Some Results On Normal Homogeneous Ideals

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SOME RESULTS ON NORMAL HOMOGENEOUS IDEALS

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Abstract. In this article we investigate when a homogeneous ideal in a graded ring is normal, that is, when all positive powers of the ideal are integrally closed. We are particularly interested in homogeneous ideals in an $\mathbb{N}$-graded ring A of the form $A_{\geq m} := \bigoplus_{\ell \geq m} A_\ell$ and monomial ideals in a polynomial ring over a field. For ideals of the form $A_{\geq m}$ we generalize a recent result of Faridi. We prove that a monomial ideal in a polynomial ring in n indeterminates over a field is normal if and only if the first $n-1$ positive powers of the ideal are integrally closed. We then specialize to the case of ideals of the form $I(\boldsymbol{\lambda}) := \overline{J(\boldsymbol{\lambda})}$, where $J(\boldsymbol{\lambda}) = (x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}) \subseteq K[x_1, \ldots, x_n]$. To state our main result in this setting, we let $\ell = \operatorname{lcm}(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots \lambda_n)$, for $1 \leq i \leq n$, and set $\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \ell, \lambda_{i+1}, \ldots, \lambda_n)$. We prove that if $I(\boldsymbol{\lambda}')$ is normal then $I(\boldsymbol{\lambda})$ is normal and that the converse holds with a small additional assumption.

1. Introduction

In this paper we explore when a homogeneous ideal in a graded ring is normal, that is, when all positive powers of the ideal are integrally closed. In particular, we are interested in homogeneous ideals of an $\mathbb{N}$-graded ring A of the form $A_{\geq m} := \bigoplus_{\ell \geq m} A_\ell$ and monomial ideals in a polynomial ring over a field. In the first setting, we generalize a recent theorem of Faridi [7]. As for monomial ideals, our first new result is that a monomial ideal I in a polynomial ring $K[x_1, \ldots, x_n]$ over a field K is normal if and only if the first $n-1$ positive powers of I are normal. We then specialize to the case of monomials ideals of the form $\overline{J(\boldsymbol{\lambda})}$, where $J(\boldsymbol{\lambda}) := (x_1^{\lambda_1}, \ldots, x_n^{\lambda_n})$ is an ideal in $R := K[x_1, \ldots, x_n]$, $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n)$ is a vector of positive integers, and $\overline{J(\boldsymbol{\lambda})}$ is the integral closure of $J(\boldsymbol{\lambda})$ in R.

In [1] Bruns and Gubeladze studied the normality of the polytopal semigroup ring $K[S(\boldsymbol{\lambda})]$, where K is a field and $S(\boldsymbol{\lambda})$ is the submonoid of $\mathbb{N}^{n+1}$ generated by

$$\{(a_1, \ldots, a_n, d) \in \mathbb{N}^{n+1} \mid a_1/\lambda_1 + \cdots + a_n/\lambda_n \leq d \text{ for } d \leq 1\}.$$

Bruns and Gubeladze defined $\boldsymbol{\lambda}$ to be normal provided that $K[S(\boldsymbol{\lambda})]$ is normal. One striking result in [1] is the following theorem.

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Theorem 1.1. [1, Theorem 1.6] *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be a vector of positive integers and set $\ell = \operatorname{lcm}(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots \lambda_n)$. Then $\boldsymbol{\lambda}$ is normal if and only if $\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \ell, \lambda_{i+1}, \ldots, \lambda_n)$ is normal; in other words the normality of $\boldsymbol{\lambda}$ depends only on the residue class of λ_i modulo the least common multiple of the λ_j with $i \neq j$.*

Notice that Theorem 1.1 says that the semigroup ring $K[S(\boldsymbol{\lambda})]$ is normal if and only if $K[S(\boldsymbol{\lambda}')]$ is normal.

The normality of the ideal $I(\boldsymbol{\lambda}) := \overline{J(\boldsymbol{\lambda})}$ is equivalent to the normality of the semigroup ring $K[S'(\boldsymbol{\lambda})]$, where $S'(\boldsymbol{\lambda})$ is the submonoid of $\mathbb{N}^{n+1}$ generated by

$$\{(a_1, \ldots, a_n, d) \in \mathbb{N}^{n+1} \mid a_1/\lambda_1 + \cdots + a_n/\lambda_n \geq d \text{ for } d \leq 1\}.$$

Due to the similarity between the semigroups $S(\boldsymbol{\lambda})$ and $S'(\boldsymbol{\lambda})$ one might ask the following questions.

Question 1.2. Is $K[S(\boldsymbol{\lambda})]$ normal if and only if $K[S'(\boldsymbol{\lambda})]$ is normal (that is, if and only if $I(\boldsymbol{\lambda})$ is normal as an ideal)?

Question 1.3. Is $I(\boldsymbol{\lambda})$ normal if and only if $I(\boldsymbol{\lambda}')$ is normal?

The answer to both of the above questions is no by Example 5.2. Since we are interested in the normality of the ideal $I(\boldsymbol{\lambda})$ rather than the normality of the polytopal semigroup ring $K[S(\boldsymbol{\lambda})]$ and the normality of one does not imply the normality of the other, we will no longer refer to the normality of the vector $\boldsymbol{\lambda}$. Later in the paper we identify the semigroup ring $K[S'(\boldsymbol{\lambda})]$ with the Rees algebra $R[I(\boldsymbol{\lambda})t]$ and drop further references to $K[S'(\boldsymbol{\lambda})]$. The normality of $I(\boldsymbol{\lambda})$ for specific $\boldsymbol{\lambda}$ can be determined readily using the **normaliz** program [5] of Bruns and Koch.

We now describe the organization of this paper. In section 2 we review some background material for our work, including integral closure of monomial algebras and ideals, normality of ideals, and polytopal semigroup rings. In section 3 we prove several results on normal ideals in polynomial rings and $\mathbb{N}$-graded rings of the form $A_{\geq m}$, generalizing recent results of Faridi [7]. In section 4 we develop for $I(\boldsymbol{\lambda})$ an analogue of [1, Proposition 1.3]. We introduce the concept of quasinormality for an additive semigroup of the nonnegative rational numbers. We show that if $I(\boldsymbol{\lambda})$ is normal then the semigroup Λ of $\mathbb{Q}_{\geq}$ generated by $1/\lambda_1, \ldots, 1/\lambda_n$ is quasinormal (Lemma 4.6), and if the λ_i are pairwise relatively prime then the converse holds (Proposition 4.7). In section 5 we show that the two aforementioned questions have negative answers. Neither implication of Question 1.2 holds. This is shown in Example 5.2. However the implication $I(\boldsymbol{\lambda}')$ normal implies $I(\boldsymbol{\lambda})$ normal of Question 1.3 always holds and the converse holds with an additional hypothesis (Theorem 5.1).

Conventions. All rings are assumed to be commutative with identity. We let $\mathbb{Z}_+$ denote the set of positive integers, $\mathbb{N}$ the set of nonnegative integers,

$\mathbb{Q}_{\geq}$ the set of nonnegative rational numbers, $\mathbb{Q}_{+}$ the set of positive rational numbers, $\mathbb{R}_{\geq}$ the set of nonnegative real numbers, and $\mathbf{e}_1, \dots, \mathbf{e}_n$ the standard basis vectors in $\mathbb{R}^n$. We write $\boldsymbol{\alpha} \leq_{pr} \boldsymbol{\beta}$ for vectors $\boldsymbol{\alpha} = (a_1, \dots, a_n), \boldsymbol{\beta} = (b_1, \dots, b_n) \in \mathbb{R}^n$ provided that $a_i \leq b_i$ for $1 \leq i \leq n$. Thus $\boldsymbol{\alpha} <_{pr} \boldsymbol{\beta}$ means that $a_i \leq b_i$ for all $1 \leq i \leq n$ and $a_j < b_j$ for some $1 \leq j \leq n$. For a subset X of $\mathbb{R}^n$ we let $\operatorname{conv}(X)$ denote the convex hull of X. Throughout this paper R will denote the polynomial ring $K[x_1, \dots, x_n]$ over a field K and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ a vector of positive integers. In this context, for a vector $\boldsymbol{\alpha} = (a_1, \dots, a_n) \in \mathbb{N}^n$ we let $x^{\boldsymbol{\alpha}}$ denote the monomial $x_1^{a_1} \cdots x_n^{a_n}$.

2. Background

In this section we recall some of the background to our investigation. The integral closure of rings and ideals are as defined, for example, in [6, Chapter 4]. We will be working primarily with monomial ideals $I \subset R = K[x_1, \dots, x_n]$ and subalgebras $A \subset R$ that are generated by a finite number of monomials. In these cases we recall some definitions and notation that appeared in [9] and [10].

