cases can be designed rather freely by the user. On the other hand, for a nonlinear closed-loop system, it is usually very hard to guarantee that the distribution of the input signal belongs to a certain class. Hence, most parts of the remaining chapters in this thesis deal with open-loop systems.

As has been mentioned previously, the fact that a certain system operates in open loop can be used to disprove that the GE-LTI-SOE represents the true system. This can be done because it may be necessary to view the GE-LTI-SOE as being a part of a closed-loop system in order to explain a nonzero $\Phi_{u\varepsilon_0}(z)$ (cf. Example 4.3).

This concludes our introductory discussion about the notion of LTI Second Order Equivalents. In this chapter, it has been shown that the OE-LTI-SOE and GE-LTI-SOE of a nonlinear system are well-defined for rather general classes of input and output signals. Furthermore, properties and interpretations of these LTI approximations that hold for all signals in these classes have been discussed. In the remaining chapters of this thesis, we will consider more restricted and specialized classes of input signals for which further properties of the OE-LTI-SOE:s and GE-LTI-SOE:s can be shown.

5

Basic Properties of LTI-SOE:s

In Example 4.2 in the previous chapter, it was shown that the OE-LTI-SOE of a static nonlinear system can be nonstatic. This observation can be viewed as an indication that some caution is needed when conclusions are drawn about the behavior of OE-LTI-SOE:s. A behavior that intuitively seems correct, for example that the OE-LTI-SOE of a static nonlinear system should be static, can actually be erroneous.

In this chapter, we will make some rather straightforward assumptions about the input signals and investigate what these assumptions imply. More specifically, we will assume that the input signal is symmetrically distributed and show that this implies that the OE-LTI-SOE only depends on the odd part of the system. We will also consider input signals generated by filtering white noise through a minimum phase filter. It turns out that for such an input signal, spectral and residual analysis can be used for validation just like in the LTI case. Furthermore, minimum phase filtered white noise can be useful if an LTI system is identified in closed-loop and a nonlinear controller is used. This signal type guarantees also that the $G_{0,OE}(z)$ and $G_{0,GE}(z)$ will be equal. This will be discussed at the end of this chapter. First, however, we will investigate how output additive noise affects the OE-LTI-SOE.

5.1 Additive Noise

Noise can affect a nonlinear system in many ways. For example, the noise can be added to or multiplied with the input before it enters the actual system, or affect the output of the system. Although a large variation of noise dependency can be found in applications, we will here mainly study one type of noise, namely output additive noise that is uncorrelated with the input and the noise-free output as in Assumption A3. Actually, the following lemma shows that this type of noise does not affect the OE-LTI-SOE of a nonlinear system.

Lemma 5.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A3 are fulfilled. Then the OE-LTI-SOE is not influenced by the additive noise process $w(t)$ in (4.44).

Proof: *Since the noise w by Assumption A3 is uncorrelated with u and y_{nf}, we have for any stable LTI model $G(q)$ that*

$$\begin{aligned}\mathrm{E}((y(t) - G(q)u(t))^2) &= \mathrm{E}((y_{nf}(t) + w(t) - G(q)u(t))^2) \\ &= \mathrm{E}(y_{nf}(t) - G(q)u(t)^2) + \mathrm{E}(w(t)^2) \end{aligned} \tag{5.1}$$

Hence, the criterion in Definition 4.1 can be rewritten

$$G_{0,OE}(q) = \arg\min_{G\in\mathcal{G}} \mathrm{E}((y(t) - G(q)u(t))^2) = \arg\min_{G\in\mathcal{G}} \mathrm{E}((y_{nf}(t) - G(q)u(t))^2)$$

and this shows that the OE-LTI-SOE is not influenced by the additive noise. □

The result of the previous lemma is the rather convenient fact that the OE-LTI-SOE is independent of the noise if Assumption A3 holds. Although this assumption might be considered rather restrictive, it will be used frequently in this thesis. Some results about OE-LTI-SOE:s where Assumption A3 is not required will be presented later in this chapter (see Sections 5.3 and 5.4). First, however, we will draw some conclusions about the different influence odd and even nonlinearities have on the OE-LTI-SOE.

5.2 Even and Odd Nonlinearities

Assume that Assumption A3 holds for a certain system and that the noise-free output y_{nf} can be written as $y_{nf}(t) = f((u(t-k))_{k=0}^M)$ for some nonnegative integer M. This means that the system is an NFIR system (see Section 2.2), and in this case, we can divide the real-valued function f in an even part f_e and an odd part f_o

$$\begin{aligned} f((u(t-k))_{k=0}^M) = &\underbrace{\frac{f((u(t-k))_{k=0}^M) + f(-(u(t-k))_{k=0}^M)}{2}}_{f_e((u(t-k))_{k=0}^M)} \\ &+ \underbrace{\frac{f((u(t-k))_{k=0}^M) - f(-(u(t-k))_{k=0}^M)}{2}}_{f_o((u(t-k))_{k=0}^M)} \end{aligned} \tag{5.2}$$

such that

$$f_e(-(u(t-k))_{k=0}^M) = f_e((u(t-k))_{k=0}^M)$$
$$f_o(-(u(t-k))_{k=0}^M) = -f_o((u(t-k))_{k=0}^M)$$

If all simultaneous probability density functions for the process u are even functions, the OE-LTI-SOE of an NFIR system will only depend on the odd part f_o of the system. Hence, we have the following lemma.

Lemma 5.2
Consider an NFIR system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A3 are fulfilled. The noise-free output y_{nf} from (4.44) can then, for some nonnegative integer M, be written

$$y_{nf}(t) = f((u(t-k))_{k=0}^M) = f_e((u(t-k))_{k=0}^M) + f_o((u(t-k))_{k=0}^M) \tag{5.3}$$

where f_e and f_o are even and odd functions, respectively. Assume that all simultaneous probability density functions for the process u are even functions. Then the OE-LTI-SOE depends only on the odd part f_o of the system, i.e.,

$$G_{0,OE}(q) = \arg\min_{G\in\mathcal{G}} \mathrm{E}\big(\big(f_o((u(t-k))_{k=0}^M) - G(q)u(t)\big)^2\big) \tag{5.4}$$

Proof: *From Lemma 5.1 we have*

$$G_{0,OE}(q) = \arg\min_{G\in\mathcal{G}} \mathrm{E}((y_{nf}(t) - G(q)u(t))^2)$$

Using (5.3) this mean-square error criterion can be rewritten

$$\begin{aligned}&\mathrm{E}((y_{nf}(t) - G(q)u(t))^2) = \mathrm{E}\big(\big(f_o((u(t-k))_{k=0}^M) - G(q)u(t)\big)^2\big)\\ &+ \mathrm{E}\big(\big(f_e((u(t-k))_{k=0}^M)\big)^2\big) + 2\mathrm{E}(f_o((u(t-k))_{k=0}^M)f_e((u(t-k))_{k=0}^M))\\ &- 2\sum_{j=0}^{\infty} g(j)\mathrm{E}(u(t-j)f_e((u(t-k))_{k=0}^M))\end{aligned}$$

The cross-terms in this expansion are equal to zero since

$$f_o((u(t-k))_{k=0}^M)f_e((u(t-k))_{k=0}^M)$$

and

$$u(t-j)f_e((u(t-k))_{k=0}^M),$$

$j \in \mathbb{N}$, are odd functions of $(u(t-k))_{k=0}^{\max(M,j)}$ and since all probability density functions of u are even functions. Hence

$$\begin{aligned}\mathrm{E}((y_{nf}(t) - G(q)u(t))^2) &= \mathrm{E}\big(\big(f_o((u(t-k))_{k=0}^M) - G(q)u(t)\big)^2\big)\\ &+ \mathrm{E}\big(\big(f_e((u(t-k))_{k=0}^M)\big)^2\big)\end{aligned} \tag{5.5}$$

and (5.4) follows. □

The fact that the OE-LTI-SOE is independent of even nonlinearities when a symmetrically distributed input signal is used implies that if two nonlinear systems only differ by even nonlinearities, they will have the same OE-LTI-SOE. However, the variance of the model residuals will of course be larger if a system contains large even nonlinearities. The influence of the even and odd parts of a system on the variance and spectral density of the model residuals is shown by the following lemma.

Lemma 5.3
Consider an NFIR system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A3 are fulfilled. The noise-free output $y_{nf}(t)$ from (4.44) can then, for some nonnegative integer M, be written as in (5.3). Assume that all simultaneous probability density functions for the process u are even functions. Then

$$\Phi_{\eta_0}(z) = \Phi_{d_o}(z) + \Phi_{y_e}(z) + \Phi_w(z) \tag{5.6}$$

where $d_o(t) = f_o((u(t-k))_{k=0}^{M}) - G_{0,OE}(q)u(t)$ and $y_e(t) = f_e((u(t-k))_{k=0}^{M})$.

Proof: *Assumption A3 gives that*

$$\begin{aligned} R_{\eta_0}(\tau) = {} & \mathrm{E}((y_{nf}(t) - G_{0,OE}(q)u(t))(y_{nf}(t-\tau) - G_{0,OE}(q)u(t-\tau))) \\ & + R_w(\tau) \quad \forall \tau \in \mathbb{Z} \end{aligned}$$

and using (5.3), we get

$$\begin{aligned} R_{\eta_0}(\tau) &= \mathrm{E}((d_o(t) + y_e(t))(d_o(t-\tau) + y_e(t-\tau))) + R_w(\tau) \\ &= R_{d_o}(\tau) + R_{d_o y_e}(\tau) + R_{y_e d_o}(\tau) + R_{y_e}(\tau) + R_w(\tau) \quad \forall \tau \in \mathbb{Z} \end{aligned}$$

Since

$$\begin{aligned} R_{d_o y_e}(\tau) = {} & \mathrm{E}(f_o((u(t-k))_{k=0}^{M}) f_e((u(t-k-\tau))_{k=0}^{M})) \\ & - \sum_{j=0}^{\infty} g_{0,OE}(j) \mathrm{E}(u(t-j) f_e((u(t-k-\tau))_{k=0}^{M})) \end{aligned}$$

contains only expectations of odd functions and since all probability density functions of u are even, we get $R_{d_o y_e}(\tau) = R_{y_e d_o}(\tau) = 0$ for all $\tau \in \mathbb{Z}$. Hence,

$$R_{\eta_0}(\tau) = R_{d_o}(\tau) + R_{y_e}(\tau) + R_w(\tau) \quad \forall \tau \in \mathbb{Z} \tag{5.7}$$

and (5.6) follows. □

In particular, (5.7) shows that the variance of the OE-LTI-SOE residuals $\eta_0(t)$ can be written

$$\begin{aligned} R_{\eta_0}(0) = \mathrm{E}(\eta_0(t)^2) = {} & \mathrm{E}(\big(f_o((u(t-k))_{k=0}^{M}) - G(q)u(t)\big)^2) \\ & + \mathrm{E}(\big(f_e((u(t-k))_{k=0}^{M})\big)^2) + \mathrm{E}(w(t)^2) \end{aligned} \tag{5.8}$$

This expression is valid when Assumption A3 holds and when the input signal has even probability density functions, and it shows that there are three conceptually different contributions to $R_{\eta_0}(0)$. The first term in (5.8) is the variance of the unmodeled part of the odd nonlinearities, while the second and third term is the variance of the even part of the system and of the noise, respectively.

Usually, it is not obvious how the input signal should be designed in order to minimize the variance of the residuals. In the next section, we will consider inputs that have been generated by filtering white noise through a minimum phase filter. Later in Section 5.4.1, it will be shown that these inputs reduce the variance of the residuals.

5.3 Minimum Phase Input Filters

A common way to generate a signal u such that its spectral density is equal to some predefined function is to filter white noise e through an LTI filter $L(z)$. Since, by Lemma 2.1, the result of this procedure will be a signal with spectral density $\Phi_u(e^{i\omega}) = |L(e^{i\omega})|^2 R_e(0)$, it is often convenient to consider only $|L(e^{i\omega})|$ when the filter $L(z)$ is designed and let the phase $\arg(L(e^{i\omega}))$ become whatever it becomes. For example, this works well if u is going to be used as input to an LTI system in a linear identification experiment.

However, if the signal u is to be used for an LTI approximation of a nonlinear system, the phase of the prefilter is crucial for the behavior of this approximation. In the following theorem, it will be shown that in this case, it is beneficial to generate the input signal by filtering white noise through an LTI filter which has the minimum phase property.

Theorem 5.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. Assume that the input signal has been generated by filtering white, possibly non-Gaussian, noise $e(t)$ through a minimum phase filter $L_m(z)$. Assume also that any other external signals that affect the output are independent of u. Then the OE-LTI-SOE is

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \frac{\Phi_{ye}(z)}{L_m(z)R_e(0)} \tag{5.9}$$

Proof: *The canonical spectral factorization of $\Phi_u(z)$ (cf. (4.1)) is*

$$\begin{aligned} L(z) &= \frac{L_m(z)}{l_m(0)} \\ r_u &= l_m(0)^2 R_e(0) \end{aligned}$$

Using (4.3) from Theorem 4.1 this gives

$$\begin{aligned}G_{0,OE}(z) &= \frac{l_m(0)}{l_m(0)^2 R_e(0) L_m(z)} \left[\frac{l_m(0)\Phi_{yu}(z)}{L_m(z^{-1})}\right]_{\text{causal}} \\ &= \frac{1}{R_e(0)L_m(z)} \left[\frac{\Phi_{yu}(z)}{L_m(z^{-1})}\right]_{\text{causal}} \\ &= \frac{1}{R_e(0)L_m(z)} \left[\frac{\Phi_{ye}(z)L_m(z^{-1})}{L_m(z^{-1})}\right]_{\text{causal}} \\ &= \frac{1}{R_e(0)L_m(z)} \left[\Phi_{ye}(z)\right]_{\text{causal}}\end{aligned}$$

where we have used Lemma 2.1 in the third equality. The nonlinear system is, by our standard assumption, causal and u is independent of all other external signals that affect the output. This, together with the fact that e is a white noise process, implies that $y(t)$ is independent of $e(t-\tau)$ for all $\tau < 0$. Hence $R_{ye}(\tau) = 0$ for all $\tau < 0$ and

$$\Phi_{ye}(z) = \sum_{\tau=0}^{\infty} R_{ye}(\tau) z^{-\tau}$$

Since the series $\Phi_{ye}(z)$ contains no positive powers of z, taking the causal part does not remove anything. Hence, we have

$$G_{0,OE}(z) = \frac{1}{R_e(0)L_m(z)} \left[\Phi_{ye}(z)\right]_{\text{causal}} = \frac{\Phi_{ye}(z)}{L_m(z)R_e(0)} = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

and we have shown (5.9). □

The assumption in Theorem 5.1 that any other external signals that affect the output should be independent of the input u implies that this theorem usually cannot be applied if the system is a closed-loop LTI system. However, for open-loop systems, the conditions in Theorem 5.1 are not very restrictive.

Actually, the reason why the OE-LTI-SOE in Example 4.2 on page 26 was equal to $\frac{\Phi_{yu}(z)}{\Phi_u(z)}$ is that the input signal in this example was generated by filtering white noise through a minimum phase filter. It is interesting to see what happens with the OE-LTI-SOE if the input filter in Example 4.2 is replaced by a non-minimum phase filter giving the same $\Phi_u(z)$. This is done in the following example.