Definition 2.1. Let X be any subset of $R = K[x_1, \dots, x_n]$. Then set

$$\Gamma(X) \ = \ \{\boldsymbol{\alpha} \in \mathbb{N}^n \ \mid x^{\boldsymbol{\alpha}} \in X\}.$$

We refer to $\Gamma(X)$ as the *exponent set of* X. If I is a monomial ideal then $\Gamma(I)$ is an ideal of the monoid $\mathbb{N}^n$ [8, page 3]. If A is a subalgebra of R generated by monomials then $\Gamma(A)$ is a submonoid of $\mathbb{N}^n$, and A is isomorphic to the monoid ring $K[\Gamma(A)]$.

Definition 2.2. For an arbitrary subset Λ of $\mathbb{R}^n$ and a positive integer m we let

$$\begin{aligned} m \cdot \Lambda \ &= \ \{\lambda_1 + \cdots + \lambda_m \ \mid \lambda_i \in \Lambda \ \ (i = 1, \dots, m)\}; \ \text{ and} \\ m\Lambda \ &= \ \{m\lambda \mid \lambda \in \Lambda\}. \end{aligned}$$

If $\Lambda = \Gamma(I)$ (respectively $\Gamma(A)$) then $\operatorname{conv}(\Lambda)$ will be denoted $\operatorname{NP}(I)$ (respectively $\operatorname{NP}(A)$), and will be referred to as the *Newton polyhedron of* I (respectively, of A).

The integral closures of monomial ideals and subalgebras now have the following geometric descriptions.

Theorem 2.3. (a) *Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$. Then the integral closure $\overline{I}$ of I in R is the ideal defined by $\Gamma(\overline{I}) = \operatorname{NP}(I) \cap \mathbb{N}^n$ (so that* $\operatorname{NP}(I) = \operatorname{NP}(\overline{I})$*). Furthermore*

$$\Gamma(\overline{I}) = \{\boldsymbol{\alpha} \in \mathbb{N}^n | m\boldsymbol{\alpha} \in m \cdot \Gamma(I) \textit{ for some } m \geq 1\}.$$

(b) *Let A be a subalgebra of R generated by a finite number of monomials. Then the integral closure $\overline{A}$ of A in R is the monoid ring defined by $\Gamma(\overline{A}) = \mathrm{NP}(A) \cap \mathbb{N}^n$. Furthermore $\mathrm{NP}(A)$ is the cone spanned by $\Gamma(A)$ (or by the exponents of a (finite) set of algebra generators of A) and*

$$\Gamma(\overline{A}) = \{\boldsymbol{\alpha} \in \mathbb{N}^n | m\boldsymbol{\alpha} \in \Gamma(A) \text{ for some } m \geq 1\}.$$

Proof. (a) See Exercises 4.22, 4.23 in [6].

(b) This is [4, Proposition 6.1.2]. See also [9, 3.1] for a form closer to what we want here. □

Polytopal semigroup rings, introduced in [2], are examples of such monomial algebras. A polytopal semigroup ring is a monoid algebra $K[S_P]$, where P is a polytope in $\mathbb{R}^n$ whose vertices have integer coordinates and S_P is the submonoid of $\mathbb{R}^{n+1}$ generated by the points $\{(\boldsymbol{\alpha}, 1) \mid \boldsymbol{\alpha} \in P \cap \mathbb{Z}^n\}$. The polytopal semigroup ring of Bruns and Gubeladze that we referred to as $K[S(\boldsymbol{\lambda})]$ in the Introduction is denoted in [1] by $K[S_{\Delta(\boldsymbol{\lambda})}]$, where the polytope $\Delta(\boldsymbol{\lambda}) \subseteq \mathbb{R}^n$ has vertices $(0, \ldots, 0), (\lambda_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \lambda_n)$.

An ideal I in an integral domain A is defined to be normal if I^m is integrally closed for all $m \in \mathbb{Z}_+$. The following result is well known (for example, see [11]).

Theorem 2.4. *Let I be an ideal in an integral domain A. Then the integral closure of the Rees algebra $A[It]$ in $A[t]$ is $\oplus_{i \geq 0} \overline{I^i} t^i$.*

Thus in case the containing ring is a normal integral domain, the normality of I is equivalent to the normality of the Rees ring $A[It]$. Note that "normal" and "integrally closed (in its quotient field)" are synonyms for reduced Noetherian rings but not for ideals. The following observation may be helpful when contemplating normal ideals in $R = K[x_1, \ldots, x_n]$.

Lemma 2.5. *Let $I = (x^{\boldsymbol{\beta}_1}, \ldots, x^{\boldsymbol{\beta}_k}) \subseteq R$ be a monomial ideal, let $m \geq 1$, and $J = (x^{m\boldsymbol{\beta}_1}, \ldots, x^{m\boldsymbol{\beta}_k})$. Then*

(a) $\mathrm{NP}(J) = \mathrm{NP}(I^m) = m\mathrm{NP}(I) = m \cdot \mathrm{NP}(I) =$
$\{\boldsymbol{\alpha} \in \mathbb{R}^n_{\geq} \mid \boldsymbol{\alpha} \geq_{pr} \sum_{i=1}^k c_i \boldsymbol{\beta}_i$ *for some* $c_i \in \mathbb{R}_{\geq}, \sum c_i = m\}$.

(b) *If $\boldsymbol{\alpha} \in \mathrm{NP}(I^m)$ there exist r affinely independent vectors $\boldsymbol{\beta}_{i(1)}, \ldots, \boldsymbol{\beta}_{i(r)}$ in $\{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_k\}$ ($r \leq n$) such that $\boldsymbol{\alpha} \in m\mathrm{conv}(\boldsymbol{\beta}_{i(1)}, \ldots, \boldsymbol{\beta}_{i(r)}) + \mathbb{R}^n_{\geq}$.*

Proof. (a) Obviously $\mathrm{NP}(J) \subseteq \{\boldsymbol{\alpha} \in \mathbb{R}^n_{\geq} \mid \boldsymbol{\alpha} \geq_{pr} \sum_{i=1}^k c_i \boldsymbol{\beta}_i$, for some $c_i \geq 0, \sum c_i = m\}$ and the other sets mentioned in the lemma lie in between so it suffices to prove that $\{\boldsymbol{\alpha} \in \mathbb{R}^n_{\geq} \mid \boldsymbol{\alpha} \geq_{pr} \sum_{i=1}^k c_i \boldsymbol{\beta}_i, c_i \geq 0, \sum c_i = m\} \subseteq \mathrm{NP}(J)$. If $\boldsymbol{\beta} = \sum_{i=1}^k c_i \boldsymbol{\beta}_i + \boldsymbol{\gamma}$ with $c_i \geq 0$, $\sum c_i = m$ and $\boldsymbol{\gamma} \in \mathbb{R}^n_{\geq}$ then $\boldsymbol{\beta} = \sum_{i=1}^k (c_i/m)(m\boldsymbol{\beta}_i) + \boldsymbol{\gamma}$ with $\sum_{i=1}^k c_i/m = 1$, $m\boldsymbol{\beta}_i \in \mathrm{NP}(J)$ and $\boldsymbol{\gamma} \in \mathbb{R}^n_{\geq}$ so $\boldsymbol{\beta} \in \mathrm{NP}(J)$.

(b) If $\boldsymbol{\alpha} \in \mathrm{NP}(I^m)$ then by (a) $\boldsymbol{\alpha} = \sum_{i=1}^k c_i\boldsymbol{\beta}_i + \boldsymbol{\gamma} = m\sum_{i=1}^k (c_i/m)\boldsymbol{\beta}_i + \boldsymbol{\gamma}$ where $c_i \geq 0$, $\sum c_i = m$ (so that $\sum(c_i/m) = 1$) and $\boldsymbol{\gamma} \in \mathbb{R}^n_{\geq}$. By Carathéodory's Theorem $\boldsymbol{\delta} := \sum_{i=1}^k (c_i/m)\boldsymbol{\beta}_i$ is in the interior of $\mathrm{conv}(\boldsymbol{\beta}_{i(1)}, \ldots, \boldsymbol{\beta}_{i(r)})$ where $\{\boldsymbol{\beta}_{i(1)}, \ldots, \boldsymbol{\beta}_{i(r)}\}$ is an affinely independent subset of $\{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_k\}$ (so that $r \leq n+1$). If $r \leq n$ we are done. Otherwise $r = n+1$ and there exists $t > 0$ such that $\boldsymbol{\delta} - t\mathbf{e}_1 \in \mathrm{conv}(\boldsymbol{\beta}_{i(1)}, \ldots, \widehat{\boldsymbol{\beta}_{i(j)}}, \ldots, \boldsymbol{\beta}_{i(n+1)})$ for some $1 \leq j \leq n+1$. Thus we may write $m\boldsymbol{\delta} = \boldsymbol{\rho} + \boldsymbol{\nu}$, where $\boldsymbol{\rho} \in m\,\mathrm{conv}(\boldsymbol{\beta}_{i(1)}, \ldots, \widehat{\boldsymbol{\beta}_{i(j)}}, \ldots, \boldsymbol{\beta}_{i(n+1)})$ and $\boldsymbol{\nu} \in \mathbb{R}^n_{\geq}$. Thus $\boldsymbol{\alpha} = \boldsymbol{\rho} + (\boldsymbol{\nu} + \boldsymbol{\gamma}) \in m\,\mathrm{conv}(\boldsymbol{\beta}_{i(1)}, \ldots, \widehat{\boldsymbol{\beta}_{i(j)}}, \ldots, \boldsymbol{\beta}_{i(n+1)}) + \mathbb{R}^n_{\geq}$. □

Remark 2.6. It follows from the above discussion that the normalization $\overline{R[I(\boldsymbol{\lambda})t]}$ of $R[I(\boldsymbol{\lambda})t]$ is the subalgebra of $R[t] = K[x_1, \ldots, x_n, t]$ generated by all $x^{\boldsymbol{\alpha}}t^d$ where $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$, $a_i, d \in \mathbb{N}$, and

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_n}{\lambda_n} \geq d.$$

On the other hand $\overline{K[S(\boldsymbol{\lambda})]}$ is isomorphic to the subalgebra of $K[x_1, \ldots, x_n, t]$ generated by all $x^{\boldsymbol{\alpha}}t^d$ where $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$, $a_i, d \in \mathbb{N}$, and

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_n}{\lambda_n} \leq d.$$

A crucial difference between the two cases is that $\overline{R[I(\boldsymbol{\lambda})t]}$ contains $x_1, \ldots, x_n$ but not t whereas $\overline{K[S(\boldsymbol{\lambda})]}$ contains t but not $x_1, \ldots, x_n$.

3. First Results on Normal Ideals

In this section we obtain our first new result, namely that a monomial ideal I in $R = K[x_1, \ldots, x_n]$ is normal if and only if the powers I^m for $1 \leq m \leq n-1$ are integrally closed. For the case $n = 2$ this follows from the celebrated theorem of Zariski [12, Appendix 5] that asserts that the product of integrally closed ideals in a 2-dimensional regular ring is again integrally closed. Then we obtain a number of results on the normality of the ideal $A_{\geq d}$ of all elements of degree $\geq d$ in the $\mathbb{N}$-graded ring A. Note that the ideal $I(\boldsymbol{\lambda})$ is of this form for a suitable grading on R.

Proposition 3.1. *Let $I \subseteq R = K[x_1, \ldots, x_n]$ be a monomial ideal. If I^m is integrally closed for $m = 1, \ldots, n-1$, then I is normal.*

Proof. By Lemma 2.3 part(a) it suffices to show that if $\Gamma(I^m) = \mathrm{NP}(I^m) \cap \mathbb{N}^n$ for $m = 1, \ldots, n-1$, then $\Gamma(I^m) = \mathrm{NP}(I^m) \cap \mathbb{N}^n$ for all $m \geq 1$.

Let $m \geq n$ and assume that $\Gamma(I^i) = \mathrm{NP}(I^i) \cap \mathbb{N}^n$ for all i, $1 \leq i \leq m-1$. Clearly $\Gamma(I^m) \subseteq \mathrm{NP}(I^m) \cap \mathbb{N}^n$. Suppose that $\boldsymbol{\gamma} \in \mathrm{NP}(I^m) \cap \mathbb{N}^n$. By Lemma 2.5 part (b), $\boldsymbol{\gamma} \in m\,\mathrm{conv}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r) + \mathbb{R}^n_{\geq} \subseteq \mathrm{NP}(I)$ where $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r \in \Gamma(I)$ are affinely independent and we may assume that $r \leq n$. Write $\boldsymbol{\gamma} = \sum_{i=1}^r c_i\boldsymbol{\beta}_i + \boldsymbol{\nu}$, where each $c_i \geq 0$, $\sum c_i = m$, and $\boldsymbol{\nu} \geq_{pr} 0$. Since $m \geq n \geq r$, some $c_i \geq 1$. We may and shall assume that $c_1 \geq 1$. Then, $\boldsymbol{\gamma} - \boldsymbol{\beta}_1 = \sum_{i=1}^r (c_i - \delta_{i1})\boldsymbol{\beta}_i + \boldsymbol{\nu}$ and $\sum_{i=1}^r (c_i - \delta_{i1}) = m - 1$. By the induction hypothesis, $\boldsymbol{\gamma} - \boldsymbol{\beta}_1 \in \Gamma(I^{m-1})$ and hence $\boldsymbol{\gamma} \in \Gamma(I^m)$. □

Remark 3.2. The special case of Proposition 3.1 when I is integral over the subideal generated by all monomials of the least total degree follows from [3, Theorem 3.3].

Another useful observation is the following.

Lemma 3.3. *Let A be a normal integral domain, $I \subseteq A$ be an ideal, and let $J = aI$, for $a \in A$. Then, the following hold.*

(a) *I is integrally closed if and only if J is integrally closed.*
(b) *I is normal if and only if J is normal.*

Proof. Notice that an element $x \in A$ is integral over J if and only if $x/a \in A$ and is integral over I. Part (a) and part (b) follow immediately. □

Notation 3.4. *For an $\mathbb{N}$-graded ring A and a positive integer m, we let $A_{\geq m}$ denote the homogeneous ideal defined by $A_{\geq m} = \bigoplus_{\ell \geq m} A_\ell$.*

Lemma 3.5 and Proposition 3.7 below generalize recent results of Faridi [7, Lemma and Theorem 3].

Lemma 3.5. *Let A be an $\mathbb{N}$-graded ring generated over A_0 by homogeneous elements $x_1, \ldots, x_n$ of positive degrees $\omega_1, \ldots, \omega_n$ and $w = \operatorname{lcm}(\omega_1, \ldots, \omega_n)$. Consider the ideal $I = A_{\geq kw}$ for a positive integer k. If $I^p = A_{\geq pkw}$ for $1 \leq p \leq \frac{n-2}{k} + 1$, then $I^p = A_{\geq pkw}$ for all $p \geq 1$. In particular, if $k \geq n - 1$, then $I^p = A_{\geq pkw}$ for all $p \geq 1$.*

Proof. We proceed by induction on $p \geq 1$, the case $p \leq \frac{n-2}{k} + 1$ being a priori true.

Suppose that $p > \frac{n-2}{k} + 1$ and that $I^{p-1} = A_{\geq (p-1)kw}$. Let $\boldsymbol{\mu} = x_1^{c_1} \ldots x_n^{c_n}$ be a monomial of degree at least pkw. We must show that $\boldsymbol{\mu} \in I^p$.

Set $\lambda_i = w/\omega_i$ $(i = 1, \ldots, n)$ and let $q_i = \lfloor c_i/\lambda_i \rfloor$ $(i = 1, \ldots, n)$. Then

$$\begin{aligned} pkw \leq \deg(\boldsymbol{\mu}) &= \sum_{i=1}^n c_i\omega_i \\ &< \sum_{i=1}^n (q_i + 1)\lambda_i\omega_i \\ &= \left(\sum_{i=1}^n q_i\right) w + nw. \end{aligned}$$

This implies that $\sum q_i \geq pk - n + 1$. Since we assumed that $p > \frac{n-2}{k} + 1$, we have $(p-1)k \geq n-1$ and hence $pk - n + 1 \geq k$. Thus $\sum q_i \geq k$.

Choose integers $0 \leq s_i \leq q_i$ $(i = 1, \ldots, n)$ with $s_1 + \cdots + s_n = k$. Then, $\boldsymbol{\mu} = (x_1^{s_1\lambda_1} \ldots x_n^{s_n\lambda_n})\boldsymbol{\nu}$ where $\deg \boldsymbol{\nu} = \deg(\boldsymbol{\mu}) - (s_1 + \cdots + s_n)w = \deg(\boldsymbol{\mu}) - kw \geq (p-1)kw$. Thus, $\boldsymbol{\nu} \in I^{p-1}$ by the induction hypothesis and $\boldsymbol{\mu} \in I^p$, as desired. Since the inclusion $I^p \subseteq A_{\geq pkw}$ is immediate, the assertion is proven. $\square$

We suspect that the following result is well known but we do not know a reference so we provide a brief proof. We point out that this result holds if $\mathbb{N}$ is replaced by any totally ordered abelian group G and we assume that A is positively graded. The proof goes through without any changes.