Example 5.1

Consider the static nonlinear system

$$y(t) = u(t)^3 \tag{5.10}$$

with the input

$$u(t) = \frac{1}{2}e(t) + e(t-1)$$

where $e(t)$ is a sequence of independent random variables with uniform distribution over the interval $[-1, 1]$. For the moment, let $\tilde{R}_{yu}(\tau)$ denote the cross-covariance function in Example 4.2. Then

$$\begin{aligned}
R_{yu}(0) &= \mathrm{E}(u(t)^4) = \mathrm{E}(\left(\frac{1}{2}e(t) + e(t-1)\right)^4) = \mathrm{E}(\left(e(t) + \frac{1}{2}e(t-1)\right)^4) \\
&= \tilde{R}_{yu}(0) = \frac{91}{240} \\
R_{yu}(1) &= \mathrm{E}(u(t)^3 u(t-1)) \\
&= \mathrm{E}(\left(\frac{1}{2}e(t) + e(t-1)\right)^3 \cdot \left(\frac{1}{2}e(t-1) + e(t-2)\right)) \\
&= \mathrm{E}(\left(e(t) + \frac{1}{2}e(t-1)\right)^3 \cdot \left(e(t+1) + \frac{1}{2}e(t)\right)) \\
&= \tilde{R}_{yu}(-1) = \frac{34}{240} \\
R_{yu}(-1) &= \mathrm{E}(u(t)^3 u(t+1)) \\
&= \mathrm{E}(\left(\frac{1}{2}e(t) + e(t-1)\right)^3 \cdot \left(\frac{1}{2}e(t+1) + e(t)\right)) \\
&= \mathrm{E}(\left(e(t) + \frac{1}{2}e(t-1)\right)^3 \cdot \left(e(t-1) + \frac{1}{2}e(t-2)\right)) \\
&= \tilde{R}_{yu}(1) = \frac{46}{240} \\
R_{yu}(\tau) &= 0 \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}
\end{aligned}$$

This gives

$$\Phi_{yu}(z) = \frac{1}{240}(46z + 91 + 34z^{-1}) = \frac{1}{240}\left(1 + \frac{1}{2}z^{-1}\right)(68 + 46z)$$

Furthermore, Lemma 2.1 gives

$$\Phi_u(z) = \left(\frac{1}{2} + z^{-1}\right) \cdot \frac{1}{3} \cdot \left(\frac{1}{2} + z\right) = \frac{1}{12}\left(2z + 5 + 2z^{-1}\right)$$

The canonical spectral factor of $\Phi_u(z)$ is $L(z) = 1 + \frac{1}{2}z^{-1}$ and $r_u = \frac{1}{3}$. Theorem 4.1 gives

$$\begin{aligned}
G_{0,OE}(z) &= \frac{1}{r_u L(z)}\left[\frac{\Phi_{yu}(z)}{L(z^{-1})}\right]_{\text{causal}} = \frac{3}{1 + \frac{1}{2}z^{-1}}\left[\frac{46z + 91 + 34z^{-1}}{240(1 + \frac{1}{2}z)}\right]_{\text{causal}} \\
&= \frac{1}{80} \cdot \frac{1}{1 + \frac{1}{2}z^{-1}}\left[\frac{9z}{1 + \frac{1}{2}z} + 74 + 34z^{-1}\right]_{\text{causal}} \\
&= \frac{1}{80} \cdot \frac{74 + 34z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{1}{40} \cdot \frac{37 + 17z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{0.925 + 0.425z^{-1}}{1 + 0.5z^{-1}}
\end{aligned}$$

Here, just like in Example 4.2, the OE-LTI-SOE of the static nonlinear system $y(t) = u(t)^3$ is nonstatic. However, the OE-LTI-SOE in Example 4.2 is not equal to the OE-LTI-SOE here, since the two input signals have different distributions.

The input signals in Examples 4.2 and 5.1 are similar in the sense that they have equal z-spectra. Furthermore, the probability density function for one input signal component, (or the amplitude distribution of every single $u(t)$), is the same in both examples. Since the nonlinear system is static, this also implies that the probability density functions of a single $y(t)$ are equal in these examples.

Despite these similarities between the input signals in Example 4.2 and 5.1, these inputs generate different OE-LTI-SOE:s because the simultaneous probability density functions of $u(t)$ and $u(t-1)$ are different in the two examples. As we have not calculated $R_{\eta_0}(0)$ in these examples, it is not obvious which OE-LTI-SOE that is most successful in approximating the true system. However, in the next section we will show that it is always better to use a minimum phase generated input than a non-minimum phase generated.

5.4 Properties when $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$

The fact that $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ for, for example, input signals generated by filtering white noise through a minimum phase filter is convenient, since it makes it possible to calculate the OE-LTI-SOE without spectral factorization of the input z-spectrum. In addition, OE-LTI-SOE:s of this kind exhibit a number of interesting properties.

5.4.1 Optimality Properties

The perhaps most obvious property that holds when $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ concerns the residuals $\eta_0(t)$. In the following lemma it will be shown that for such an OE-LTI-SOE the residuals will be uncorrelated with *all* input signal components (cf. Corollary 4.2).

Lemma 5.4
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumption A1 is fulfilled. Assume that the OE-LTI-SOE can be written as

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} \tag{5.11}$$

and let

$$\eta_0(t) = y(t) - G_{0,OE}(q)u(t) \tag{5.12}$$

Then it follows that

$$\Phi_{\eta_0 u}(z) = \Phi_{yu}(z) - G_{0,OE}(z)\Phi_u(z) = 0 \tag{5.13a}$$

$$\Phi_{\eta_0}(z) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}) \tag{5.13b}$$

Proof: *The expression for $\Phi_{\eta_0 u}(z)$ in (5.13a) follows directly from (5.11) and (5.12). Furthermore, using Lemma 2.1, (5.11) and (5.12) also give*

$$\begin{aligned}\Phi_{\eta_0}(z) &= \Phi_y(z) - G_{0,OE}(z)\Phi_{uy}(z) - \Phi_{yu}(z)G_{0,OE}(z^{-1}) \\ &\quad + G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1}) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1})\end{aligned}$$

and hence (5.13b) has been shown. □

The fact that (5.13a) holds when $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ shows that the OE-LTI-SOE in this case really is the “best” noncausal LTI model since the model residuals are uncorrelated with all input signal components. This is no surprise since it can be shown that the ratio $\Phi_{yu}(z)/\Phi_u(z)$ is always the mean-square error optimal noncausal LTI model, i.e., the noncausal LTI-SOE (see Section 3.2.2). If this ratio is causal, it is of course equal to the OE-LTI-SOE.

Intuitively, it seems that it should always be a good idea to use input signals for which the OE-LTI-SOE is equal to the noncausal LTI-SOE. As a matter of fact, input signals for which the OE-LTI-SOE of a nonlinear system can be written as $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ exhibit the following optimality property.

Theorem 5.2
Consider a nonlinear system with input $u_1(t)$ and output $y_1(t)$ such that Assumption A1 is fulfilled. Let $G_{0,OE,1}(z)$ denote the OE-LTI-SOE of the nonlinear system with respect to u_1 and assume that it can be written as

$$G_{0,OE,1}(z) = \frac{\Phi_{y_1 u_1}(z)}{\Phi_{u_1}(z)}$$

Furthermore, let $\eta_{0,1}(t) = y_1(t) - G_{0,OE,1}(q)u_1(t)$.

Consider also another input signal $u_2(t)$ to the same nonlinear system. Assume that this signal generates the output $y_2(t)$ and that $(u_2(t), y_2(t))$ satisfy Assumption A1. Let $G_{0,OE,2}(z)$ denote the OE-LTI-SOE of the nonlinear system with respect to u_2 and let $\eta_{0,2}(t) = y_2(t) - G_{0,OE,2}(q)u_2(t)$. Assume that

$$\begin{aligned}\Phi_{u_2}(e^{i\omega}) &= \Phi_{u_1}(e^{i\omega}), \quad \forall\omega \in [-\pi, \pi] \\ |\Phi_{y_2 u_2}(e^{i\omega})| &= |\Phi_{y_1 u_1}(e^{i\omega})|, \quad \forall\omega \in [-\pi, \pi] \\ R_{y_2}(0) &= R_{y_1}(0)\end{aligned}$$

Then the model residual variance for the OE-LTI-SOE corresponding to u_2 cannot be smaller than it is for the one corresponding to u_1, i.e.,

$$R_{\eta_{0,2}}(0) \geq R_{\eta_{0,1}}(0) \tag{5.14}$$

Proof: *From (4.10) we have for any OE-LTI-SOE that*

$$R_{\eta_0}(0) = R_y(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega})|^2 \Phi_u(e^{i\omega})d\omega$$

For any input signal, the noncausal LTI-SOE is always $\frac{\Phi_{yu}(z)}{\Phi_u(z)}$*. It is easy to verify that (4.10) holds also for the noncausal LTI-SOE if* $G_{0,OE}(e^{i\omega})$ *is replaced by* $\frac{\Phi_{yu}(e^{i\omega})}{\Phi_u(e^{i\omega})}$*. As the stable and causal LTI systems are a subset of the stable and noncausal, it follows that the OE-LTI-SOE will always have a minimum mean-square error that is greater than or equal to the minimum mean-square error that is obtained for the noncausal LTI-SOE. Hence,*

$$\begin{aligned} R_{\eta_{0,2}}(0) &\geq R_{y_2}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \frac{\Phi_{y_2u_2}(e^{i\omega})}{\Phi_{u_2}(e^{i\omega})} \right|^2 \Phi_{u_2}(e^{i\omega}) d\omega \\ &= R_{y_1}(0) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \frac{\Phi_{y_1u_1}(e^{i\omega})}{\Phi_{u_1}(e^{i\omega})} \right|^2 \Phi_{u_1}(e^{i\omega}) d\omega = R_{\eta_{0,1}}(0) \end{aligned}$$

since $G_{0,OE,1}(z) = \frac{\Phi_{y_1u_1}(z)}{\Phi_{u_1}(z)}$. □

Theorem 5.2 shows that, for example, a given minimum phase generated input signal is optimal over a set of other inputs in the sense that it minimizes the variance of the model residuals. Usually, it is not easy to describe this set of input signals and in some cases it might actually be empty. However, the input signals in Examples 4.2 and 5.1 fulfill the assumptions in Theorem 5.2. Hence it follows from (5.14) that the variance of the model residual in Example 4.2 is less than or equal to the corresponding variance in Example 5.1. Actually it can be shown that the variance in the first example is strictly less than the variance in the second.

It should be noted that there is no guarantee that the OE-LTI-SOE will be a useful model of a nonlinear system just because $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$. The only thing that is guaranteed is that it will be easier to see if an estimated model is close to the OE-LTI-SOE since some useful validation methods will work in this case. This will be discussed more in the next section.

5.4.2 Spectral and Residual Analysis

A common way to validate an estimated model of an open-loop LTI system is to compare the frequency response of the model with a nonparametric frequency response estimate obtained by spectral analysis. If these frequency responses are similar this indicates that the order of the parametric model is sufficiently high and that the numerical computation of the estimate has been successful. In Ljung (1999, Sec. 6.4) it is shown that the spectral analysis frequency response estimate $\hat{G}_N(e^{i\omega_0})$ based on N measurements can be written

$$\hat{G}_N(e^{i\omega_0}) = \frac{\hat{\Phi}_{yu}^N(e^{i\omega_0})}{\hat{\Phi}_u^N(e^{i\omega_0})}$$

where $\hat{\Phi}_u^N(e^{i\omega_0})$ and $\hat{\Phi}_{yu}^N(e^{i\omega_0})$ are estimates of the spectral and cross-spectral densities that can be written as smoothed periodograms.

If an LTI model is estimated for an open-loop nonlinear system, it might be tempting to use spectral analysis as a validation method also in this case. However, the spectral analysis frequency response estimate can be quite different from the frequency response of the OE-LTI-SOE and is thus in general useless for validation purposes. Only when the OE-LTI-SOE can be computed as $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ spectral analysis can be used as a validation method.

Example 5.2
Consider once again the system and input signals from Examples 4.2 and 5.1. In these examples, it was shown that the OE-LTI-SOE:s of this system are

$$G_{0,OE,1}(z) = \frac{0.85 + 0.575z^{-1}}{1 + 0.5z^{-1}}$$

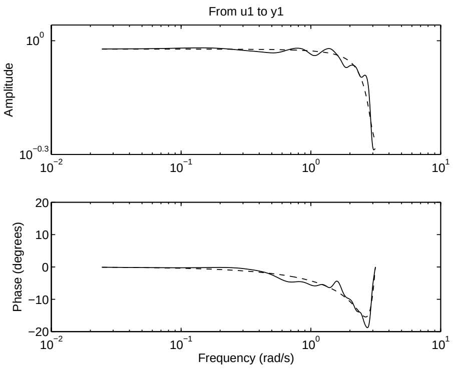
and

$$G_{0,OE,2}(z) = \frac{0.925 + 0.425z^{-1}}{1 + 0.5z^{-1}},$$

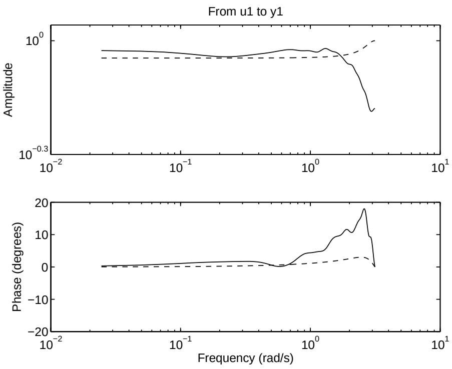
respectively. Here, two data sets with 10000 noise-free input and output measurements have been generated. The first of these data sets was generated with a realization of the minimum phase filtered signal from Example 4.2 as input while the second data set was generated with a realization of the non-minimum phase filtered signal from Example 5.1 as input.
Nonparametric frequency response estimates have been computed from these data sets using spectral analysis with a Hamming window of lag size 30. These estimates are shown in Figure 5.1 together with the frequency responses of the corresponding OE-LTI-SOE:s. The MATLAB code that has been used to generate this figure is available in Appendix B.1.
In Figure 5.1 it can be seen that there is a close match between the OE-LTI-SOE and the nonparametric frequency response estimate when the input has been generated by a minimum phase filter. However, when the input has been generated by a non-minimum phase filter, the OE-LTI-SOE is quite different from the nonparametric estimate.

The conclusion that can be drawn from the previous example is that for LTI approximations of nonlinear systems, spectral analysis can be used as a validation method only when an input signal that guarantees that $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ has been used. An additional property of such input signals is that they make the result of another validation method, residual analysis, easier to interpret. Residual analysis can be used to check if there is remaining correlation between the input and the residuals for a certain model. Such remaining correlation indicates that the model order might be too low. If an accurate model of an open-loop LTI system has been found, the residuals will be uncorrelated with the input signal.

However, if the OE-LTI-SOE of a nonlinear system has been estimated, it will by (4.9) in general only have residuals that are uncorrelated with past and present



a. The OE-LTI-SOE (dashed) and the spectral analysis estimate (solid) for an input generated by a minimum phase filter.



b. The OE-LTI-SOE (dashed) and the spectral analysis estimate (solid) for an input generated by a non-minimum phase filter.

Figure 5.1 *A nonparametric frequency response estimate will be a good approximation of the OE-LTI-SOE only when* $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$.

input signal components. If it is not known that the true system is nonlinear, the remaining correlation between the residuals and future input components might be taken as an indication that the system actually is a closed-loop system. However, if an input that guarantees that $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ has been used, this cannot happen since the residuals then by (5.13a) will be uncorrelated with all input components. In the next section, this property will be used for another purpose.

5.4.3 Closed-loop Identification

One application where LTI approximations of nonlinear systems are useful is closed-loop identification of LTI systems which operate under nonlinear feedback. Consider the closed loop system in Figure 5.2. This system consists of an unknown LTI plant G_0 and a nonlinear controller, and its output can be described as

$$y(t) = G_0(q)u(t) + w(t) \tag{5.15}$$

where u and w are correlated due to the feedback loop. The reference signal r is here assumed to be independent of the process noise w.