Lemma 3.6. *Let A be a reduced $\mathbb{N}$-graded ring and let $I = A_{\geq d}$ for some positive integer d. Then, I is an integrally closed ideal.*

Proof. Assume that $x \in A$ is integral over I so that $x^n + a_1x^{n-1} + \cdots + a_n = 0$, for some $n \geq 1$ and $a_k \in I^k$ $(k = 1, \ldots, n)$. Just suppose that the smallest component $x(i)$ of x has degree $i < d$. Since $a_k \in I^k \subseteq A_{\geq kd}$, the smallest component of a_kx^{n-k} has degree strictly greater than ni for $k = 1, \ldots, n$. Hence we must have $x(i)^n = 0$, contradicting the assumption that A is reduced. Thus $x \in I$. $\square$

Proposition 3.7. *Let A be a reduced $\mathbb{N}$-graded ring generated over A_0 by homogeneous elements $x_1, \ldots, x_n$ of positive degrees $\omega_1, \ldots, \omega_n$ and $w = \operatorname{lcm}(\omega_1, \ldots, \omega_n)$. Consider the ideal $I = A_{\geq kw}$ for a positive integer k. If $I^p = A_{\geq pkw}$ for $1 \leq p \leq \frac{n-2}{k} + 1$, then I is a normal ideal. In particular, if $k \geq n-1$, then I is a normal ideal. In this case, if A is a normal domain, then the Rees ring $A[It]$ is again a normal domain.*

Proof. This follows immediately from Lemmas 3.5 and 3.6. $\square$

Remark 3.8. Notice that for $I = A_{\geq kw} \subseteq A$ as in Lemma 3.5, we always have $\overline{I^p} = A_{\geq pkw}$. The containment $\overline{I^p} \subseteq A_{\geq pkw}$ follows from Lemma 3.6. To see the opposite containment, suppose that $x^{\boldsymbol{\gamma}} \in A_{\geq pkw}$, where $\boldsymbol{\gamma} = (c_1, \ldots, c_n)$. Then, $(x^{\boldsymbol{\gamma}})^{kw} = \prod x_i^{k\lambda_i\omega_i c_i} = \prod (x_i^{k\lambda_i})^{\omega_i c_i} \in I^{\omega_1 c_1} \cdots I^{\omega_n c_n} = I^{\boldsymbol{\omega}\cdot\boldsymbol{\gamma}}$. However $x^{\boldsymbol{\gamma}} \in A_{\geq pkw}$ implies $\boldsymbol{\omega} \cdot \boldsymbol{\gamma} \geq pkw$ so that $(x^{\boldsymbol{\gamma}})^{kw} \in (I^p)^{kw}$ and we are done.

4. $\mathfrak{m}$-Primary Monomial Ideals

Conventions. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ denote the maximal homogeneous ideal of $R = K[x_1, \ldots, x_n]$. Furthermore let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, $J(\boldsymbol{\lambda}) = (x_1^{\lambda_1}, \ldots, x_n^{\lambda_n})$, and $I(\boldsymbol{\lambda}) = \overline{J(\boldsymbol{\lambda})}$, as in the Introduction. We ask when the integrally closed $\mathfrak{m}$-primary monomial ideal $I(\boldsymbol{\lambda})$ of R is normal.

Notation 4.1. *Let $L = \operatorname{lcm}(\lambda_1, \ldots, \lambda_n)$, $\omega_i = L/\lambda_i$, $1/\boldsymbol{\lambda} = (1/\lambda_1, \ldots, 1/\lambda_n)$ and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$, so that $L/\boldsymbol{\lambda} = \boldsymbol{\omega}$. We will denote $\Gamma(I(\boldsymbol{\lambda}))$ (Definition 2.1) simply by Γ.*

Observe that $\mathrm{NP}(I(\boldsymbol{\lambda})) = \mathrm{NP}(J(\boldsymbol{\lambda}))$ has one bounded facet with vertices $\lambda_1\mathbf{e}_1, \ldots, \lambda_n\mathbf{e}_n$. For $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$ the hyperplane $(1/\boldsymbol{\lambda}) \cdot \boldsymbol{\alpha} = a_1/\lambda_1 + \cdots + a_n/\lambda_n = 1$ passes through these vertices, and upon multiplication by L, the equation of this hyperplane becomes $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = L$. This explains the lemma below.

Lemma 4.2. $\Gamma = \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid (1/\boldsymbol{\lambda}) \cdot \boldsymbol{\alpha} \geq 1\} = \{\boldsymbol{\alpha} \in \mathbb{N}^n \mid \boldsymbol{\omega} \cdot \boldsymbol{\alpha} \geq L\}$.

Assigning $\deg(x_i) = \omega_i$ $(i = 1, \ldots, n)$ we now have $I(\boldsymbol{\lambda}) = R_{\geq L}$. Furthermore, the following gives necessary and sufficient conditions for $I(\boldsymbol{\lambda})$ to be normal.

Lemma 4.3. *For the ideal $I(\boldsymbol{\lambda})$ defined above the following are equivalent.*

(a) *$I(\boldsymbol{\lambda})$ is normal.*
(b) *Whenever $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} \geq pL$ for $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $p \in \mathbb{N}$, there exist vectors $\boldsymbol{\beta}_j \in \Gamma$ $(j = 1, \ldots, p)$ such that $\boldsymbol{\alpha} = \sum \boldsymbol{\beta}_j$.*
(c) *Whenever $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} \geq pL$ for $\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ with $\lambda_i > a_i$ and $1 \leq p < n$, there exist vectors $\boldsymbol{\beta}_j \in \Gamma$ $(j = 1, \ldots, p)$ such that $\boldsymbol{\alpha} = \sum \boldsymbol{\beta}_j$.*

Proof. The equivalence of (a) and (b) is an immediate consequence of Lemma 2.5 and Theorem 2.3. Clearly (b) implies (c) so it remains only to show that (c) implies (b).

Suppose (c) holds. We need only verify condition (b) for $2 \leq p < n$ by Lemma 3.1 and the observation that (b) automatically holds for $p = 1$. We argue by decreasing induction on p. Assume that $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} \geq pL$ for $\boldsymbol{\alpha} \in \mathbb{N}^n$. If $a_i < \lambda_i$ for all i we can apply (c) directly. If $a_i \geq \lambda_i$ for some $i \in \{1, \ldots, n\}$ (we may assume that $a_1 \geq \lambda_1$) then $\boldsymbol{\alpha} = (\boldsymbol{\alpha} - \lambda_1\mathbf{e}_1) + \lambda_1\mathbf{e}_1$. Dotting with $\boldsymbol{\omega}$ we obtain $\boldsymbol{\omega} \cdot ((\boldsymbol{\alpha} - \lambda_1\mathbf{e}_1) + \lambda_1\mathbf{e}_1) \geq pL$, which implies $\boldsymbol{\omega} \cdot (\boldsymbol{\alpha} - \lambda_1\mathbf{e}_1) \geq (p-1)L$ (since $\boldsymbol{\omega} \cdot \lambda_1\mathbf{e}_1 = L$). By induction, there exist vectors $\boldsymbol{\beta}_j \in \Gamma$ $(j = 1, \ldots, p-1)$ with $\boldsymbol{\alpha} - \lambda_1\mathbf{e}_1 = \sum \boldsymbol{\beta}_j$. Thus $\boldsymbol{\alpha} = \sum \boldsymbol{\beta}_j + \lambda_1\mathbf{e}_1$ and condition (b) is satisfied. □

Due to this characterization, we will say that Γ is *normal* if either condition (b) or (c) above holds (so that Γ is normal if and only if $I(\boldsymbol{\lambda})$ is normal).

To put this section into context with the preceding section notice that if $w = \mathrm{lcm}(\omega_1, \ldots, \omega_n)$ and $d = \gcd(\lambda_1, \ldots, \lambda_n)$ then $L = dw$ (this equality is easily checked by showing that any prime number p has the same exponent in L and in dw) so that $I(\boldsymbol{\lambda}) = R_{\geq dw}$. From this point of view we obtain the following corollary.

Corollary 4.4. *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n$, $n \geq 3$, and suppose that $\gcd(\lambda_1, \ldots, \lambda_n) > n - 2$. Then the monomial ideal $I(\boldsymbol{\lambda}) \subseteq K[x_1, \ldots, x_n]$ is normal.*

Proof. This is an immediate consequence of Proposition 3.7 since, in the notation of that result, $k \geq n-1$. □

In [1] Bruns-Gubeladze define a submonoid S of $\mathbb{Q}_{\geq}$ to be *1-normal* if whenever $x \in S$ and $x \leq p$ for some $p \in \mathbb{N}$, there exist rational numbers $y_1, \dots, y_p$ in S with $y_i \leq 1$ for all i such that $x = y_1 + \cdots + y_p$. Then they relate the normality of $K[S(\boldsymbol{\lambda})]$ to the 1-normality of the submonoid Λ (defined below) of $\mathbb{Q}_{\geq}$. We modify this program as follows.

Definition 4.5. A submonoid S of $\mathbb{Q}_{\geq}$ is *quasinormal* provided that whenever $x \in S$ and $x \geq p$ for some $p \in \mathbb{N}$, there exist rational numbers $y_1, \dots, y_p$ in S with $y_i \geq 1$ for all i such that $x = y_1 + \cdots + y_p$.

We now have the following.

Lemma 4.6. *Let $\Lambda = \langle 1/\lambda_1, \dots, 1/\lambda_n \rangle$, the additive submonoid of $\mathbb{Q}_{\geq}$ generated by $1/\lambda_1, \dots, 1/\lambda_n$, and $\Lambda_{\geq 1} = \{x \in \Lambda \mid x \geq 1\}$. If $I(\boldsymbol{\lambda})$ is normal then Λ is quasinormal.*

Proof. Suppose $I(\boldsymbol{\lambda})$ is normal and $x \in \Lambda,\ x \geq p$. Then $x = (1/\boldsymbol{\lambda}) \cdot \boldsymbol{\alpha}$ for $\boldsymbol{\alpha} \in \mathbb{N}^n$. As noted in Lemma 4.2, $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} \geq pL$. Therefore by Lemma 4.3 there exist vectors $\boldsymbol{\beta}_i \in \Gamma,\ 1 \leq i \leq p$, so that $\boldsymbol{\alpha} = \boldsymbol{\beta}_1 + \cdots + \boldsymbol{\beta}_p$. Thus $x = (1/\boldsymbol{\lambda}) \cdot \boldsymbol{\alpha} = (1/\boldsymbol{\lambda}) \cdot \boldsymbol{\beta}_1 + \cdots + (1/\boldsymbol{\lambda}) \cdot \boldsymbol{\beta}_p$. Again by the description of Γ in Lemma 4.2, $(1/\boldsymbol{\lambda}) \cdot \boldsymbol{\beta}_i \in \Lambda_{\geq 1}$. Hence Λ is quasinormal. □

When we assume that the integers $\lambda_1, \dots, \lambda_n$ are pairwise relatively prime the converse is true. So in this special case, the normality condition on the n-dimensional monoid Γ is reduced to the quasinormality condition on the 1-dimensional monoid Λ.

Proposition 4.7. *Suppose that $\lambda_1, \dots, \lambda_n$ are pairwise relatively prime positive integers and let Λ be as in Lemma 4.6. With notation and assumptions as above, $I(\boldsymbol{\lambda})$ is normal if and only if Λ is quasinormal.*

Proof. By Lemma 4.6 it suffices to show that if Λ is quasinormal then $I(\boldsymbol{\lambda})$ is normal. We assume that Λ is quasinormal and establish the criterion for normality of $I(\boldsymbol{\lambda})$ in Lemma 4.3(c).

First, as in [1, Proposition 1.3], we consider the natural surjection

$$\pi : \mathbb{Z}^n \to \operatorname{grp}(\Lambda),$$

defined by $\mathbf{e}_i \mapsto 1/\lambda_i$ $(i = 1, \dots, n)$, where $\operatorname{grp}(\Lambda)$ is the subgroup of $\mathbb{Q}$ generated by Λ. Suppose $(a_1, \dots, a_n) \in \ker(\pi)$. Clearing denominators in the equation $a_1/\lambda_1 + \cdots + a_n/\lambda_n = 0$ and using that the λ_i are pairwise relatively prime we observe that λ_i divides a_i. In particular any nonzero element of $\ker(\pi)$ has i^{th} coordinate greater than or equal to λ_i for some i.

Suppose that $\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ satisfies $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} \geq pL$, with $\boldsymbol{\alpha} <_{pr} \boldsymbol{\lambda}$, as in the hypotheses of Lemma 4.3(c). We have

$$x := (1/\boldsymbol{\lambda}) \cdot \boldsymbol{\alpha} = a_1/\lambda_1 + \cdots + a_n/\lambda_n \geq p.$$

Since Λ is quasinormal there exist numbers $y_i \in \Lambda, y_i \geq 1\,(i = 1, \ldots, p)$ such that $x = y_1 + \cdots + y_p$. Write $y_i = (1/\boldsymbol{\lambda}) \cdot \boldsymbol{\beta}_i$ with $\boldsymbol{\beta}_i \in \mathbb{N}^n$ $(i = 1, \ldots, p)$. By Lemma 4.2, $\boldsymbol{\beta}_i \in \Gamma$ for all i. Then $\boldsymbol{\beta} := \boldsymbol{\beta}_1 + \cdots + \boldsymbol{\beta}_p \in \boldsymbol{\alpha} + \ker(\pi)$. By the above discussion of $\ker(\pi)$, $\boldsymbol{\alpha}$ is the only element in its coset with nonnegative coordinates. Hence $\boldsymbol{\alpha} = \boldsymbol{\beta}$, which completes the proof. □

Proposition 4.8. *Let $\boldsymbol{\lambda} \in \mathbb{Z}_+^n$. If the monoid $\Lambda = \langle 1/\lambda_1, \ldots, 1/\lambda_n \rangle$ is quasinormal, then $1 + 1/L \in \Lambda$.*

Proof. Assume that Λ is quasinormal. Notice that $\omega_1, \ldots, \omega_n$ are relatively prime and 1 is the smallest positive integer in $\operatorname{grp}(\omega_1, \ldots, \omega_n)$. Hence $1/L$ is the smallest positive number in $\operatorname{grp}(\Lambda)$. Choose an integer $N \gg 0$ in Λ such that $N + 1/L \in \Lambda$. Write $N + 1/L = (1/\boldsymbol{\lambda}) \cdot \boldsymbol{\beta}$, where $\boldsymbol{\beta} \in \mathbb{N}^n$. Since Λ is quasinormal, $N + 1/L = y_1 + \cdots + y_N$, where $y_i \in \Lambda$, $y_i \geq 1$, $(i = 1, \ldots, N)$.

We claim that if $x \in \Lambda$ and $x > 1$, then $x \geq 1 + 1/L$. Now suppose that $1 < x \in \Lambda$. Since $1 = \lambda_1(1/\lambda_1) \in \Lambda$, $x - 1 \in \operatorname{grp}(\Lambda)$. Furthermore $x - 1 > 0$ implies $x - 1 \geq 1/L$. Thus $y_1 + \cdots + y_N = N + 1/L$ forces $N - 1$ of the y_i to be one and the remaining to be $1 + 1/L$. In particular, $1 + 1/L \in \Lambda$. □

Multiplying by L we obtain the following version of the corollary.

Proposition 4.9. *Let $\boldsymbol{\lambda} \in \mathbb{Z}_+^n$. If $\Lambda = \langle 1/\lambda_1, \ldots, 1/\lambda_n \rangle$ is quasinormal, then $L + 1 \in \langle \omega_1, \ldots, \omega_n \rangle$.*

5. Normality of $I(\boldsymbol{\lambda})$ and $I(\boldsymbol{\lambda}')$

In this section we discussion the relationship between the normality of $I(\boldsymbol{\lambda})$ and that of $I(\boldsymbol{\lambda}')$. Our notation continues as usual: $R = K[x_1, \ldots, x_n]$ for a field K, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ for arbitrary positive integers λ_j, $L = \operatorname{lcm}(\lambda_1, \ldots, \lambda_n)$, and $\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \ell, \lambda_{i+1} \ldots, \lambda_n)$, where $\ell = \operatorname{lcm}(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_n)$. There is no loss of generality in taking $i = n$, so that $\ell = \operatorname{lcm}(\lambda_1, \ldots, \lambda_{n-1})$ and $\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + \ell)$. We now state our main result.

Theorem 5.1. *If $I(\boldsymbol{\lambda}')$ is normal then $I(\boldsymbol{\lambda})$ is normal. If $\lambda_n \geq \ell$ and $I(\boldsymbol{\lambda})$ is normal so is $I(\boldsymbol{\lambda}')$.*

Before beginning the proof of Theorem 5.1 we will give an example to show that Question 1.3 has a negative answer.