Suppose that a model of G_0 is desired and that measurements of r, u and y are available. In this case, the most natural way to estimate such a model is often to

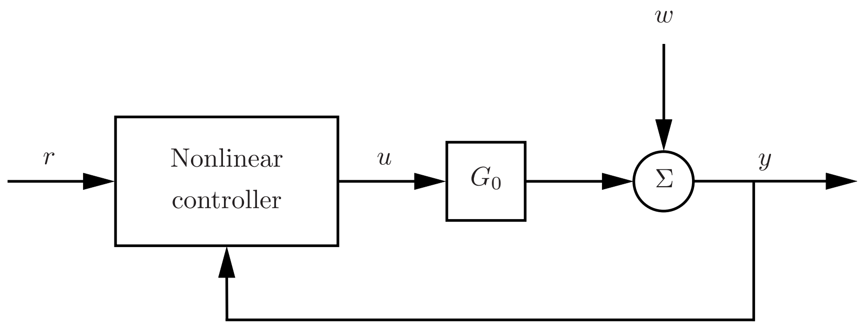


Figure 5.2 *A nonlinear closed-loop system.*

use the *direct prediction error method* (Ljung, 1999, Sec. 13.5). Provided that the model structure is flexible enough to contain the true system, including the true noise description, this approach defines a consistent estimator of G_0.

However, if the model of G_0 is to be used for controller design, there might be one drawback with the direct approach. When an approximate model is used for controller design, it is usually appropriate that it is as accurate as possible in a frequency interval around the desired crossover frequency. In the open-loop case, this accuracy can be increased by the use of a frequency weighting. Unfortunately, in the closed-loop case, such a frequency weighting cannot be used in the direct approach without making the estimator of G_0 biased.

A solution to this problem when the complete closed-loop system is linear is given by the *two-step method* (Van den Hof and Schrama, 1993) or, when a nonlinear controller is present, a version of this method called the *projection method* (Forssell and Ljung, 2000). Both these methods can be viewed as ways to translate the closed-loop identification problem to an open-loop problem where frequency weighting can be used. The main idea used in the two-step and projection methods is to first estimate an LTI model $S(z)$ from r to u and then to use this model to construct a simulated input signal $\hat{u}(t) = S(q)r(t)$. Finally, the identification of G_0 is performed using measurements of y and of the simulated input $\hat{u}$ instead of the true input u.

The reason why it is beneficial to use $\hat{u}$ instead of u in the identification of G_0 is that this enables the use of frequency weighting since $\hat{u}$ and the noise will be uncorrelated, just like for an open-loop system. This is due to the fact that if an output error model

$$u(t) = S(q)r(t) + \eta(t)$$

is used to describe the mapping from r to u and if the identification of $S(z)$ is successful, the result will, in the case of linear feedback, be that

$$S(z) = \frac{\Phi_{ur}(z)}{\Phi_r(z)} \tag{5.16}$$

Let $\tilde{u}(t) = u(t) - \hat{u}(t)$. Then (5.16) implies that

$$\Phi_{\tilde{u}\hat{u}}(z) = \Phi_{u\hat{u}}(z) - \Phi_{\hat{u}}(z) = \Phi_{ur}(z)S(z^{-1}) - S(z)\Phi_r(z)S(z^{-1}) = 0$$

and (5.15) can thus be rewritten as

$$y(t) = G_0(q)\hat{u}(t) + \tilde{w}(t) \tag{5.17}$$

where $\tilde{w}(t) = w(t) + G_0(q)\tilde{u}(t)$ is uncorrelated with $\hat{u}$ since both w and $\tilde{u}$ are uncorrelated with $\hat{u}$. Hence, the system (5.17) can be viewed as an open-loop system and frequency weighting can thus be used.

For a linear closed-loop system, $S(z)$ is simply the ordinary sensitivity function, which of course is causal. However, when a nonlinear controller is present in the closed-loop system, the mapping from r to u is nonlinear. In that case, the ratio $\frac{\Phi_{ur}(z)}{\Phi_u(z)}$ might be noncausal if we want a stable interpretation of it. This is also pointed out by Forssell and Ljung (2000). The main difference between the projection method and the two-step method is that in the former, a noncausal FIR model is used to model the mapping from r to u, while in the latter only a causal $S(z)$ is used. Hence, the projection method is applicable to closed-loop systems with nonlinear controllers while the two-step method in general is not.

However, from the discussion in Section 5.3 we know that if, for example, the reference signal has been generated by filtering white noise through a minimum phase filter, then $\frac{\Phi_{ur}(z)}{\Phi_u(z)}$ will be causal since the nonlinear mapping from r to u is causal and r and w are independent. This means that if a minimum phase generated input signal is used, there is no need to use a noncausal $S(z)$, and hence the two-step method can be applied instead of the projection method.

One advantage of the two-step method is that $S(z)$ can be a rational function. In the projection method in Forssell and Ljung (2000), $S(z)$ is a noncausal FIR model, which means that the true sensitivity function usually cannot be modeled exactly even when the controller is linear. With this observation in mind, it seems that the two-step method is at least an as good alternative for closed-loop identification of an LTI system with nonlinear feedback as the projection method, provided that the reference signal has been designed such that $\frac{\Phi_{ur}(z)}{\Phi_u(z)}$ will be stable and causal independently of the structure of the controller. One example of a class of such reference signals is minimum phase filtered white noise.

Finally, it should be emphasized that the discussion here is based on properties of OE-LTI-SOE:s and that it hence is valid mostly for large data sets. As a matter of fact, the use of a noncausal $S(z)$ in the projection method might be useful for smaller data sets also in cases where $S(z)$ asymptotically will be causal (Forssell and Ljung, 2000).

5.5 LTI-SOE:s with a General Error Model

In the previous section, it was shown that the OE-LTI-SOE exhibit a number of interesting properties for input signals which guarantee that it can be written

$G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$. It is thus a legitimate question whether these input signals also generate GE-LTI-SOE:s with special properties.

Hence, we will now shift focus and discuss GE-LTI-SOE:s of nonlinear systems with input signals such that $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$. In Example 4.3, it was shown that in general $G_{0,OE}(z)$ and $G_{0,GE}(z)$ are not equal even for open-loop nonlinear systems. However, when $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ it follows that $G_{0,GE}(z)$ will be equal to this ratio too. This is shown in the following theorem.

Theorem 5.3
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Assume that the input signal is such that

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} \tag{5.18}$$

Assume also that

$$\Phi_{\eta_0}(z) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1})$$

from (5.13b) has a canonical spectral factorization

$$\Phi_{\eta_0}(z) = L_{\eta_0}(z) r_{\eta_0} L_{\eta_0}(z^{-1}) \tag{5.19}$$

with $r_{\eta_0} > 0$. Then the GE-LTI-SOE is

$$G_{0,GE}(z) = G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} \tag{5.20a}$$

$$H_{0,GE}(z) = L_{\eta_0}(z) \tag{5.20b}$$

Proof: *Let*

$$T(z) = \begin{pmatrix} L(z) & 0 \\ z^{-1}G_{0,OE}(z)L(z) & L_{\eta_0}(z) \end{pmatrix}$$

$$Q_\zeta = \begin{pmatrix} r_u & 0 \\ 0 & r_{\eta_0} \end{pmatrix}$$

where $L(z)$ and r_u are factors in the canonical spectral factorization of $\Phi_u(z)$ according to (4.1). Then

$$T(z)Q_\zeta T^T(z^{-1}) = \begin{pmatrix} L(z) & 0 \\ z^{-1}G_{0,OE}(z)L(z) & L_{\eta_0}(z) \end{pmatrix} \begin{pmatrix} r_u & 0 \\ 0 & r_{\eta_0} \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} L(z^{-1}) & zG_{0,OE}(z^{-1})L(z^{-1}) \\ 0 & L_{\eta_0}(z^{-1}) \end{pmatrix} = \begin{pmatrix} \Phi_u(z) & z\Phi_{uy}(z) \\ z^{-1}\Phi_{yu}(z) & \Phi_y(z) \end{pmatrix} = \Phi_\zeta(z)$$

where we have used (5.18) in the second equality. Since $T(z)$ and

$$T^{-1}(z) = \begin{pmatrix} \frac{1}{L(z)} & 0 \\ -\frac{z^{-1}G_{0,OE}(z)}{L_{\eta_0}(z)} & \frac{1}{L_{\eta_0}(z)} \end{pmatrix}$$

both are analytic in $\{z \in \mathbb{C} : |z| \geq 1\}$, $T(+\infty) = I$ and $Q_\zeta > 0$, we have found the canonical spectral factorization of $\Phi_\zeta(z)$, and from (4.15) in Theorem 4.3 we obtain

$$G_{0,GE}(z) = \frac{zT_{21}(z)}{T_{11}(z)} = G_{0,OE}(z)$$
$$H_{0,GE}(z) = \frac{T_{11}(z)T_{22}(z) - T_{12}(z)T_{21}(z)}{T_{11}(z)} = L_{\eta_0}(z)$$

□

Theorem 5.3 shows that $G_{0,OE}(z)$ and $G_{0,GE}(z)$ will be equal if $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ and that $H_{0,GE}(z)$ in this case will be equal to the canonical spectral factor of $\Phi_{\eta_0}(z)$. Hence, the GE-LTI-SOE will be stable when $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ and a canonical spectral factorization of a matrix-valued z-spectrum will not be needed for the calculation of the GE-LTI-SOE. Furthermore, Theorem 5.3 can be used to describe how even and odd nonlinearities will affect the GE-LTI-SOE.

Consider an NFIR system with an input signal that has even probability density functions and is such that Lemmas 5.2 and 5.3 and Theorem 5.3 can be applied. In this case, $G_{0,GE}(z) = G_{0,OE}(z)$ will, by Lemma 5.2, depend only on the odd nonlinearities. Furthermore, (5.6) shows that

$$\Phi_{\eta_0}(z) = \Phi_{d_o}(z) + \Phi_{y_e}(z) + \Phi_w(z)$$

Hence, (5.20b) implies that there will be three different contributions to $H_{0,GE}(z)$. The first, $\Phi_{d_o}(z)$, is the z-spectrum of the unmodeled odd nonlinear part of the system output, while $\Phi_{y_e}(z)$ and $\Phi_w(z)$ are the z-spectra of the contributions to the output from the even part and the noise, respectively. Furthermore, Theorem 5.3 also gives the following corollary.

Corollary 5.1
Consider a nonlinear system with input $u(t)$ and output $y(t)$ such that Assumptions A1 and A2 are fulfilled. Assume that the input signal is such that

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

Assume also that

$$\Phi_{\eta_0}(z) = \Phi_y(z) - G_{0,OE}(z)\Phi_u(z)G_{0,OE}(z^{-1})$$

from (5.13b) has a canonical spectral factorization

$$\Phi_{\eta_0}(z) = L_{\eta_0}(z)r_{\eta_0}L_{\eta_0}(z^{-1})$$

with $r_{\eta_0} > 0$. Then it follows that $\varepsilon_0(t)$ (see (4.30)) has the following properties

$$\Phi_{\varepsilon_0 u}(z) = 0 \tag{5.21a}$$
$$\Phi_{\varepsilon_0}(z) = r_{\eta_0} \tag{5.21b}$$

Proof: *The fact that $G_{0,GE}(z) = G_{0,OE}(z)$ according to Theorem 5.3 implies that (4.30) can be rewritten as*

$$\begin{aligned}\varepsilon_0(t) &= H_{0,GE}^{-1}(q)(y(t) - G_{0,GE}(q)u(t)) \\ &= H_{0,GE}^{-1}(q)(y(t) - G_{0,OE}(q)u(t)) = H_{0,GE}^{-1}(q)\eta_0(t)\end{aligned}$$

Using Lemmas 2.1 and 5.4, this gives

$$\begin{aligned}\Phi_{\varepsilon_0 u}(z) &= H_{0,GE}^{-1}(z)\Phi_{\eta_0 u}(z) = 0 \\ \Phi_{\varepsilon_0}(z) &= H_{0,GE}^{-1}(z)\Phi_{\eta_0}(z)H_{0,GE}^{-1}(z^{-1}) = r_{\eta_0}\end{aligned}$$

where (5.19) and (5.20b) have been used in the last equality. The results in (5.21) have thus been shown. □

Corollary 5.1 shows that residual analysis without any further considerations can be used as a validation method also for GE-LTI-SOE:s when

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)},$$

since there will be no spurious correlations between $u(t)$ and past $\varepsilon_0(t-k)$. In addition, this shows that there will be no need for a closed-loop interpretation of the GE-LTI-SOE in this case.

In this chapter, the basic properties of OE-LTI-SOE:s have been discussed. It has been shown that an OE-LTI-SOE is independent of output additive noise that is uncorrelated with the input and the noise-free output. Furthermore, it has been shown that the OE-LTI-SOE often depends only on the odd nonlinearities in a system. A class of input signals guaranteeing that $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$, and some properties that the OE-LTI-SOE and GE-LTI-SOE exhibit in this case have also been discussed. In the next chapter, we will turn our attention to a class of nonlinear systems where LTI approximations are very natural.

6

Almost Linear Systems

When system identification is used to model real-life systems, it is very common to neglect the presence of small nonlinearities in the true system. This works well in many cases but, as we will see in the beginning in this chapter, LTI approximations of almost linear systems can sometimes exhibit a rather strange behavior. Later in this chapter, a continuity result that holds when the nonlinearities tend to zero will also be shown. Finally, a bound on the distance between the OE-LTI-SOE of an almost linear NFIR system and the linear part of that system will be given.

6.1 Almost Linear Systems

The use of a linear model is very natural when the true system is close to being linear. In many cases, the behavior of an almost linear system can be understood, at least intuitively, from the theory of linear systems. Hence, it is a legitimate question to ask whether this linear intuition can be extended also to OE-LTI-SOE:s for almost linear systems. An almost linear system will here be defined as a system that for a certain input signal can be written

$$y(t) = G_l(q)u(t) + \alpha y_n(t) + w(t)$$

where the linear part, $G_l(q)u(t)$, of the system is much larger than the nonlinear part $\alpha y_n(t)$. Here, the parameter α defines the size of the nonlinear part of the system and $w(t)$ is a noise term.

If the nonlinear contribution to the output is small for a certain input, one might assume that the corresponding OE-LTI-SOE would be close to the linear part of the system in some sense. However, as we will see in the following example, this is not always the case.

Example 6.1
Consider the nonlinear system

$$\begin{aligned} y(t) &= y_l(t) + \alpha y_n(t) \\ y_l(t) &= u(t) \\ y_n(t) &= u(t)^3 \end{aligned}$$

The output from this system consists of a linear part, $y_l(t)$, and a nonlinear part, $\alpha y_n(t)$, whose size is controlled by the parameter α. Here, the transfer function $G_l(q)$ of the linear part is equal to one. For bounded input signals, small values of α will give a system output that is close to the output from G_l. In this sense, α defines how close the nonlinear system is to the linear system $G_l(q)$. Let the input signal be

$$u(t) = L_m(q, c)e(t) \tag{6.1}$$

where

$$L_m(q, c) = \left(1 - cq^{-1}\right)^2 = 1 - 2cq^{-1} + c^2q^{-2}, \quad 0 < c < 1 \tag{6.2}$$

and where $e(t)$ is a white noise process with uniform distribution over the interval $[-1, 1]$. For all c with $0 < c < 1$, the input is bounded, $-4 < u(t) < 4$. For this input, a small value of α like, for example, $\alpha = 0.01$ will give an output that is very similar to the output from G_l, i.e., the output when $\alpha = 0$. This can be seen in Figure 6.1a for a particular realization of the input signal. However, the small differences between these output signals will sometimes give rise to totally different OE-LTI-SOE:s.
Since the input is generated by filtering white noise through a minimum phase filter, Theorem 5.1 gives that the OE-LTI-SOE can be written

$$G_{0,OE}(z, \alpha, c) = \frac{\Phi_{yu}(z, \alpha, c)}{\Phi_u(z, c)} = \underbrace{G_l(z)}_{=1} + \alpha \frac{\Phi_{y_n e}(z, c)}{L_m(z, c)R_e(0)} \tag{6.3}$$

If $y_n(t)$ is expanded we get

$$\begin{aligned} y_n(t) = e(t)^3 &- 6ce(t)^2e(t-1) + 3c^2e(t)^2e(t-2) + 12c^2e(t)e(t-1)^2 \\ &- 12c^3e(t)e(t-1)e(t-2) + 3c^4e(t)e(t-2)^2 - 8c^3e(t-1)^3 \\ &+ 12c^4e(t-1)^2e(t-2) - 6c^5e(t-1)e(t-2)^2 + c^6e(t-2)^3 \end{aligned}$$

Using the fact that $\mathrm{E}(e(t)^2) = \frac{1}{3}$ and $\mathrm{E}(e(t)^4) = \frac{1}{5}$ and that $e(t)$ and $e(t-k)$ are *independent when $k \neq 0$, we can derive the cross-covariance function $R_{y_n e}(\tau, c)$*

$$\begin{aligned}
R_{y_n e}(0,c) &= \mathrm{E}(y_n(t)e(t)) = \mathrm{E}(e(t)^4) + 12c^2\mathrm{E}(e(t)^2 e(t-1)^2) \\
&\quad + 3c^4\mathrm{E}(e(t)^2 e(t-2)^2) = \frac{1}{5} + 12c^2\frac{1}{9} + 3c^4\frac{1}{9} = \frac{1}{15}\left(3 + 20c^2 + 5c^4\right) \\
R_{y_n e}(1,c) &= \mathrm{E}(y_n(t)e(t-1)) = -6c\mathrm{E}(e(t)^2 e(t-1)^2) - 8c^3\mathrm{E}(e(t-1)^4) \\
&\quad - 6c^5\mathrm{E}(e(t-1)^2 e(t-2)^2) = -6c\frac{1}{9} - 8c^3\frac{1}{5} - 6c^5\frac{1}{9} \\
&= -\frac{1}{15}\left(10c + 24c^3 + 10c^5\right) \\
R_{y_n e}(2,c) &= \mathrm{E}(y_n(t)e(t-2)) = 3c^2\mathrm{E}(e(t)^2 e(t-2)^2) + 12c^4\mathrm{E}(e(t-1)^2 e(t-2)^2) \\
&\quad + c^6\mathrm{E}(e(t-2)^4) = 3c^2\frac{1}{9} + 12c^4\frac{1}{9} + c^6\frac{1}{5} = \frac{1}{15}\left(5c^2 + 20c^4 + 3c^6\right) \\
R_{y_n e}(\tau,c) &= 0 \quad \forall \tau \in \mathbb{Z} \setminus \{0,\ 1,\ 2\}
\end{aligned}$$

Inserted in (6.3) this gives

$$\begin{aligned}
&G_{0,OE}(z,\alpha,c) = \\
&= 1 + \alpha\frac{1}{5} \cdot \frac{\left(3 + 20c^2 + 5c^4\right) - \left(10c + 24c^3 + 10c^5\right)z^{-1} + \left(5c^2 + 20c^4 + 3c^6\right)z^{-2}}{1 - 2cz^{-1} + c^2 z^{-2}}
\end{aligned} \tag{6.4}$$

Let $\Delta_G(z,\alpha,c) = \sum_{k=0}^{\infty} \delta_G(k,\alpha,c)z^{-k} = G_{0,OE}(z,\alpha,c) - G_l(z)$. Then the static gain of Δ_G is

$$\Delta_G(1,\alpha,c) = \frac{\alpha}{5} \cdot \frac{3 - 10c + 25c^2 - 24c^3 + 25c^4 - 10c^5 + 3c^6}{(1-c)^2} \tag{6.5}$$

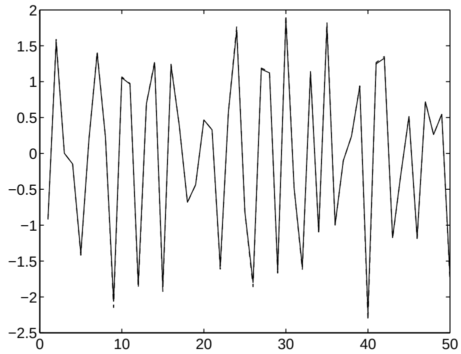
From (6.5) we see that the numerator of $\Delta_G(1,\alpha,c)$ approaches 12α when $c \to 1$, i.e., for c close to 1 we have $\Delta_G(1,\alpha,c) \approx \frac{12\alpha}{5(1-c)^2}$. This implies that no matter how small $\alpha > 0$ we select, we can always make $\Delta_G(1,\alpha,c)$ arbitrarily large by choosing a c sufficiently close to 1. That is, no matter how linear the system is, there is always a bounded input signal such that its OE-LTI-SOE is far from $G_l(e^{i\omega})$ for $\omega = 0$. The difference between $|G_{0,OE}(e^{i\omega}, 0.01, 0.99)|$ and $|G_l(e^{i\omega})|$ is shown in Figure 6.1b. Furthermore, since

$$|\Delta_G(1,\alpha,c)| = |\sum_{k=0}^{\infty} \delta_G(k,\alpha,c)| \leq \sum_{k=0}^{\infty} |\delta_G(k,\alpha,c)|$$

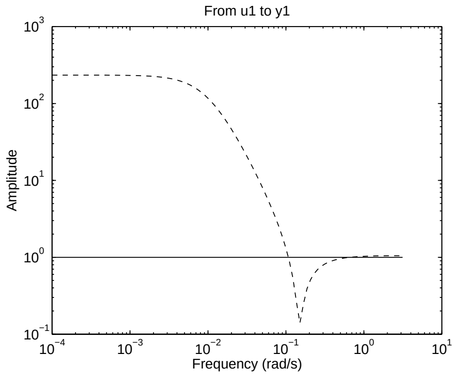
it follows that for any $\alpha > 0$ the l_1-norm of the impulse response of Δ_G can be made arbitrarily large by an appropriate choice of c. Furthermore, it can be shown that, for any fixed $\alpha > 0$, also

$$\sum_{k=0}^{\infty} \delta_G(k,\alpha,c)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_G(e^{i\omega},\alpha,c)|^2 d\omega$$

can be made arbitrarily large by taking a c close to 1.



a. The output $y(t)$ (dashed) of the nonlinear system (with $\alpha = 0.01$ and $c = 0.99$) in Example 6.1 and the output $y_l(t) = u(t)$ (solid) of the linear part of that system for a particular realization of the input signal.



b. Bode plot showing the difference between the OE-LTI-SOE $G_{0,OE}(e^{i\omega}, 0.01, 0.99)$ (dashed) and the linear part $G_l(e^{i\omega})$ (solid) of the system in Example 6.1

Figure 6.1 *The frequency response of the OE-LTI-SOE can be far from the response of the linear part of the system also when the nonlinear contributions to the output are small.*

The previous example is a clear indication that the OE-LTI-SOE not always can be understood from linear theory. There is no corresponding behavior in the linear case, since a small linear, time-invariant deviation from G_l in Example 6.1 only would have given rise to exactly the same small deviation in the OE-LTI-SOE. It should be noted that for small α values, G_l is a good model of the true system and it can predict the output very well from the input signal. Despite this, the prediction error estimate will not converge to G_l but to $G_{0,OE}$ when the number of measurements tends to infinity.

We have seen that the OE-LTI-SOE in some cases can be far from the linear part of the system. This can in some circumstances be an undesirable property, e.g. if the OE-LTI-SOE is supposed to be used as a basis for robust control design. Such a design puts restrictions on the control laws in order to guarantee the stability of the resulting true closed-loop system, despite the presence of model errors.

Assume that the true system is almost linear in the sense that it deviates from an LTI system G_l by a nonlinearity with a small gain. In this case, G_l is a very good basis for robust control design, since the small nonlinearity often only gives rise to rather mild restrictions on the controller. However, if an OE-LTI-SOE that is far from G_l is used for the controller design, the restrictions on the controller might become much harder, since the gain of the model error now is large. Hence, it is interesting to investigate under what circumstances we can guarantee that the OE-LTI-SOE will be close to the linear part of the system when the nonlinearities are small. Some answers to this question will be given in this chapter.

Consider a system $y(t) = f((u(t-k))_{k=0}^{\infty}, \alpha)$ where α, just like in the previous example, is a parameter that defines the size of the nonlinear part of f. Let S_{A1} denote the set of all inputs such that they, and the outputs from the system f they generate, fulfill Assumption A1. Assume that f is continuous at $\alpha = 0$ and that $f((u(t-k))_{k=0}^{\infty}, 0)$ is a stable and causal LTI system G_l, i.e.,

$$f((u(t-k))_{k=0}^{\infty}, 0) = \sum_{k=0}^{\infty} g_l(k)u(t-k) = G_l(q)u(t) \tag{6.6}$$

Let $G_{0,OE}(z, \alpha)$ denote the OE-LTI-SOE that is obtained for a certain input signal u and a certain α. The conclusion that we can draw from Example 6.1 is that we cannot in general assume that, for example,

$$\sup_{u \in S_{A1}} |G_{0,OE}(e^{i\omega}, \alpha) - G_l(e^{i\omega})| \tag{6.7}$$

will approach 0 when $\alpha \to 0$ for a fixed $\omega \in [-\pi, \pi]$. For some systems we can, whenever there is a small nonlinear term in the system output, find a u for which the OE-LTI-SOE is far from G_l. This property of the OE-LTI-SOE makes it different from, for example, a linearization based on a Taylor series expansion. Since such a linearization is based only on local properties of the nonlinear system, e.g. for $u(t) \equiv 0$, it will have a continuous behavior when $\alpha \to 0$. However, as we will see in the next section, it is possible to show that also OE-LTI-SOE:s exhibit some, less general, continuity properties.

6.2 A Continuity Result

Despite the fact that we even for an almost linear system cannot prove that the OE-LTI-SOE:s are close to the linear part of the system for all inputs that fulfill Assumption A1, it is possible to say something about the behavior of the OE-LTI-SOE for a more restricted class of input signals, whose main feature is that their spectral densities are strictly positive. This is done in the following theorem.

Theorem 6.1
Consider a nonlinear system $y(t) = f((u(t-k))_{k=0}^{\infty}, \alpha) + w(t)$ such that the function $f((x(t-k))_{k=0}^{\infty}, \alpha) \to f((x(t-k))_{k=0}^{\infty}, 0) = \sum_{k=0}^{\infty} g_l(k)x(t-k)$ uniformly on the set of sequences $M_f = \{(x(t-k))_{k=0}^{\infty} : \ |x(t-k)| < u_{max} \ \forall k \in \mathbb{N}\}$ when $\alpha \to 0$. Assume that the limit $G_l(q)$ is a stable and causal LTI system. Let S_f denote the set of stochastic input signals that fulfill the following criteria

(i) $P(|u(t)| \geq u_{max}) = 0$

(ii) $\mathrm{E}(|u(t)|) \leq m_c < \infty$

(iii) $\Phi_u(e^{i\omega}) \geq \mu_c > 0$ *for all* $\omega \in [-\pi, \pi]$

(iv) $u(t)$ *is such that* $G_{0,OE}(z,\alpha) = \frac{\Phi_{yu}(z,\alpha)}{\Phi_u(z)}$ *for all* α *with* $|\alpha| < \alpha_{max}$

(v) $u(t)$ *and* $y(t) = f((u(t-k))_{k=0}^{\infty}, \alpha) + w(t)$ *fulfill Assumptions A1 and A3 for all* α *with* $|\alpha| < \alpha_{max}$

(vi) $\exists M_c \in \mathbb{Z}_+,\ \lambda_c,\ 0 \leq \lambda_c < 1,\ K_c > 0$ *such that* $|R_{yu}(\tau,\alpha)| < K_c \lambda_c^{|\tau|}$ *when* $|\tau| > M_c\ \forall \alpha$ *with* $|\alpha| < \alpha_{max}$.

where u_{max}, m_c, μ_c, α_{max}, M_c, λ_c *and* K_c *are given constants. Then*

$$\sup_{u \in S_f} \int_{-\infty}^{\infty} |G_{0,OE}(e^{i\omega},\alpha) - G_l(e^{i\omega})|^n d\omega \to 0, \quad \alpha \to 0, \quad n = 1,2 \tag{6.8}$$

Proof: Take an arbitrary $\tau \in \mathbb{Z}$. Then

$$\begin{aligned}
&\sup_{u \in S_f} |R_{yu}(\tau,\alpha) - R_{yu}(\tau,0)| \\
&= \sup_{u \in S_f} \left| \mathrm{E}\left(\left(f((u(t-k))_{k=0}^{\infty},\alpha) - \sum_{k=0}^{\infty} g_l(k) u(t-k) \right) u(t-\tau) \right) \right| \\
&\leq \sup_{u \in S_f} \mathrm{E}\left(\left| f((u(t-k))_{k=0}^{\infty},\alpha) - \sum_{k=0}^{\infty} g_l(k) u(t-k) \right| |u(t-\tau)| \right) \\
&\leq \sup_{u \in S_f} \mathrm{E}\left(\sup_{x \in M_f} \left| f((x(t-k))_{k=0}^{\infty},\alpha) - \sum_{k=0}^{\infty} g_l(k) x(t-k) \right| |u(t-\tau)| \right) \\
&= \sup_{x \in M_f} \left| f((x(t-k))_{k=0}^{\infty},\alpha) - \sum_{k=0}^{\infty} g_l(k) x(t-k) \right| \sup_{u \in S_f} \mathrm{E}(|u(t-\tau)|) \\
&\leq \sup_{x \in M_f} \left| f((x(t-k))_{k=0}^{\infty},\alpha) - \sum_{k=0}^{\infty} g_l(k) x(t-k) \right| m_c \to 0, \quad \alpha \to 0
\end{aligned}$$

Here, we have used (i) and (ii) in the last inequality. Since τ *was arbitrary it follows that*

$$\sup_{u \in S_f} |R_{yu}(\tau,\alpha) - R_{yu}(\tau,0)| \to 0, \quad \alpha \to 0, \quad \forall \tau \in \mathbb{Z} \tag{6.9}$$

Now we need to show that $\sup_{u \in S_f} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega},\alpha) - \Phi_{yu}(e^{i\omega},0)|^2 d\omega \to 0$ *when* $\alpha \to 0$. *Take an arbitrary* $\varepsilon > 0$. *By Parseval's identity, which holds for all* α *with*

$|\alpha| < \alpha_{max}$ according to (vi) we get

$$\begin{aligned}
&\sup_{u\in S_f} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega},\alpha) - \Phi_{yu}(e^{i\omega},0)|^2 d\omega \\
&= \sup_{u\in S_f} 2\pi \sum_{\tau=-\infty}^{\infty} |R_{yu}(\tau,\alpha) - R_{yu}(\tau,0)|^2 \\
&= \sup_{u\in S_f} 2\pi \sum_{\tau=-C_0}^{C_0} |R_{yu}(\tau,\alpha) - R_{yu}(\tau,0)|^2 + Q(C_0,\alpha)
\end{aligned}$$

Choose C_0 such that the term $Q(C_0,\alpha)$ is less than $\varepsilon/2$ for all α with $|\alpha| < \alpha_{max}$. (This is possible according to (vi)). Then, from (6.9) it follows that $\exists \delta_\varepsilon > 0$ such that $|\alpha| < \delta_\varepsilon \Rightarrow \sup_{u\in S_f} 2\pi \sum_{\tau=-C_0}^{C_0} |R_{yu}(\tau,\alpha) - R_{yu}(\tau,0)|^2 < \varepsilon/2$. Thus $\sup_{u\in S_f} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega},\alpha) - \Phi_{yu}(e^{i\omega},0)|^2 d\omega < \varepsilon$ if $|\alpha| < \delta_\varepsilon$ and since ε was arbitrary we get

$$\sup_{u\in S_f} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega},\alpha) - \Phi_{yu}(e^{i\omega},0)|^2 d\omega \to 0, \quad \alpha \to 0 \tag{6.10}$$

Using (iii) and (iv) together with (6.10) we obtain

$$\begin{aligned}
&\sup_{u\in S_f} \int_{-\infty}^{\infty} |G_{0,OE}(e^{i\omega},\alpha) - G_l(e^{i\omega})|^2 d\omega \\
&= \sup_{u\in S_f} \int_{-\infty}^{\infty} \frac{|\Phi_{yu}(e^{i\omega},\alpha) - \Phi_{yu}(e^{i\omega},0)|^2}{\Phi_u(e^{i\omega})^2} d\omega \\
&\le \sup_{u\in S_f} \frac{1}{\mu_c^2} \int_{-\infty}^{\infty} |\Phi_{yu}(e^{i\omega},\alpha) - \Phi_{yu}(e^{i\omega},0)|^2 d\omega \to 0, \quad \alpha \to 0
\end{aligned}$$

and thus we have shown (6.8) for $n = 2$. Schwarz' inequality now gives

$$\begin{aligned}
&\sup_{u\in S_f} \int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega},\alpha) - G_l(e^{i\omega})| d\omega \\
&\le \sup_{u\in S_f} \sqrt{2\pi} \left(\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega},\alpha) - G_l(e^{i\omega})|^2 d\omega \right)^{1/2} \to 0, \quad \alpha \to 0
\end{aligned}$$

Hence we have shown (6.8) also for $n = 1$. □

Theorem 6.1 gives conditions on the system and set of input signals that guarantee a uniformly continuous behavior of the OE-LTI-SOE:s in the point where $\alpha = 0$, i.e., when the system is linear. The reason why we cannot apply this theorem in Example 6.1 is that the set of input signals generated by (6.1) for all c with $0 < c < 1$ does not fulfill criterion (iii) since $\Phi_u(1) = \frac{(1-c)^2}{3}$. Hence, there is no constant $\mu_c > 0$ such that *all* considered input signals fulfill $\Phi_u(e^{i\omega}) \ge \mu_c$ for all ω in the interval $[-\pi, \pi]$.