Example 5.2. The ideal $I = I(2,3,7) = \overline{(x^2, y^3, z^7)} \subseteq K[x, y, z]$ is not normal. In this case, $L = 42$ and $L + 1 = 43$ is not in the monoid generated by $\omega_1 = 21, \omega_2 = 14$, and $\omega_3 = 6$. Hence the monoid Λ is not quasinormal, which implies, by Lemma 4.6 that the ideal I is not normal. Alternatively, $\boldsymbol{\alpha} = (1, 2, 6)$ satisfies $(1/\boldsymbol{\lambda}) \cdot \boldsymbol{\alpha} \geq 2$ but $\boldsymbol{\alpha}$ is not the sum of two elements of $\Gamma(I)$. Thus, by the discussion of section 2, xy^2z^6 is integral over I^2 but not in I^2. Hence I^2 is not integrally closed, so I is not normal. However by [1, Theorem 1.6], the ring $K[S(2, 3, 7)]$ is normal because the ring $K[S(2, 3, 1)]$ is normal. Thus $\boldsymbol{\lambda} = (2, 3, 7)$ is an example where $K[S(\boldsymbol{\lambda})]$ is normal but $I(\boldsymbol{\lambda})$ is not. Also $I(2, 3, 1)$ is normal, so we also have a counterexample to Question 1.2. In the other direction $K[S(2, 3, 5)]$ is not normal but $I(2, 3, 5)$ is normal.

Our method of proof of Theorem 5.1 is to compare the minimal generators of the integral closures of the Rees algebras $R[I(\boldsymbol{\lambda})t]$ and $R[I(\boldsymbol{\lambda}')t]$. The integral closure $\overline{R[I(\boldsymbol{\lambda})t]}$ of $R[I(\boldsymbol{\lambda})t]$, by Remark 2.6, is the subalgebra of $R[t] = K[x_1, \ldots, x_n, t]$ generated by all $x^{\boldsymbol{\alpha}}t^d$ where $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$, such that $a_i, d \in \mathbb{N}$ and

$$\text{(A)} \qquad \frac{a_1}{\lambda_1} + \cdots + \frac{a_n}{\lambda_n} \geq d.$$

The algebra $\overline{R[I(\boldsymbol{\lambda})t]}$ has a unique (finite) minimal set of monomial generators, corresponding to exponent vectors $(a_1, \cdots, a_n, d) \in \mathbb{N}^{n+1}$ of the following types:

(1) The "trivial" exponent vectors

$$\begin{matrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{matrix}$$

corresponding to $x_i \in R \subset \overline{R[I(\boldsymbol{\lambda})t]}$.

(2) Exponent vectors of the form

$$\begin{matrix} \lambda_1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 1 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & \lambda_n & 1 \end{matrix}$$

corresponding to $x_i^{\lambda_i}t \in I(\boldsymbol{\lambda})t \subset \overline{R[I(\boldsymbol{\lambda})t]}$.

(3) Exponent vectors

$$(a_1, \ldots, a_{n-1}, 0, d)$$

with $d > 0$ and $a_ia_j > 0$ for some $0 < i < j < n$.

(4) Exponent vectors

$$(a_1, \ldots, a_n, d)$$

with $d > 0$ and $a_ia_n > 0$ for some $0 < i < n$.

The exponent vectors of type (2) have been written down separately because they are the initial data of the problem. The condition that an exponent vector correspond to a minimal generator is that it cannot be written as the sum of two nonzero vectors satisfying condition (A).

In the sequel we will informally refer to the exponent vectors themselves as being generators of $\overline{R[I(\boldsymbol{\lambda})t]}$. In this language generators of types (1) and (2) are obviously minimal. The condition that $(a_1, \ldots, a_n, d)$ of type (3) or (4) be minimal is that in addition to satisfying condition (A) it also satisfy:

(B) If any one of $a_1, ..., a_n$ which is greater than 0 is decreased, then inequality (A) fails to hold, and

(C) If $d > 1$ then $(a_1, \ldots, a_n, d)$ cannot be written in the form $(a_1, \ldots, a_n, d) = (b_1, \ldots, b_n, d_1) + (c_1, \ldots, c_n, d_2)$ with $0 < d_1, d_2 < d$, where $(b_1, \ldots, b_n, d_1)$ and $(c_1, \ldots, c_n, d_2)$ both satisfy (A).

Condition (B) says that $(a_1, \ldots, a_n, d)$ cannot be written as the sum of a vector of type (1) and another vector satisfying (A). Condition (C) says that $(a_1, \ldots, a_n, d)$ cannot be written as the sum of two vectors of types (2), (3), or (4).

In this context Theorem 2.4 can be restated as follows.

Lemma 5.3. *$I(\boldsymbol{\lambda})$ is normal if and only if the minimal generators of types (3) and (4) all have $d = 1$.*

The lemma below will be useful.

Lemma 5.4. *In any minimal generator of type (3) or (4) we have $0 \leq a_i < \lambda_i$ for all i.*

Proof. We argue by contradiction. Let $(a_1, \ldots, a_n, d)$ be a minimal generator of type (3) or (4) with $a_i \geq \lambda_i$ for some i. We may suppose without loss of generality that $a_1 \geq \lambda_1$. From condition (A) we must have $d \geq 1$. If we subtract the equality

$$\frac{\lambda_1}{\lambda_1} = 1$$

from the inequality (A) we obtain

$$\frac{a_1 - \lambda_1}{\lambda_1} + \frac{a_2}{\lambda_2} + \cdots + \frac{a_n}{\lambda_n} \geq d - 1,$$

that is, $(a_1 - \lambda_1, a_2, \ldots, a_n, d - 1)$ satisfies condition (A) and $(a_1 - \lambda_1, a_2, \ldots, a_n, d - 1) + (\lambda_1, 0, \ldots, 0, 1) = (a_1, \ldots, a_n, d)$ so $(a_1, \ldots, a_n, d)$ is not minimal, a contradiction. ☐

Corollary 5.5. *For any minimal generator $(a_1, \ldots, a_n, d)$ we have $d < n$.*

Corollary 5.5 reproves Proposition 3.1, but only in the special case of $I(\boldsymbol{\lambda})$.

We now compare the minimal generators of $\overline{R[I(\boldsymbol{\lambda})t]}$ and $\overline{R[I(\boldsymbol{\lambda}')t]}$. There is obviously a bijection between minimal generators of types (1), (2), and (3) for these two algebras. If $a_n = 0$ then $(a_1, \ldots, a_n, d)$ corresponds to itself, as does $(0, 0, \ldots, 1, 0)$, and the generator $(0, \ldots, 0, \lambda_n, 1)$ for $\overline{R[I(\boldsymbol{\lambda})t]}$ corresponds to the generator $(0, \ldots, 0, \lambda_n + \ell, 1)$ for $\overline{R[I(\boldsymbol{\lambda}')t]}$. Now let $(a_1, \ldots, a_n, d)$ be a minimal generator of $\overline{R[I(\boldsymbol{\lambda})t]}$ of type (4), i.e., with $a_n > 0, d > 0$. Then, by condition (B), a_n is the smallest integer such that (for fixed $a_1, \ldots, a_{n-1}, d$)

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_n}{\lambda_n} \geq d.$$

Now define a_n' to be the smallest integer such that

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a_n'}{\lambda_n + \ell} \geq d,$$

so that $(a_1, \ldots, a_{n-1}, a_n', d)$ is the exponent vector of an element of $\overline{R[I(\boldsymbol{\lambda}')t]}$.