Furthermore, Theorem 6.1 shows that it does not matter if $G_{0,OE}(z)$ and $G_l(z)$ have different orders. For input signals in S_f, $G_{0,OE}(z)$ will still approach $G_l(z)$ in a well-behaved way according to (6.8) when the nonlinearities tend to zero.

6.3 Almost Linear NFIR Systems

The theorem in the previous section tells us that the OE-LTI-SOE will converge to the linear part of the system when the nonlinearities tend to zero, but not how fast this convergence is. In order to be able to derive an upper bound on the distance between the OE-LTI-SOE and the linear part of a system with a nonzero nonlinearity of a certain size, we will have to make some new restrictions on the types of systems and excitation signals.

Hence, we will in this section only consider nonlinear FIR systems with white input signals that can be written like $y(t) = f((u(t-k))_{k=0}^{M}) + w(t)$ and that are close to a linear system $z(t) = \sum_{k=0}^{M} g_l(k)u(t-k) + w(t)$. The following theorem gives an upper bound on the distance between the OE-LTI-SOE and the linear part of such a nonlinear system.

Theorem 6.2
Let $u(t)$ be a white input process with zero mean and let

$$y(t) = f((u(t-k))_{k=0}^{M}) + w(t)$$

be an NFIR system such that

$$\left| f((u(t-k))_{k=0}^{M}) - \sum_{k=0}^{M} g_l(k)u(t-k) \right| < a \tag{6.11}$$

Assume that the output $y(t)$ together with $u(t)$ fulfills the conditions in Assumptions A1 and A3. Assume also that $\mathrm{E}(|u(t)|) < +\infty$. Then

$$\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})| d\omega < a2\pi\sqrt{(M+1)} \left(\frac{\mathrm{E}(|u(t)|)}{\mathrm{E}(u(t)^2)} \right) \tag{6.12a}$$

$$\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})|^2 d\omega < a^2 2\pi(M+1) \left(\frac{\mathrm{E}(|u(t)|)}{\mathrm{E}(u(t)^2)} \right)^2 \tag{6.12b}$$

Proof: *We start by proving the following inequality*

$$\left| R_{yu}(\tau) - \sum_{k=0}^{M} g_l(k) R_u(\tau-k) \right| < a\mathrm{E}(|u(t-\tau)|) \quad \forall \tau \in \mathbb{Z} \tag{6.13}$$

Take an arbitrary $\tau \in \mathbb{Z}$. Then Assumption A3 gives

$$\begin{aligned}
&\left| R_{yu}(\tau) - \sum_{k=0}^{M} g_l(k) R_u(\tau - k) \right| \\
&= \left| \mathrm{E}(y_{nf}(t)u(t-\tau)) - \sum_{k=0}^{M} g_l(k)\mathrm{E}(u(t-k)u(t-\tau)) \right| \\
&= \left| \mathrm{E}((y_{nf}(t) - \sum_{k=0}^{M} g_l(k)u(t-k))u(t-\tau)) \right| \\
&\leq \mathrm{E}(|y_{nf}(t) - \sum_{k=0}^{M} g_l(k)u(t-k)||u(t-\tau)|) \\
&< a\mathrm{E}(|u(t-\tau)|)
\end{aligned}$$

Since τ was arbitrary, (6.13) follows. The assumption that $u(t)$ consists of independent random variables implies that $\Phi_u(e^{i\omega}) = R_u(0)$,

$$\sum_{k=0}^{M} g_l(k) R_u(\tau - k) = g_l(\tau) R_u(0), \quad 0 \leq \tau \leq M,$$

and that $R_{yu}(\tau) = 0$ when $\tau > M$ or $\tau < 0$. This, together with Parseval's identity and Equation (6.13) give

$$\begin{aligned}
&\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})|^2 d\omega \\
&= \frac{1}{R_u(0)^2} \int_{-\pi}^{\pi} |\Phi_{yu}(e^{i\omega}) - G_l(e^{i\omega}) R_u(0)|^2 d\omega \\
&= \frac{2\pi}{R_u(0)^2} \sum_{\tau=0}^{M} |R_{yu}(\tau) - g_l(\tau) R_u(0)|^2 \\
&< a^2 2\pi (M+1) \left(\frac{\mathrm{E}(|u(t)|)}{\mathrm{E}(u(t)^2)} \right)^2
\end{aligned}$$

Finally, Schwarz' inequality gives

$$\begin{aligned}
&\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})| d\omega \leq \left(\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})|^2 d\omega \right)^{1/2} \sqrt{2\pi} \\
&< a 2\pi \sqrt{(M+1)} \left(\frac{\mathrm{E}(|u(t)|)}{\mathrm{E}(u(t)^2)} \right)
\end{aligned}$$

□

Theorem 6.2 tells us that the distance

$$\int_{-\pi}^{\pi} |G_{0,OE}(e^{i\omega}) - G_l(e^{i\omega})| d\omega$$

between the OE-LTI-SOE and G_l is less than a bound that is proportional to a (where a is the size of the nonlinearity in (6.11)). Furthermore, this theorem shows the effect on the OE-LTI-SOE of a scaling of the input signal.

The use of $\tilde{u} = \nu u$ as input instead of u will result in a new OE-LTI-SOE. The distance in (6.12a) between this new OE-LTI-SOE and G_l will have an upper bound that is $\frac{1}{|\nu|}$ times the original bound. When a white input signal is used, it is thus possible to reduce the distance between the OE-LTI-SOE and the linear part of a nonlinear FIR system that fulfill (6.11) simply by scaling the input signal. This is natural, since the relative error gets smaller for large inputs.

The main objective of this chapter has been to describe the behavior of OE-LTI-SOE:s for almost linear systems. It has been shown that the OE-LTI-SOE sometimes can be far from the linear part of such a system. Furthermore, a continuity result has been derived and, finally, a bound on the distance between the OE-LTI-SOE of an almost linear NFIR system with a white input signal has been presented. In the next chapter, we will consider a special kind of minimum phase generated signals that have the property that their OE-LTI-SOE:s preserve parts of the structure of the nonlinear system.

7

NFIR Systems with Gaussian Inputs

Random variables and processes with Gaussian distributions play a rather special role in probability theory and applications. They exhibit a large number of properties that simplify many statistical problems or make the solutions to the problems more general. One interesting property that holds if two random variables have a simultaneous Gaussian distribution, is that these variables are independent if and only if they are uncorrelated.

This property implies that a Gaussian process u, whose z-spectrum has a canonical spectral factorization, can always be viewed as if it has been generated by filtering white Gaussian noise through a minimum phase LTI filter. This results follows from the fact that any signal can be viewed as if it has been generated by filtering an *uncorrelated* signal with a minimum phase filter. Hence, the OE-LTI-SOE of an arbitrary causal nonlinear system with a Gaussian input will be

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)}$$

provided that the input and output signals fulfill the conditions in Theorem 5.1. This implies that all properties from Sections 5.4 and 5.5 will hold also for Gaussian input signals.

However, in this chapter we will show that OE-LTI-SOE:s for Gaussian inputs also have properties that other minimum phase generated inputs do not have. For example, OE-LTI-SOE:s of NFIR systems with Gaussian inputs will always be linear FIR models. It will be shown later in this chapter that this fact can be used for structure identification of NFIR systems and for identification of classes of systems that here will be called *generalized Wiener* or *Hammerstein systems*. First, however, we will show a generalization of Bussgang's classic theorem.

7.1 A Generalization of Bussgang's Theorem

In Section 3.2.1 it was mentioned that Bussgang's theorem (Bussgang, 1952) has turned out to be quite useful for the theory of nonlinear system identification. We will now present a result that can be viewed as a generalization of Bussgang's theorem. Similar results can be found in, for example, Scarano et al. (1993) and have also previously been used in the research area of stochastic mechanical vibrations (see, for example, Lutes and Sarkani, 1997, Chap. 9). We will however present a complete, independent derivation here and we begin with the following technical assumptions.

Assumption A4: Assume that the real-valued functions $f(x)$ and $\varphi(\tilde{x})$, where $x \in \mathbb{R}^N$ and $\tilde{x} = (x^T, v)^T \in \mathbb{R}^{N+1}$, are such that $f \cdot \varphi$, $f'_{x_i} \cdot \varphi$ and $f \cdot \tilde{x}_i \cdot \varphi$, $i = 1, \dots, (N+1)$ all belong to $L^1(\mathbb{R}^{N+1})$ and that $f(x)\varphi(\tilde{x}) \to 0$ when $|\tilde{x}| \to +\infty$. (Here, f'_{x_i} is the partial derivative of f with respect to x_i).

Assumption A5: Consider two stationary stochastic processes $(u(t))_{t=-\infty}^{\infty}$ and $(y(t))_{t=-\infty}^{\infty}$ such that $y(t) = f((u(t-k))_{k=0}^{M_0})$. Assume that $(u(t))_{t=-\infty}^{\infty}$ is a Gaussian process with zero mean and that $\mathrm{E}(y(t)) = 0$. Form random vectors

$$\omega_\sigma = (u(t), u(t-1), \dots, u(t-M_0), u(t-\sigma))^T \tag{7.1}$$

with $\sigma < 0$ or $\sigma > M_0$. Let C_σ and φ_σ denote the covariance matrices and joint probability density functions of these vectors, respectively. Assume that $\det C_\sigma \neq 0$ and that f and φ_σ satisfy Assumption A4 for all $\sigma < 0$ or $\sigma > M_0$.

Assumption A4 holds if, for example, f is a polynomial and φ is a Gaussian probability density function. This assumption is used in the following lemma.

Lemma 7.1
Let

$$\tilde{x} = (x^T, v)^T = (x_1, x_2, \dots, x_N, v)^T \tag{7.2}$$

be a jointly Gaussian distributed random vector with zero mean and covariance matrix C with $\det C \neq 0$. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a differentiable function of x with $\mathrm{E}(f(x)) = 0$ and let φ denote the probability density function of $\tilde{x}$. Furthermore, assume that f and φ satisfy Assumption A4. Then

$$\mathrm{E}(f(x)\tilde{x}) = Cw \tag{7.3}$$

where

$$w = \begin{pmatrix} \mathrm{E}(f'_{x_1}(x)) \\ \mathrm{E}(f'_{x_2}(x)) \\ \vdots \\ \mathrm{E}(f'_{x_N}(x)) \\ 0 \end{pmatrix}$$

Proof: *Factorize C as $C = \tilde{M}\tilde{M}^T$ and define a new stochastic vector z as $z = \tilde{M}^{-1}\tilde{x}$. Then z is jointly normally distributed with zero mean and a covariance matrix that is equal to the identity matrix. Let M denote the matrix that is obtained from $\tilde{M}$ by removing the last row. Then $x = Mz$ and we get*

$$\begin{aligned} \mathrm{E}(f(x)\tilde{x}) &= \tilde{M}\mathrm{E}(f(x)\tilde{M}^{-1}\tilde{x}) = \tilde{M}\mathrm{E}(f(Mz)z) \\ &= \tilde{M}\begin{pmatrix} \mathrm{E}(\frac{\partial f(Mz)}{\partial z_1}) \\ \mathrm{E}(\frac{\partial f(Mz)}{\partial z_2}) \\ \vdots \\ \mathrm{E}(\frac{\partial f(Mz)}{\partial z_{N+1}}) \end{pmatrix} = \tilde{M}\tilde{M}^T \begin{pmatrix} \mathrm{E}(f'_{x_1}(x)) \\ \mathrm{E}(f'_{x_2}(x)) \\ \vdots \\ \mathrm{E}(f'_{x_N}(x)) \\ 0 \end{pmatrix} = Cw \end{aligned}$$

The third equality follows from the fact that $\mathrm{E}(h(z)z_i) = \mathrm{E}(h'_{z_i}(z))$ when z has an $N(0, I)$ distribution. This equality holds since

$$\int_{-\infty}^{\infty} g(r)re^{-r^2/2}dr = \left[-g(r)e^{-r^2/2}\right]_{r=-\infty}^{\infty} + \int_{-\infty}^{\infty} g'(r)e^{-r^2/2}dr$$

Furthermore, the fourth equality in the derivation above follows from the chain rule, which here can be written as

$$\frac{\partial f(Mz)}{\partial z_i} = \frac{\partial f(Mz)}{\partial x_1}M_{1i} + \frac{\partial f(Mz)}{\partial x_2}M_{2i} + \ldots + \frac{\partial f(Mz)}{\partial x_N}M_{Ni}$$

□

Lemma 7.1 is used in the proof of the following generalization of Bussgang's theorem.

Theorem 7.1
Let $y(t) = f((u(t-k))_{k=0}^{M_0})$ be an NFIR system with a stationary Gaussian process $(u(t))_{t=-\infty}^{\infty}$ as input. Assume that u and y satisfy Assumption A5. Then it follows that

$$R_{yu}(\tau) = \sum_{k=0}^{M_0} b(k)R_u(\tau - k) \quad \forall \tau \in \mathbb{Z} \tag{7.4}$$

where

$$b(k) = \mathrm{E}(f'_{u(t-k)}((u(t-j))_{j=0}^{M_0}))$$

Proof: *Choose an arbitrary $\sigma < 0$ or $\sigma > M_0$ and let*

$$x = (u(t), u(t-1), \ldots, u(t-M_0))^T$$

and $v = u(t-\sigma)$ in Lemma 7.1. Then Equation (7.3) gives

$$\mathrm{E}(y(t)\begin{pmatrix} u(t) \\ u(t-1) \\ \vdots \\ u(t-M_0) \\ u(t-\sigma) \end{pmatrix}) = \begin{pmatrix} R_u(0) & R_u(1) & \dots & R_u(M_0) & R_u(\sigma) \\ R_u(1) & R_u(0) & \dots & R_u(M_0-1) & R_u(\sigma-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_u(M_0) & R_u(M_0-1) & \dots & R_u(0) & R_u(\sigma-M_0) \\ R_u(\sigma) & R_u(\sigma-1) & \dots & R_u(\sigma-M_0) & R_u(0) \end{pmatrix} w \tag{7.5}$$

where $w_{i+1} = \mathrm{E}(f'_{u(t-i)}((u(t-k))_{k=0}^{M_0}))$ for $i = 0, \dots, M_0$ and $w_{M_0+2} = 0$. Equation (7.5) can be written more compactly as

$$R_{yu}(\tau) = \sum_{k=0}^{M_0} b(k) R_u(\tau - k), \quad \tau = 0, 1, \dots, M_0, \sigma$$

where $b(k) = w_{k+1} = \mathrm{E}(f'_{u(t-k)}((u(t-j))_{j=0}^{M_0}))$. As σ was chosen arbitrarily, this relation holds for all $\tau \in \mathbb{Z}$. □

As mentioned above, the previous theorem can be viewed as a generalization of Bussgang's theorem to NFIR systems. Using z-transforms, the result (7.4) can also be written as

$$\Phi_{yu}(z) = B(z)\Phi_u(z) \tag{7.6}$$

where $B(z) = \sum_{k=0}^{M_0} b(k) z^{-k}$. The relation between Theorem 7.1 and OE-LTI-SOE:s of NFIR systems will be discussed in the next section.

7.2 OE-LTI-SOE:s of NFIR Systems with Gaussian Inputs

The generalization of Bussgang's theorem in the previous section can be used to characterize the OE-LTI-SOE of an NFIR system with Gaussian inputs. As has been previously mentioned, the OE-LTI-SOE is in general obtained by the Wiener filter construction in (4.3). However, from (7.6) we see that the ratio $\Phi_{yu}(z)/\Phi_u(z)$ is stable and causal if the nonlinear system is an NFIR system with a Gaussian input. Hence, with Corollary 4.1 in mind we can state the following theorem.

Theorem 7.2
Consider an NFIR system

$$y(t) = f((u(t-k))_{k=0}^{M_0}) + w(t)$$

with a Gaussian input $u(t)$ such that Assumptions A1, A3 and A5 are satisfied. Then the OE-LTI-SOE of this system is the linear FIR model

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \sum_{k=0}^{M_0} b(k) z^{-k} \tag{7.7}$$

where $b(k) = \mathrm{E}(f'_{u(t-k)}((u(t-j))_{j=0}^{M_0}))$.

In general, it is quite possible that the OE-LTI-SOE of an NFIR system with a non-Gaussian input will have an infinite impulse response (cf. Example 4.2) and it is usually hard to give a detailed characterization of it. However, as we have shown here, when the input is Gaussian the OE-LTI-SOE is always an FIR model and the coefficients of this model can be characterized exactly by (7.7).

The property that makes Gaussian processes special is that for a Gaussian process the conditional expectation $\mathrm{E}(u(t-\tau)|u(t),\ldots,u(t-M_0))$ is always a linear combination of $u(t),\ldots,u(t-M_0)$. Note that there also exist non-Gaussian processes that have this property. Such processes have been studied by Nuttall (1958) and they will be discussed more in Chapter 8.

In many cases, it is possible to shed some light on a theoretical result by interpreting it in a geometrical framework. This can as a matter of fact be done also in our case. For a fixed t, we can view the output $y(t)$ and the components of the input signal $u(\tau)$, $\tau \in \mathbb{Z}$ as vectors in an infinite dimensional inner-product space with the inner product $< u, v >= \mathrm{E}(uv)$ (see Brockwell and Davis, 1987).

The OE-LTI-SOE of the NFIR system will in this framework be the orthogonal projection of $y(t)$ into the linear subspace that is spanned by $u(t)$, $u(t-1)$, ..., $u(t-\infty)$. From (7.7) we can draw the conclusion that this projection actually lies in the finite dimensional linear subspace that is spanned by $u(t), u(t-1), \ldots, u(t-M_0)$.

7.3 Applications

The characterization (7.7) of the OE-LTI-SOE of an NFIR system with a Gaussian input is not only theoretically interesting, but can also be useful in some applications of system identification. We will here briefly discuss three such applied identification problems.

7.3.1 Structure Identification of NFIR Systems

The most obvious application of the result (7.7) is perhaps to use it for guidance when an NFIR system is going to be identified. However, linear models are not useful for all types of NFIR systems. As was mentioned in Section 5.2, any NFIR system can be written as a sum of an even and an odd function. Since all Gaussian probability density functions with zero mean are even functions, Lemma 5.2 shows that the OE-LTI-SOE of an NFIR system is only influenced by the odd part of the system.

Hence we will here only consider odd NFIR systems, i.e., NFIR systems $y(t) = f((u(t-n_k-j))_{j=0}^{M_0})$ where

$$f((-u(t-n_k-j))_{j=0}^{M_0}) = -f((u(t-n_k-j))_{j=0}^{M_0})$$

When such an odd NFIR system is going to be identified, it is in general not obvious how the time delay n_k and order M_0 should be estimated in an efficient way. However, if the input is Gaussian and sufficiently many measurements can be collected, n_k and M_0 can both be obtained from an impulse response estimate. Such an estimate can be computed very efficiently by means of the least squares method.

Furthermore, if only a few of the input terms $u(t-n_k)$, $u(t-n_k-1), \dots,$ $u(t-n_k-M_0)$ enter the system in a nonlinear way, it might be interesting to know which these terms are. If a nonlinear model of the system is desired, this knowledge can be used to reduce the complexity of the proposed model. A coefficient $b(j)$ in (7.7) will be invariant of the input properties if the corresponding input term $u(t-j)$ only affects the system linearly, while an input term that affects the system in a nonlinear way will have an input dependent b-coefficient in (7.7).

This fact makes it possible to extract information about which nonlinear terms that are present in the system simply by looking at the differences between FIR models that have been estimated with different Gaussian input signals. The coefficients that correspond to an input term that enters the system in a nonlinear way will be different in these estimates, provided that the covariance functions of the inputs are different. This idea is used in the following example.

Example 7.1
Consider the nonlinear system $y(t) = u(t) + u(t-1)^3$ and assume that the input to this system is Gaussian and such that the conditions in Theorem 7.2 are fulfilled. Then the OE-LTI-SOE of this system will be

$$G_{0,OE}(q) = b(0) + b(1)q^{-1}$$

where $b(0) = 1$ and $b(1) = 3R_u(0)$. If the variance of the input is changed, $b(1)$ will change too, while $b(0)$ will remain equal to one. Hence, it is easy to see which input signal component that affects $y(t)$ in a nonlinear way.

7.3.2 Identification of Generalized Hammerstein Systems

In Section 3.2.1, we mentioned that Bussgang's theorem has been used to show important results concerning the identification of Hammerstein and Wiener systems. In principle, these results state that an estimated LTI model will converge to a scaled version of the linear part of a Hammerstein or Wiener system when the number of measurements tends to infinity, provided that the input is Gaussian. These results simplify the identification of Wiener and Hammerstein systems significantly.

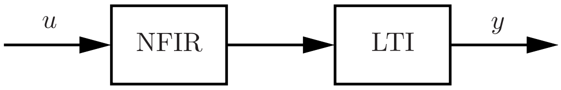


Figure 7.1 *A generalized Hammerstein system.*

Hence, it is interesting to investigate if the result (7.7) about the OE-LTI-SOE:s of NFIR systems can be used to prove similar results for extended classes of systems. In this section we will study a type of systems that we will call generalized Hammerstein systems, while we will consider generalized Wiener systems in the next section.

More specifically, we will call a nonlinear system a generalized Hammerstein system if it consists of an NFIR system $v(t) = f((u(t-k))_{k=0}^{M_0})$ followed by an LTI system $y(t) = G(q)v(t)$ as is shown in Figure 7.1. The following corollary to Theorem 7.2 shows that the OE-LTI-SOE of such a system has a certain structure.

Corollary 7.1
Consider a generalized Hammerstein system $y(t) = G(q)v(t) + w(t)$ where $v(t) = f((u(t-k))_{k=0}^{M_0})$ and where $G(q)$ is a stable and causal LTI system. Assume that $u(t)$ is Gaussian and that it, together with $y(t)$, fulfill Assumptions A1 and A3. Assume also that $u(t)$ and $v(t)$ fulfill Assumptions A1 and A5. Then the OE-LTI-SOE of this system is

$$G_{0,OE}(z) = G(z)B(z) \tag{7.8}$$

where $B(z) = \sum_{k=0}^{M_0} b(k)z^{-k}$ and $b(k) = \mathrm{E}(f'_{u(t-k)}((u(t-j))_{j=0}^{M_0}))$.

Proof: *Lemma 2.1 gives that $\Phi_{yu}(z) = G(z)\Phi_{vu}(z)$, and using Theorem 7.2 and Corollary 4.1, the result (7.8) follows.* □

Corollary 7.1 shows that the OE-LTI-SOE of a generalized Hammerstein system with a Gaussian input will be $G(z)B(z)$, and hence an estimated output error model will approach this model when the number of measurements tends to infinity. In particular, as $B(z)$ is an FIR model, this shows that the denominator of the estimated model will approach the denominator of G if the degree of the model denominator polynomial is correct.

We will thus get consistent estimates of the poles of G despite the presence of the NFIR system. This result is verified numerically in Example 7.2.

Example 7.2
Consider a generalized Hammerstein system

$$y(t) = G(q)f(u(t), u(t-1)) + w(t)$$

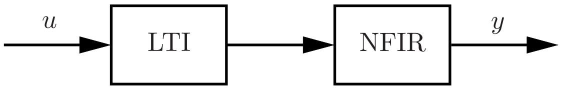


Figure 7.2 *A generalized Wiener system.*

where

$$G(q) = \frac{1}{1 + 0.6q^{-1} + 0.1q^{-2}}$$
$$f(u(t), u(t-1)) = \arctan(u(t)) \cdot u(t-1)^2$$

and where $w(t)$ is white Gaussian noise with $\mathrm{E}(w(t)) = 0$ and $\mathrm{E}(w(t)^2) = 1$.
Let the input $u(t)$ be generated by linear filtering of a white Gaussian process $e(t)$ with $\mathrm{E}(e(t)) = 0$ and $\mathrm{E}(e(t)^2) = 1$ such that

$$u(t) = \frac{1 - 0.8q^{-1} + 0.1q^{-2}}{1 - 0.2q^{-1}} e(t)$$

and assume that $e(t)$ and $w(s)$ are independent for all $t, s \in \mathbb{Z}$.
This input signal has been used in an identification experiment where a data set consisting of 100000 measurements of $u(t)$ and $y(t)$ was collected. A linear output error model $\hat{G}_{OE}$ with $n_b = n_f = 2$ and $n_k = 0$ has been estimated from this data set and the result was

$$\hat{G}_{OE}(q) = \frac{0.762 - 0.682q^{-1}}{1 + 0.613q^{-1} + 0.102q^{-2}} \tag{7.9}$$

As can easily be seen from (7.9), the denominator of $\hat{G}_{OE}(q)$ is indeed close to the denominator of $G(q)$. This is exactly what one would expect as the previous theoretical discussion give that the OE-LTI-SOE of the generalized Hammerstein system is the product between $G(q)$ and an FIR model $B(q)$. The MATLAB code that has been used in this example is available in Appendix B.2.

7.3.3 Identification of Generalized Wiener Systems

We will call a nonlinear system a generalized Wiener system if it consists of an LTI system $n(t) = G(q)u(t)$ followed by an NFIR system $y(t) = f((n(t-k))_{k=0}^{M_0})$ as shown in Figure 7.2. For such systems, it is possible to prove a similar result as in Corollary 7.1.

Corollary 7.2
Consider a generalized Wiener system $y(t) = f((n(t-k))_{k=0}^{M_0}) + w(t)$ where $n(t) = G(q)u(t)$ and where $G(q)$ is a stable and causal LTI system. Assume that $u(t)$ is Gaussian and that it, together with $y(t)$, fulfill Assumptions A1 and A3. Assume

also that $n(t)$ and $y(t)$ fulfill Assumptions A1, A3 and A5. Then the OE-LTI-SOE of this system is

$$G_{0,OE}(z) = G(z)B(z) \tag{7.10}$$

where $B(z) = \sum_{k=0}^{M_0} b(k)z^{-k}$ and $b(k) = \mathrm{E}(f'_{n(t-k)}((n(t-j))_{j=0}^{M_0}))$.

Proof: *Lemma 2.1 gives that*

$$\Phi_{yn}(z) = \Phi_{yu}(z)G(z^{-1}) \tag{7.11}$$

and

$$\Phi_n(z) = G(z)\Phi_u(z)G(z^{-1}) \tag{7.12}$$

In addition, Theorem 7.2 gives that

$$\Phi_{yn}(z) = B(z)\Phi_n(z) \tag{7.13}$$

Inserting (7.11) and (7.12) in (7.13) gives

$$\Phi_{yu}(z) = G(z)B(z)\Phi_u(z)$$

and hence (7.10) follows from Corollary 4.1. □

Corollary 7.2 implies, just as in the case with generalized Hammerstein systems, that consistent estimates of the poles of $G(q)$ can be obtained by estimating an output error model. The following example verifies this result for a particular generalized Wiener system.

Example 7.3

Consider a generalized Wiener system consisting of the same linear and nonlinear blocks as the generalized Hammerstein system in Example 7.2 but with the linear block before the nonlinear, i.e.,

$$\begin{aligned} y(t) &= f(n(t), n(t-1)) + w(t) \\ n(t) &= G(q)u(t) \end{aligned}$$

where

$$\begin{aligned} G(q) &= \frac{1}{1 + 0.6q^{-1} + 0.1q^{-2}} \\ f(n(t), n(t-1)) &= \arctan(n(t)) \cdot n(t-1)^2 \end{aligned}$$

and where $w(t)$ is white Gaussian noise with $\mathrm{E}(w(t)) = 0$ and $\mathrm{E}(w(t)^2) = 1$.

Let the input $u(t)$ be generated in the same way as in Example 7.2, i.e.,

$$u(t) = \frac{1 - 0.8q^{-1} + 0.1q^{-2}}{1 - 0.2q^{-1}} e(t)$$

where $e(t)$ is a white Gaussian process with $\mathrm{E}(e(t)) = 0$ and $\mathrm{E}(e(t)^2) = 1$ such that $e(t)$ and $w(s)$ are independent for all $t, s \in \mathbb{Z}$.
An identification experiment has been performed on this generalized Wiener system with a realization of this $u(t)$ as input and 100000 measurements of $u(t)$ and $y(t)$ have been collected. A linear output error model $\hat{G}_{OE}(q)$ with $n_b = n_f = 2$ and $n_k = 0$ has been estimated from the measurements and the result was

$$\hat{G}_{OE}(q) = \frac{0.929 - 2.053q^{-1}}{1 + 0.596q^{-1} + 0.0971q^{-2}} \tag{7.14}$$

From (7.14) we can see that the denominator of $\hat{G}_{OE}(q)$ is close to the denominator of $G(q)$ also when the data has been generated by a generalized Wiener system. The MATLAB code that has been used in this example is available in Appendix B.3.

In this chapter we have studied OE-LTI-SOE:s of NFIR systems with Gaussian inputs. We have shown that the OE-LTI-SOE of such a system is always an FIR model and that this fact can be used for structure identification of NFIR systems. More specifically, it can be used to tell which input signal components that actually affect the output in a nonlinear way. Furthermore, we have shown that the OE-LTI-SOE of a generalized Wiener or Hammerstein system always will be $G(z)B(z)$, where $G(z)$ is the LTI part of the system and where $B(z)$ is an FIR model. Hence, it is possible to estimate the denominator polynomial of $G(z)$ consistently without compensating for any nonlinearities.