Proposition 5.6. *Let $(a_1, \ldots, a_n, d)$ be a minimal generator of $\overline{R[I(\boldsymbol{\lambda})t]}$ of type (4), and let a_n' be as defined above. Then $a_n' = a_n + d\ell - \frac{\ell}{\lambda_1} a_1 - \cdots - \frac{\ell}{\lambda_{n-1}} a_{n-1}$ and $(a_1, \ldots, a_{n-1}, a_n', d)$ is a minimal generator of $\overline{R[I(\boldsymbol{\lambda}')t]}$ of type (4).*

Proof. Let δ be any integer, and consider the following chain of equivalent inequalities:

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a_n + \delta}{\lambda_n + \ell} \geq d \Leftrightarrow$$

$$\ell\frac{a_1}{\lambda_1} + \cdots + \ell\frac{a_{n-1}}{\lambda_{n-1}} + \frac{\ell(a_n + \delta)}{\lambda_n + \ell} \geq d\ell \Leftrightarrow$$

$$\frac{\ell(a_n + \delta)}{\lambda_n + \ell} \geq d\ell - \frac{\ell}{\lambda_1} a_1 - \cdots - \frac{\ell}{\lambda_{n-1}} a_{n-1} \Leftrightarrow$$

$$\ell a_n + \ell\delta \geq (\lambda_n + \ell)(d\ell - \frac{\ell}{\lambda_1} a_1 - \cdots - \frac{\ell}{\lambda_{n-1}} a_{n-1}).$$

In the rest of this proof we will set

$$\delta = d\ell - \frac{\ell}{\lambda_1} a_1 - \cdots - \frac{\ell}{\lambda_{n-1}} a_{n-1}$$

(which is an integer by the definition of ℓ). Then the last inequality becomes

$$\ell a_n \geq \lambda_n (d\ell - \frac{\ell}{\lambda_1} a_1 - \cdots - \frac{\ell}{\lambda_{n-1}} a_{n-1}),$$

which is equivalent to

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a_n}{\lambda_n} \geq d.$$

Putting these equivalences together we conclude that for *any* integers $a_1, \dots, a_n$ and

$$\delta = d\ell - \frac{\ell}{\lambda_1}a_1 - \cdots - \frac{\ell}{\lambda_{n-1}}a_{n-1}$$

we have

$$(*) \qquad \frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a_n + \delta}{\lambda_n + \ell} \geq d \Leftrightarrow \frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a_n}{\lambda_n} \geq d.$$

If

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a_n + \delta - 1}{\lambda_n + \ell} \geq d$$

then we have, by $(*)$ applied to $a_1, \dots, a_{n-1}, a_n - 1$, that

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a_n - 1}{\lambda_n} \geq d$$

which contradicts condition (B) for the minimality of $(a_1, \dots, a_n, d)$ as a generator of $\overline{R[I(\boldsymbol{\lambda})t]}$. Thus we have $a'_n = a_n + d\ell - \frac{\ell}{\lambda_1}a_1 - \cdots - \frac{\ell}{\lambda_{n-1}}a_{n-1}$ as claimed. By construction $(a_1, \dots, a_{n-1}, a'_n, d)$ satisfies condition (A) for minimality as a generator of $\overline{R[I(\boldsymbol{\lambda}')t]}$. If in $(a_1, \dots, a_{n-1}, a'_n, d)$ we replace a'_n by $a'_n - 1$ then condition (A) fails to hold (by the definition of a'_n). Because a'_n has the largest denominator in the inequality (A), decreasing any of $a_1, \dots, a_{n-1}$ will also violate condition (A). Therefore $(a_1, \dots, a_{n-1}, a'_n, d)$ also satisfies condition (B) for minimality as a generator of $\overline{R[I(\boldsymbol{\lambda}')t]}$.
Now we consider condition (C) for minimality of $(a_1, \dots, a_{n-1}, a'_n, d)$ as a generator of $\overline{R[I(\boldsymbol{\lambda}')t]}$. Note that since $(a_1, \dots, a_n, d)$ is a minimal generator of $\overline{R[I(\boldsymbol{\lambda})t]}$ with $a_n > 0$ we must have

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} < d$$

and hence that $\delta > 0$. Furthermore δ is linear in $a_1, \dots, a_{n-1}$, and d. Hence we have an isomorphism of abelian groups $f : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ defined by

$$f(u_1, \dots, u_n, u_{n+1}) = (u_1, \dots, u_{n-1}, u_n + d\ell - \frac{\ell}{\lambda_1}u_1 - \cdots - \frac{\ell}{\lambda_{n-1}}u_{n-1}, u_{n+1})$$

which satisfies $f(a_1, \dots, a_n, d) = (a_1, \dots, a_{n-1}, a'_n, d)$. Suppose that $(a_1, \dots, a_{n-1}, a'_n, d)$ fails to satisfy (C). Then we can write

$$(a_1, \dots, a_{n-1}, a'_n, d) = (b_1, \dots, b_{n-1}, b_n, d_1) + (c_1, \dots, c_{n-1}, c_n, d_2)$$

with $b_i, c_i \geq 0$, $0 < d_1, d_2 < d$ and

$$\frac{b_1}{\lambda_1} + \cdots + \frac{b_{n-1}}{\lambda_{n-1}} + \frac{b_n}{\lambda_n + \ell} \geq d_1,$$
$$\frac{c_1}{\lambda_1} + \cdots + \frac{c_{n-1}}{\lambda_{n-1}} + \frac{c_n}{\lambda_n + \ell} \geq d_2.$$

Applying f^{-1} we get

$$(a_1, \dots, a_{n-1}, a_n, d) = (b_1, \dots, b_{n-1}, b_n - \delta_1, d_1) + (c_1, \dots, c_{n-1}, c_n - \delta_2, d_2)$$

where

$$\delta_1 = d_1\ell - \frac{\ell}{\lambda_1}b_1 - \cdots - \frac{\ell}{\lambda_{n-1}}b_{n-1}$$

and

$$\delta_2 = d_2\ell - \frac{\ell}{\lambda_1}c_1 - \cdots - \frac{\ell}{\lambda_{n-1}}c_{n-1}.$$

This will contradict the minimality of $(a_1, \dots, a_n, d)$ as a generator of $\overline{R[I(\boldsymbol{\lambda})t]}$ if we can show that $b_n - \delta_1 \geq 0$ and $c_n - \delta_2 \geq 0$, and that $(b_1, \dots, b_{n-1}, b_n - \delta_1, d_1)$, $(c_1, \dots, c_{n-1}, c_n - \delta_2, d_2)$ satisfy condition (A) for $\overline{R[I(\boldsymbol{\lambda})t]}$. By $(*)$ we have

$$\frac{b_1}{\lambda_1} + \cdots + \frac{b_{n-1}}{\lambda_{n-1}} + \frac{b_n - \delta_1}{\lambda_n} \geq d_1$$

and

$$\frac{c_1}{\lambda_1} + \cdots + \frac{c_{n-1}}{\lambda_{n-1}} + \frac{c_n - \delta_2}{\lambda_n} \geq d_2,$$

which is condition (A). If $\delta_1 \leq 0$ and $\delta_2 \leq 0$ (which can happen) we will certainly have $b_n - \delta_1 \geq 0$ and $c_n - \delta_2 \geq 0$. Hence suppose that $\delta_1 > 0$. Then

$$\frac{b_n}{\lambda_n + \ell} \geq d_1 - \frac{b_1}{\lambda_1} - \cdots - \frac{b_{n-1}}{\lambda_{n-1}} = \frac{\delta_1}{\ell},$$

so $b_n \geq (\lambda_n + \ell)\delta_1/\ell > \delta_1$ and $b_n - \delta_1 > 0$. Similarly if $\delta_2 > 0$ then $c_n - \delta_2 > 0$. This shows that $(a_1, \dots, a_{n-1}, a'_n, d)$ is a minimal generator of $\overline{R[I(\boldsymbol{\lambda}')t]}$. Finally $(a_1, \dots, a_{n-1}, a'_n, d)$ is of type (4) because $\delta > 0$, and hence a fortiori $a'_n > 0$. □

Now we show that if $\lambda_n \geq \ell$ then f gives a bijection on minimal generators of type (4).

Proposition 5.7. *Let $(a_1, \dots, a_{n-1}, a'_n, d)$ be a minimal generator of $\overline{R[I(\boldsymbol{\lambda}')t]}$ of type* (4)*. Suppose that $\lambda_n \geq \ell$ and that f is as defined in the proof of Proposition 5.6. Then $f^{-1}(a_1, \dots, a_{n-1}, a'_n, d)$ is a minimal generator of $\overline{R[I(\boldsymbol{\lambda})t]}$ of type* (4)*.*

Proof. We have that $f^{-1}(a_1, \dots, a_{n-1}, a'_n, d) = (a_1, \dots, a_{n-1}, a'_n - \delta, d)$ where

$$\delta = d\ell - \frac{\ell}{\lambda_1}a_1 - \cdots - \frac{\ell}{\lambda_{n-1}}a_{n-1}.$$

By assumption

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a'_n}{\lambda_n + \ell} \geq d.$$

Hence

$$\frac{a'_n}{\lambda_n + \ell} \geq d - \frac{a_1}{\lambda_1} - \cdots - \frac{a_{n-1}}{\lambda_{n-1}} = \frac{\delta}{\ell}$$

so $a'_n \geq (\lambda_n + \ell)\delta/\ell > \delta$ and $a'_n - \delta > 0$. Furthermore, by $(*)$,