As have been mentioned previously, the property that makes Gaussian processes special is that certain conditional expectations become linear. In the next chapter, it will be shown that this is actually both a necessary and sufficient requirement for the OE-LTI-SOE of an arbitrary NFIR system to be an FIR model.

8

NFIR Systems with Separable Input Processes

In the previous chapter, it was shown that the OE-LTI-SOE of an NFIR system will be an FIR model when the input is Gaussian. In this chapter, we will continue to study NFIR systems but we will shift focus from Gaussian inputs to a more general class of signals. More specifically, we will define a necessary and sufficient condition on the input signal for the OE-LTI-SOE of an arbitrary NFIR system to be an FIR model. It will be shown that this condition is that the input process should be *separable* of a certain order (in Nuttall's sense (Nuttall, 1958)).

8.1 Separable Processes

Just like in the previous chapter, we will here consider NFIR systems with impulse response lengths $M > 0$ that can be written as

$$y(t) = f((u(t-k))_{k=0}^{M})$$

Furthermore, we will here consider input signals $u(t)$ that fulfill the input conditions in Assumption A1. For each choice of u, let D_u be a class of Lebesgue integrable functions such that

$$\begin{aligned} D_u = \{f : \mathbb{R}^{M+1} \to \mathbb{R} \ : \ & \mathrm{E}(f((u(t-k))_{k=0}^{M})) = 0,\ \mathrm{E}(f((u(t-k))_{k=0}^{M})^2) < \infty \\ & R_{yu}(\sigma) = \mathrm{E}(f((u(t-k))_{k=0}^{M}))u(t-\sigma)) \text{ exist } \forall \sigma \in \mathbb{Z}\} \end{aligned}$$

In this chapter, we will use the following notation

$$\mathbf{R}_U = \begin{pmatrix} R_u(0) & R_u(1) & \dots & R_u(M) \\ R_u(1) & R_u(0) & \dots & R_u(M-1) \\ \vdots & & & \vdots \\ R_u(M) & R_u(M-1) & \dots & R_u(0) \end{pmatrix} \tag{8.1a}$$

$$\mathbf{R}_{YU} = \begin{pmatrix} R_{yu}(0) \\ R_{yu}(1) \\ \vdots \\ R_{yu}(M) \end{pmatrix} \tag{8.1b}$$

$$\mathbf{R}_{U,\sigma} = \begin{pmatrix} R_u(\sigma) \\ R_u(\sigma-1) \\ \vdots \\ R_u(\sigma-M) \end{pmatrix} \tag{8.1c}$$

$$B = \begin{pmatrix} b(0) & b(1) & \dots & b(M) \end{pmatrix}^T \tag{8.1d}$$

$$C_\sigma = \begin{pmatrix} c_{\sigma,0} & c_{\sigma,1} & \dots & c_{\sigma,M} \end{pmatrix}^T = \mathbf{R}_U^{-1}\mathbf{R}_{U,\sigma} \tag{8.1e}$$

As was mentioned in the introduction to this chapter, we will here discuss under which conditions the OE-LTI-SOE of an NFIR system will be an FIR model. In this discussion, we will need the notion of the mean-square optimal FIR model of a system. The following lemma is a classic result (see, for example, Kailath et al., 2000, Theorems 3.2.1 and 3.2.2) and holds for each fixed choice of u.

Lemma 8.1 (FIR approximation)
Consider an input signal u that fulfills the input conditions in Assumption A1 and for which $\mathbf{R}_U > 0$. Then for each NFIR system f in the corresponding class D_u, there exists a unique linear FIR model of length M

$$G_{0,FIR}(z) = \sum_{k=0}^{M} \bar{b}_f(k) z^{-k}$$

that is an optimal FIR(M) approximation in the mean-square error sense. This FIR model has parameters

$$\bar{B}_f = \mathbf{R}_U^{-1}\mathbf{R}_{YU} \tag{8.2}$$

and satisfy

$$R_{yu}(\sigma) = \sum_{k=0}^{M} \bar{b}_f(k) R_u(\sigma - k), \quad \sigma = 0, 1, \dots, M \tag{8.3}$$

Proof: *The parameters in the optimal FIR(M) model are given by*

$$\bar{B}_f = \arg\min_{B \in \mathbb{R}^{M+1}} \mathrm{E}(\left(f((u(t-k))_{k=0}^{M}) - \sum_{k=0}^{M} b(k)u(t-k) \right)^2)$$

Differentiating this expression with respect to $b(i)$ gives the following equations for the stationary points of the mean-square error criterion

$$-2(R_{yu}(i) - \sum_{k=0}^{M} \bar{b}_f(k) R_u(i-k)) = 0, \quad i = 0, 1, \ldots, M$$

This can also be written as

$$\mathbf{R}_U \bar{B}_f = \mathbf{R}_{YU}$$

and (8.2) and (8.3) follows readily from this expression. □

From (8.3) we see that $G_{0,FIR}$ can explain the cross-covariance function $R_{yu}(\sigma)$ for $\sigma = 0, 1, \ldots, M$. However, sometimes it can actually explain the complete cross-covariance function, i.e.,

$$R_{yu}(\sigma) = \sum_{k=0}^{M} \bar{b}_f(k) R_u(\sigma - k), \quad \forall \sigma \in \mathbb{Z} \tag{8.4}$$

or, equivalently,

$$\Phi_{yu}(z) = G_{0,FIR}(z)\Phi_u(z)$$

In this case, we know from Corollary 4.1 that $G_{0,FIR}$ is not only the mean-square error optimal FIR(M) approximation of the system, but also the OE-LTI-SOE of the system. In the next theorem, we will give necessary and sufficient conditions on u for the equality (8.4) to hold for all $\sigma \in \mathbb{Z}$ and for all $f \in D_u$.

Theorem 8.1 (Separability of order $M+1$)
Consider a fixed $M > 0$ and a certain choice of input signal u that fulfills the input conditions in Assumption A1, and for which $\mathbf{R}_U > 0$ and $\mathrm{E}(|u(t)|) < \infty$. Let $\bar{B}_f$ denote the parameters of the mean-square error optimal FIR(M) approximation of each $f \in D_u$, i.e., $\bar{B}_f = \mathbf{R}_U^{-1}\mathbf{R}_{YU}$ according to Lemma 8.1. Then

$$R_{yu}(\sigma) = \sum_{k=0}^{M} \bar{b}_f(k) R_u(\sigma - k), \quad \forall \sigma \in \mathbb{Z} \text{ and } \forall f \in D_u \tag{8.5}$$

if and only if the following equality holds almost everywhere

$$\begin{aligned} &\int_{-\infty}^{\infty} x_{t-\sigma} p_\sigma(x_t, x_{t-1}, \ldots, x_{t-M}, x_{t-\sigma}) dx_{t-\sigma} \\ &= \sum_{i=0}^{M} c_{\sigma,i} x_{t-i} p(x_t, x_{t-1}, \ldots, x_{t-M}) \quad \forall \sigma > M \vee \sigma < 0 \end{aligned} \tag{8.6}$$

where $C_\sigma = \mathbf{R}_U^{-1}\mathbf{R}_{U,\sigma}$ *and where* p *and* p_σ *are the probability density functions of* $(u(t), u(t-1), \ldots, u(t-M))^T$ *and* $(u(t), u(t-1), \ldots, u(t-M), u(t-\sigma))^T$, *respectively.*

Proof:

(8.6) $\Rightarrow$ *(8.5): By the construction of* $\bar{B}_f$, *the equality (8.5) already holds for* $\sigma = 0, 1, \ldots, M$ *for all* $f \in D_u$ *(cf. (8.2)). Take an arbitrary* $\sigma > M$ *or* $\sigma < 0$. *Then*

$$
\begin{aligned}
R_{yu}(\sigma) &= \int_{\mathbb{R}^{M+2}} f(x_t, \ldots, x_{t-M}) x_{t-\sigma} p_\sigma(x_t, \ldots, x_{t-M}, x_{t-\sigma}) dx_{t-\sigma} dx_t \ldots dx_{t-M} \\
&= \int_{\mathbb{R}^{M+1}} \left(f(x_t, \ldots, x_{t-M}) - \sum_{k=0}^{M} \bar{b}_f(k) x_{t-k} \right) \\
&\qquad \cdot \sum_{i=0}^{M} c_{\sigma,i} x_{t-i} p(x_t, \ldots, x_{t-M}) dx_t \ldots dx_{t-M} \\
&\quad + \int_{\mathbb{R}^{M+2}} \left(\sum_{k=0}^{M} \bar{b}_f(k) x_{t-k} \right) x_{t-\sigma} p_\sigma(x_t, \ldots, x_{t-M}, x_{t-\sigma}) dx_{t-\sigma} dx_t \ldots dx_{t-M} \\
&= \sum_{i=0}^{M} c_{\sigma,i} \int_{\mathbb{R}^{M+1}} \left(f(x_t, \ldots, x_{t-M}) - \sum_{k=0}^{M} \bar{b}_f(k) x_{t-k} \right) \\
&\qquad \cdot x_{t-i} p(x_t, \ldots, x_{t-M}) dx_t \ldots dx_{t-M} \\
&\quad + \sum_{k=0}^{M} \bar{b}_f(k) \int_{\mathbb{R}^{M+2}} x_{t-k} x_{t-\sigma} p_\sigma(x_t, \ldots, x_{t-M}, x_{t-\sigma}) dx_{t-\sigma} dx_t \ldots dx_{t-M} \\
&= \sum_{k=0}^{M} \bar{b}_f(k) R_u(\sigma - k)
\end{aligned}
$$

where we in the last equality have used that (8.3) holds for all $f \in D_u$. *As* σ *was arbitrary, (8.5) holds for all* $\sigma \in \mathbb{Z}$.

(8.5) $\Rightarrow$ *(8.6): Using (8.1e) and (8.2) the equality (8.5) can be written as*

$$
R_{yu}(\sigma) = \bar{B}_f^T \mathbf{R}_{U,\sigma} = \mathbf{R}_{YU}^T C_\sigma = \sum_{i=0}^{M} c_{\sigma,i} R_{yu}(i), \quad \forall \sigma \in \mathbb{Z} \text{ and } \forall f \in D_u \tag{8.7}
$$

Take an arbitrary $\sigma > M$ *or* $\sigma < 0$. *Then (8.7) gives*

$$
\begin{aligned}
&\int_{\mathbb{R}^{M+1}} f(x_t, \ldots, x_{t-M}) \bigg(\int_{-\infty}^{\infty} x_{t-\sigma} p_\sigma(x_t, \ldots, x_{t-M}, x_{t-\sigma}) dx_{t-\sigma} \\
&\quad - \sum_{i=0}^{M} c_{\sigma,i} x_{t-i} p(x_t, \ldots, x_{t-M}) \bigg) dx_t \ldots dx_{t-M} = 0, \quad \forall f \in D_u
\end{aligned} \tag{8.8}
$$

Let

$$v_\sigma(x_t, \dots, x_{t-M}) = \int_{-\infty}^{\infty} x_{t-\sigma} p_\sigma(x_t, \dots, x_{t-M}, x_{t-\sigma}) dx_{t-\sigma} - \sum_{i=0}^{M} c_{\sigma,i} x_{t-i} p(x_t, \dots, x_{t-M})$$

and define a function

$$f_0(x_t, \dots, x_{t-M}) = \operatorname{sign}(v_\sigma(x_t, \dots, x_{t-M})) - \mu_0$$

where

$$\mu_0 = \mathrm{E}(\operatorname{sign}(v_\sigma((u(t-k))_{k=0}^M)))$$

Since $\mathrm{E}(f_0((u(t-k))_{k=0}^M)) = 0$, $\mathrm{E}(f_0((u(t-k))_{k=0}^M)^2) \leq 1$ *and*

$$\begin{aligned}
&|\mathrm{E}(f_0((u(t-k))_{k=0}^M) u(t-\tau))| \\
&= |\mathrm{E}(\operatorname{sign}(v_\sigma((u(t-k))_{k=0}^M)) u(t-\tau)) - \mu_0 \underbrace{\mathrm{E}(u(t-\tau))}_{=0}| \\
&= |\mathrm{E}(\operatorname{sign}(v_\sigma((u(t-k))_{k=0}^M)) u(t-\tau))| \\
&\leq \mathrm{E}(|\operatorname{sign}(v_\sigma((u(t-k))_{k=0}^M)) u(t-\tau)|) \\
&\leq \mathrm{E}(|u(t-\tau)|) < \infty \quad \forall \tau \in \mathbb{Z}
\end{aligned}$$

it follows that $f_0 \in D_u$. *Hence, (8.8) holds for* $f = f_0$ *and this implies that*

$$\begin{aligned}
&\int_{\mathbb{R}^{M+1}} |v_\sigma(x_t, \dots, x_{t-M})| dx_t \dots dx_{t-M} \\
&\quad - \mu_0 \underbrace{\mathrm{E}(u(t-\sigma))}_{=0} + \mu_0 \sum_{i=0}^{M} c_{\sigma,i} \underbrace{\mathrm{E}(u(t-i))}_{=0} = 0 \\
&\Rightarrow \int_{\mathbb{R}^{M+1}} |v_\sigma(x_t, \dots, x_{t-M})| dx_t \dots dx_{t-M} = 0 \\
&\Rightarrow v_\sigma(x_t, \dots, x_{t-M}) = 0 \quad \text{almost everywhere}
\end{aligned}$$

As σ *was arbitrary, (8.6) follows.* □

A process u that fulfills (8.6) is said to be *separable of order* $M+1$. Theorem 8.1 is an extension of the corresponding theorem about separability of order one in Nuttall (1958). There, the notion of separability of order one is discussed in detail and it is also mentioned briefly (on p. 76) that this notion might be extended to separability of higher orders by considering integrals like

$$\int_{-\infty}^{\infty} x_t p(x_t, x_{t-\tau_1}, x_{t-\tau_2}) dx_t$$

However, no further conclusions are drawn in Nuttall (1958) and to the author's knowledge, no such extension has been made elsewhere.

Using conditional expectations, the separability condition (8.6) can be written more intuitively as

$$\mathrm{E}(u(t-\sigma)|u(t), u(t-1)\ldots, u(t-M)) = \sum_{i=0}^{M} c_{\sigma,i} u(t-i) \quad \sigma > M \vee \sigma < 0 \tag{8.9}$$

Actually, separability of order $M+1$ can be defined as the property that the conditional expectation in (8.9) is any linear combination of the given input components, i.e.,

$$\mathrm{E}(u(t-\sigma)|u(t), u(t-1)\ldots, u(t-M)) = \sum_{i=0}^{M} a_{\sigma,i} u(t-i) \quad \sigma > M \vee \sigma < 0 \tag{8.10}$$

This definition gives

$$\begin{aligned} R_u(\sigma - k) &= \mathrm{E}\big(u(t-k)u(t-\sigma)\big) \\ &= \mathrm{E}\big(\mathrm{E}\big(u(t-k)u(t-\sigma)|u(t), u(t-1)\ldots, u(t-M)\big)\big) \\ &= \mathrm{E}\big(u(t-k)\mathrm{E}\big(u(t-\sigma)|u(t), u(t-1)\ldots, u(t-M)\big)\big) \\ &= \sum_{i=0}^{M} a_{\sigma,i} \mathrm{E}\big(u(t-k)u(t-i)\big) \\ &= \sum_{i=0}^{M} a_{\sigma,i} R_u(k-i) \quad k = 0, 1, \ldots, M \end{aligned}$$

Here, we have used the facts that

$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(\mathrm{E}(Y|X)) \\ \mathrm{E}(g(X)Y|X) &= g(X)\mathrm{E}(Y|X) \end{aligned}$$

(see, for example, Gut, 1995, Chap. 2). If A_σ is defined as

$$A_\sigma = \begin{pmatrix} a_{\sigma,0} & a_{\sigma,1} & \ldots & a_{\sigma,M} \end{pmatrix}^T$$

the previous expression can also be written as

$$\mathbf{R}_U A_\sigma = \mathbf{R}_{U,\sigma}$$

This shows that $A_\sigma = \mathbf{R}_U^{-1}\mathbf{R}_{U,\sigma} = C_\sigma$. Hence, separability of order $M+1$ means just that the property (8.10) holds. Since this is a well-known property of Gaussian signals (see, for example, Brockwell and Davis, 1987, p. 64), it immediately follows that such signals are separable of order $M+1$ for any $M \in \mathbb{N}$. Furthermore, it is easy to see that white, possibly non-Gaussian, signals fulfill (8.10) too. Theorem 8.1 together with Corollary 4.1 give the following theorem.

Theorem 8.2
Consider a fixed $M > 0$ and a certain input signal u that fulfills the input conditions in Assumption A1, and for which $\mathbf{R}_U > 0$ and $\mathrm{E}(|u(t)|) < \infty$. Consider NFIR systems

$$y(t) = f((u(t-k))_{k=0}^{M}) + w(t)$$

where the noise $w(t)$ is such that Assumption A3 is fulfilled for all f. Then the OE-LTI-SOE of such a system will be a linear FIR model

$$G_{0,OE}(z) = \frac{\Phi_{yu}(z)}{\Phi_u(z)} = \sum_{k=0}^{M} \bar{b}_f(k) z^{-k} \tag{8.11}$$

where $\bar{B}_f = \mathbf{R}_U^{-1}\mathbf{R}_{YU}$ for all *$f \in D_u$, if and only if u is separable of order $M+1$.*

Theorem 8.2 shows that separability of order $M+1$ is a necessary and sufficient condition for the OE-LTI-SOE and the mean-square error optimal noncausal LTI model both to be equal to an FIR model of length M. Since separability of order $M+1$ guarantees that $G_{0,OE}(z) = \Phi_{yu}(z)/\Phi_u(z)$ for an NFIR system, all properties in Sections 5.4.1, 5.4.2 and 5.5 will hold also for separable input signals provided that $\Phi_y(z)$ also is well-defined according to Assumption A1. Hence, separable inputs can in many cases be used as an alternative to Gaussian input signals.

The focus of this chapter has been on FIR approximations of NFIR systems. It has been shown that separability of order $M+1$ is a necessary and sufficient condition on the input signal for the OE-LTI-SOE of any NFIR system in a rather wide class of systems to be an FIR model. This result concludes the main part of this thesis. In the next chapter, a short summary of the previous discussion about LTI models of nonlinear systems will be given.

9

Conclusions

A number of results about LTI Second Order Equivalents have been shown in the previous chapters of this thesis. Since some of these results have practical implications, it might be appropriate to compile some guidelines for the user. This will be done in this last chapter.

It should once again be pointed out that the conclusions that have been drawn in this thesis concern *asymptotic* properties of prediction-error model estimates, i.e., properties when the number of measurements tends to infinity. Hence, the results are usually only applicable to identification problems where large data sets are used.

Previously, we have seen that there can be remaining correlation between the input and the model residuals both for the OE-LTI-SOE and the GE-LTI-SOE. If no additional knowledge about the system structure is available, this correlation might be taken as an indication that the system operates in closed loop.

However, for some classes of input signals, this cannot happen for an open-loop nonlinear system. If the input has been generated by filtering white noise through a minimum phase filter, there will be no spurious correlation between the input and the residuals of the OE-LTI-SOE and the GE-LTI-SOE. Furthermore, these input signals have the following properties.

- Each minimum phase generated input signal is optimal over a class of other input signals in the sense the variance of the residuals of the OE-LTI-SOE is minimized (see Theorem 5.2).
- Residual and spectral analysis can be used to validate an estimated model and to see if it is sufficiently close to the OE-LTI-SOE of the system (see Section 5.4.2).

- A minimum phase generated input signal is a good choice of reference signal for closed-loop identification using the two-step method since it implies that only a causal $S(z)$ has to be estimated (see Section 5.4.3).

One particular choice of minimum phase generated input signal is Gaussian noise with arbitrary color. This class of input signals has the following properties in addition to the ones above.

- For a Gaussian input signal there will be no extra dynamics in the OE-LTI-SOE. Hence, the OE-LTI-SOE of an NFIR system will be an FIR model with coefficients that are expectations of the partial derivatives of the nonlinear function (see Theorem 7.2). This simplifies structure identification of such a system significantly.
- Furthermore, the OE-LTI-SOE of a generalized Hammerstein or Wiener system will be the cascade product of the linear part of that system and an FIR model (see Corollaries 7.1 and 7.2). Hence, it is possible to estimate the denominator polynomial of the linear part consistently without compensating for the nonlinearities.

With the properties mentioned above in mind, it is possible to give a general advice concerning LTI approximations of nonlinear systems using the prediction-error method and large data sets. Of course, this is not the definite solution to the input selection problem, but merely a conclusion that can be drawn from the discussions in this thesis.

Use a Gaussian input signal or, if that is not possible, any other signal generated by filtering white noise through a minimum phase filter.

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A

Calculations

A.1 Example 4.3

Since $\mathrm{E}(e(t)^3) = 0$ and $\mathrm{E}(e(t)^4) = 3$ we get

$$\begin{aligned}
R_u(0) &= \mathrm{E}(u(t)^2) = \mathrm{E}((e(t) + e(t-1)^2 - 1)^2) \\
&= \mathrm{E}(e(t)^2) + \mathrm{E}(e(t-1)^4) + 1 - 2\mathrm{E}(e(t-1)^2) = 1 + 3 + 1 - 2 = 3 \\
R_u(\pm 1) &= \mathrm{E}(u(t)u(t-1)) = 0 \\
R_u(\tau) &= 0 \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}
\end{aligned}$$

Furthermore, since $\mathrm{E}(e(t)^5) = \mathrm{E}(e(t)^7) = 0$, $\mathrm{E}(e(t)^6) = 15$ and $\mathrm{E}(e(t)^8) = 105$, we get

$$\begin{aligned}
R_{yu}(0) &= \mathrm{E}(y(t)u(t)) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t) + e(t-1)^2 - 1\big)\big) \\
&= 2\mathrm{E}(e(t)^2 e(t-1)^2) - 2\mathrm{E}(e(t)^2) + \mathrm{E}(e(t-1)^6) - \mathrm{E}(e(t-1)^4) \\
&\quad - 2\mathrm{E}(e(t-1)^4) + 2\mathrm{E}(e(t-1)^2) \\
&= 2 - 2 + 15 - 3 - 2 \cdot 3 + 2 = 8 \\
R_{yu}(1) &= \mathrm{E}(y(t)u(t-1)) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t-1) + e(t-2)^2 - 1\big)\big) = 0
\end{aligned}$$

$$
\begin{aligned}
R_{yu}(-1) &= \mathrm{E}(y(t)u(t+1)) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t+1) + e(t)^2 - 1\big)\big) = \mathrm{E}(e(t)^4) - \mathrm{E}(e(t)^2) = 3 - 1 = 2 \\
R_{yu}(\tau) &= 0 \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}
\end{aligned}
$$

and

$$
\begin{aligned}
R_y(0) &= \mathrm{E}(y(t)^2) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big)\big) \\
&= \mathrm{E}(e(t)^4) + 2\mathrm{E}(e(t)^2 e(t-1)^4) - 4\mathrm{E}(e(t)^2 e(t-1)^2) - 4\mathrm{E}(e(t)^2) \\
&\quad + \mathrm{E}(e(t-1)^8) - 4\mathrm{E}(e(t-1)^6) - 4\mathrm{E}(e(t-1)^4) + 4\mathrm{E}(e(t)^2 e(t-1)^4) \\
&\quad - 8\mathrm{E}(e(t)^2 e(t-1)^2) + 4\mathrm{E}(e(t)^2) + 4\mathrm{E}(e(t-1)^4) + 8\mathrm{E}(e(t-1)^2) + 4 \\
&= 3 + 2\cdot 3 - 4 - 4 + 105 - 4\cdot 15 - 4\cdot 3 + 4\cdot 3 - 8 + 4 + 4\cdot 3 + 8 + 4 \\
&= 66 \\
R_y(\pm 1) &= \mathrm{E}(y(t)y(t-1)) \\
&= \mathrm{E}\big(\big(e(t)^2 + e(t-1)^4 + 2e(t)e(t-1)^2 - 2e(t) - 2e(t-1)^2 - 2\big) \\
&\quad \cdot \big(e(t-1)^2 + e(t-2)^4 + 2e(t-1)e(t-2)^2 - 2e(t-1) \\
&\quad -2e(t-2)^2 - 2\big)\big) \\
&= \mathrm{E}(e(t)^2 e(t-1)^2) + \mathrm{E}(e(t-1)^6) - 2\mathrm{E}(e(t-1)^4) \\
&\quad - 2\mathrm{E}(e(t-1)^2) + \mathrm{E}(e(t)^2 e(t-2)^4) + \mathrm{E}(e(t-1)^4 e(t-2)^4) \\
&\quad - 2\mathrm{E}(e(t-1)^2 e(t-2)^4) - 2\mathrm{E}(e(t-2)^4) - 2\mathrm{E}(e(t)^2 e(t-2)^2) \\
&\quad - 2\mathrm{E}(e(t-1)^4 e(t-2)^2) + 4\mathrm{E}(e(t-1)^2 e(t-2)^2) + 4\mathrm{E}(e(t-2)^2) \\
&\quad - 2\mathrm{E}(e(t)^2) - 2\mathrm{E}(e(t-1)^4) + 4\mathrm{E}(e(t-1)^2) + 4 \\
&= 1 + 15 - 2\cdot 3 - 2 + 3 + 9 - 2\cdot 3 - 2\cdot 3 - 2 - 2\cdot 3 + 4 + 4 - 2 \\
&\quad - 2\cdot 3 + 4 + 4 = 8 \\
R_y(\tau) &= 0 \quad \forall \tau \in \mathbb{Z} \setminus \{-1,\ 0,\ 1\}
\end{aligned}
$$

The z-spectra are thus

$$
\begin{aligned}
\Phi_u(z) &= 3 \\
\Phi_{yu}(z) &= 2z + 8 \\
\Phi_y(z) &= 8z + 66 + 8z^{-1}
\end{aligned}
$$

Inserted in (4.2b) this gives

$$
\Phi_\zeta(z) = \begin{pmatrix} 3 & 2 + 8z \\ 2 + 8z^{-1} & 8z + 66 + 8z^{-1} \end{pmatrix}
$$

In order to compute the canonical spectral factorization $\Phi_\zeta(z) = T(z)Q_\zeta T^T(z^{-1})$ we first pre- and postmultiply $\Phi_\zeta(z)$ with matrices $T_1(z)$ and $T_1^T(z^{-1})$, respectively. If $T_1(z)$ is chosen as

$$T_1(z) = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} - \frac{8}{3}z^{-1} & 1 \end{pmatrix}$$

the result is the following diagonal matrix

$$D(z) = T_1(z)\Phi_\zeta(z)T_1^T(z^{-1}) = \begin{pmatrix} 3 & 0 \\ 0 & \frac{2}{3}\left(4z^{-1} + 65 + 4z\right) \end{pmatrix}$$

The matrix element $D_{22}(z) = \frac{2}{3} \cdot \left(4z^{-1} + 65 + 4z\right)$ can be factorized as

$$D_{22}(z) = \frac{8}{3}(z - z_0)\left(1 - \frac{1}{z_0 z}\right) = -\frac{8}{3z_0}(z - z_0)\left(z^{-1} - z_0\right)$$

where $z_0 = (-65 + \sqrt{4161})/8$. Hence, the matrix $D(z)$ can be factorized as $D(z) = T_2(z)T_2^T(z^{-1})$ where

$$T_2(z) = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \kappa(z - z_0) \end{pmatrix}$$

where $\kappa = \sqrt{-\frac{8}{3z_0}}$. The two matrices $T_1(z)$ and $T_2(z)$ defines a spectral factorization, $\Phi_\zeta(z) = T_p(z)T_p^T(z^{-1})$ with

$$T_p(z) = T_1^{-1}(z)T_2(z) = \begin{pmatrix} \sqrt{3} & 0 \\ \frac{2+8z^{-1}}{\sqrt{3}} & \kappa(z - z_0) \end{pmatrix}$$

This is however not the *canonical* spectral factorization as $T_p(+\infty) \neq I$. Let

$$T_3(z) = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ -\frac{2}{3\kappa z} & \frac{1}{\kappa z} \end{pmatrix}$$

and let

$$\begin{aligned} T(z) = T_p(z)T_3(z) &= \begin{pmatrix} 1 & 0 \\ \left(\frac{8+2z_0}{3}\right)z^{-1} & 1 - z_0 z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{4161}-33}{12}z^{-1} & 1 + \frac{65-\sqrt{4161}}{8}z^{-1} \end{pmatrix} \\ Q_\zeta = T_3^{-1}(z)T_3^{-T}(z^{-1}) &= \begin{pmatrix} \sqrt{3} & 0 \\ \frac{2}{\sqrt{3}} & \kappa z \end{pmatrix}\begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & \kappa z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 \\ 2 & \frac{4}{3} + \kappa^2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 23 + \frac{\sqrt{4161}}{3} \end{pmatrix} \end{aligned}$$

This gives $\Phi_\zeta(z) = T(z)Q_\zeta T^T(z^{-1})$ and both $T(z)$ and $T^{-1}(z)$ are analytical on and outside the unit circle, $T(+\infty) = I$ and $Q_\zeta > 0$. Hence we have found the canonical spectral factorization of $\Phi_\zeta(z)$.

B

MATLAB Code

B.1 Example 5.2

MATLAB commands for Example 5.2:

```
N=10000;

% Nonminimum phase case
enmp=2*rand(N,1)-1;
unmp=filter([.5 1],1,enmp);
ynmp=unmp.^3;
znmp=iddata(ynmp,unmp,1);
spanmp=spa(znmp,30);
Gnmp=tf([.925 .425],[1 .5],1);
figure(1)
clf
bode(spanmp,'k',Gnmp,'k--')
subplot(211)
axis([0.01 10 10^(-.3) 1.1])
subplot(212)
axis([0.01 10 -20 20])

% Minimum phase case
emp=2*rand(N,1)-1;
ump=filter([1 .5],1,emp);
ymp=ump.^3;
```

```
zmp=iddata(ymp,ump,1);
spamp=spa(zmp,30);
Gmp=tf([.85 .575],[1 .5],1);
figure(2)
clf
bode(spamp,'k',Gmp,'k--')
subplot(211)
axis([0.01 10 10^(-.3) 1.1])
subplot(212)
axis([0.01 10 -20 20])
```

B.2 Example 7.2

MATLAB commands for Example 7.2:

```
N=100001;

e=randn(N,1);
u=filter([1 -.8 .1],[1 -.2],e);
v=u(1:end-1).^2.*atan(u(2:end));
w=randn(N-1,1);
y=filter(1,[1 .6 .1],v)+w;

z=iddata(y,u(2:end),1);
G=oe(z,[2 2 0],'lim',0,'cov','none')
```

B.3 Example 7.3

MATLAB commands for Example 7.3:

```
N=100001;

e=randn(N,1);
u=filter([1 -.8 .1],[1 -.2],e);
n=filter(1,[1 .6 .1],u);
w=randn(N-1,1);
y=n(1:end-1).^2.*atan(n(2:end))+w;

z=iddata(y,u(2:end),1);
G=oe(z,[2 2 0],'lim',0,'cov','none')
```

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