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a'_n - \delta}{\lambda_n} \geq d,$$

hence $(a_1, \ldots, a_{n-1}, a'_n - \delta, d)$ represents an element of $\overline{R[I(\boldsymbol{\lambda})t]}$.
Since $(a_1, \ldots, a_{n-1}, a'_n, d)$ is minimal,

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a'_n - 1}{\lambda_n + \ell} < d.$$

By $(*)$ we may conclude

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a'_n - 1 - \delta}{\lambda_n} < d.$$

Because $\lambda_n \geq \ell$ (and hence $\lambda_n \geq \lambda_i$ for $i < n$), we also have that $(a_1, \ldots, a_{n-1}, a'_n - \delta, d)$ satisfies condition (B) for $\overline{R[I(\boldsymbol{\lambda})t]}$. If condition (C) fails then write $(a_1, \ldots, a_{n-1}, a'_n - \delta, d)$ as a sum of at least two minimal generators $(a_{1,i}, a_{2,i}, ..., a_{n,i}, d_i)$ of $\overline{R[I(\boldsymbol{\lambda})t]}$. If $a_{n,i} > 0$, or if $a_{n,i} = 0$ and

$$\frac{a_{1,i}}{\lambda_1} + \cdots + \frac{a_{n-1,i}}{\lambda_{n-1}} = d_i$$

then $f(a_{1,i}, a_{2,i}, ..., a_{n,i}, d_i)$ is the exponent vector of an element of $\overline{R[I(\boldsymbol{\lambda}')t]}$. If this holds for all i, then $(a_1, \ldots, a_{n-1}, a'_n, d) = f(a_1, \ldots, a_{n-1}, a'_n - \delta, d) = \sum_i f(a_{1,i}, a_{2,i}, ..., a_{n,i}, d_i)$, contradicting the minimality of $(a_1, \ldots, a_{n-1}, a'_n, d)$ as a generator of $\overline{R[I(\boldsymbol{\lambda}')t]}$. If, for some i, $a_{n,i} = 0$ and

$$\frac{a_{1,i}}{\lambda_1} + \cdots + \frac{a_{n-1,i}}{\lambda_{n-1}} > d_i$$

then $f(a_{1,i}, a_{2,i}, ..., a_{n,i}, d_i)$ will no longer represent an element of $\overline{R[I(\boldsymbol{\lambda}')t]}$. But we then have

$$\frac{a_{1,i}}{\lambda_1} + \cdots + \frac{a_{n-1,i}}{\lambda_{n-1}} \geq d_i + \frac{1}{\ell}$$

and adding over all i we obtain

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a'_n - \delta}{\lambda_n} \geq d + \frac{1}{\ell}.$$

Since $\lambda_n \geq \ell$ it now follows that

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a'_n - \delta - 1}{\lambda_n} \geq d,$$

which contradicts the previous observation that $(a_1, \ldots, a_{n-1}, a'_n - \delta, d)$ satisfies condition (B) for $\overline{R[I(\boldsymbol{\lambda})t]}$. This shows that $(a_1, \ldots, a_{n-1}, a'_n - \delta, d)$ is a minimal generator of $\overline{R[I(\boldsymbol{\lambda})t]}$. Finally $(a_1, \ldots, a_{n-1}, a'_n - \delta, d)$ is of type (4) since $a'_n - \delta > 0$. □

We can now prove Theorem 5.1. We remarked after Corollary 5.5 that there is a bijection between minimal generators of types (1), (2), (3) for $\overline{R[I(\boldsymbol{\lambda})t]}$ and $\overline{R[I(\boldsymbol{\lambda}')t]}$. Proposition 5.6 gives an injection of generators of type (4) from $\overline{R[I(\boldsymbol{\lambda})t]}$ to $\overline{R[I(\boldsymbol{\lambda}')t]}$. Since d is preserved under these correspondences the first assertion of Theorem 5.1 follows from Lemma 5.3. If $\lambda_n \geq \ell$ then Proposition 5.7 gives a bijection of generators of type (4) (hence on all generators), so the final assertion of Theorem 5.1 follows again from Lemma 5.3. □

Example 5.8. We illustrate the above ideas by revisiting Example 5.2. If $\boldsymbol{\lambda} = (2, 3, 1)$ then the minimal generators of $\overline{R[I(\boldsymbol{\lambda})t]}$ are the rows of the following array (computed with [5]).

1	0	0	0
0	1	0	0
0	0	1	0
2	0	0	1
0	3	0	1
0	0	1	1
1	2	0	1

Here $\ell = 6$ and $f(u_1, u_2, u_3, u_4) = (u_1, u_2, u_3 + 6u_4 - 3u_1 - 2u_2, u_4)$. There are no generators of type (4). We have $\boldsymbol{\lambda}' = (2, 3, 7)$ and the minimal generators of $\overline{R[I(\boldsymbol{\lambda}')t]}$ are the 12 rows of the following array.

1	0	0	0
0	1	0	0
0	0	1	0
2	0	0	1
0	3	0	1
0	0	7	1
1	0	4	1
0	1	5	1
1	1	2	1
0	2	3	1
1	2	6	2
1	2	0	1

There are five generators of type (4), namely rows 7 through 11. The normality of $I(\boldsymbol{\lambda}')$ fails because of the row $(1, 2, 6, 2)$ (and Lemma 5.3), but $I(\boldsymbol{\lambda})$ is normal. If for example we apply f^{-1} to $(1, 2, 6, 2)$ we obtain $f^{-1}(1, 2, 6, 2) =$

$(1,2,1,2)$. The vector $(1,2,1,2)$ satisfies conditions (A) and (B) for $I(2,3,1)$. An expression for $(1,2,1,2)$ as a sum of minimal generators for $I(2,3,1)$ is $(1,2,1,2) = (1,2,0,1)+(0,0,1,1)$ and $(1,2,6,2) = f(1,2,1,2) = f(1,2,0,1)+ f(0,0,1,1) = (1,2,-1,1)+(0,0,7,1)$, which does not contradict the minimality of $(1,2,6,2)$ as a generator of $I(2,3,7)$. The argument given in the proof of Theorem 5.7 that $f(1,2,0,1)$ should represent an element of $\overline{R[I(2,3,7)t]}$ does not work since $\lambda_3 = 1$ is not large enough. On the other hand, if we pass to $I(2,3,13)$ we obtain that the minimal generators of $\overline{R[I(2,3,13)t]}$, by [5], are the rows of

$$\begin{array}{cccc}
1 & 0 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & 0 & 1 & 0\\
2 & 0 & 0 & 1\\
0 & 3 & 0 & 1\\
0 & 0 & 13 & 1\\
1 & 0 & 7 & 1\\
0 & 1 & 9 & 1\\
1 & 1 & 3 & 1\\
0 & 2 & 5 & 1\\
1 & 2 & 11 & 2\\
1 & 2 & 0 & 1
\end{array}$$

and indeed $f(1,0,4,1) = (1,0,7,1)$, $f(0,1,5,1) = (0,1,9,1)$, $f(1,1,2,1) = (1,1,3,1)$, $f(0,2,3,1) = (0,2,5,1)$ and $f(1,2,6,2) = (1,2,11,2)$, giving the bijection of Proposition 5.7 on generators of type (4) for $R[I(\boldsymbol{\lambda}')t]$ and $R[I(\boldsymbol{\lambda}'')t]$.

The following example shows that we cannot replace the hypothesis $\lambda_n \geq \ell$ by $\lambda_n \geq \lambda_i$ for all i. However we do not know if the hypothesis $\lambda_n \geq \ell$ is sharp.

Example 5.9. $I(2,3,5,6)$ is normal, but $I(2,3,5,36)$ is not.

Remark 5.10. We will conclude with the following remark. Suppose that $\lambda_n \geq \lambda_i$ for all i. Then there is a one-to-one correspondence between minimal generators of $\overline{R[It]}$ with $d = 1$ and $(n-1)$-tuples $(a_1, \ldots, a_{n-1})$ of non-negative integers a_i such that

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} < 1$$

under which $(a_1, \ldots, a_{n-1})$ corresponds to $(a_1, \ldots, a_{n-1}, a_n, 1)$, where a_n is the smallest integer (necessarily positive) such that

$$\frac{a_1}{\lambda_1} + \cdots + \frac{a_{n-1}}{\lambda_{n-1}} + \frac{a_n}{\lambda_n} \geq 1.$$

For $(a_1, \ldots, a_{n-1}, a_n, 1)$ satisfies condition (B) because $\lambda_n \geq \lambda_i$ for all i, and condition (C) because 1 cannot be written as the sum of two positive integers.

One can attempt to use this argument to recursively enumerate the minimal generators with $d = 1$.

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