Some results on odd factor of graphs

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Abstract

A $\{1, 3, \cdots, 2n-1\}$-factor of a graph G is defined to be a spanning subgraph of G, each degree of whose vertices is one of $\{1, 3, \cdots, 2n-1\}$, where n is a positive integer. In this paper, we give a sufficient condition for a graph to have a $\{1, 3, \cdots, 2n-1\}$-factor.

1 Main theorem

We consider finite graphs that have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$. For a vertex v of G, we write $deg_G(v)$ for the degree of v in G. For a subset S of $V(G)$, the neighborhood $\Gamma_G(S)$ of S is defined to be the set of vertices if G that are adjacent to at least one vertex of S. Let I be a set of nonnegative integers. A graph G is called an I-graph if $deg_G(x) \in I$ for all $x \in V(G)$. We call a spanning I-subgraph of G an I-factor of G. In particular, a $\{1, 3, \cdots, 2n-1\}$-factor of a graph G is an spanning subgraph F of G such that the degree of every vertex of F is contained in $\{1, 3, \cdots, 2n-1\}$, where n is a positive integer. A $\{k\}$-factor will be called a k-factor.

The following proposition gives a sufficient condition for the existence of a 1-factor in a graph by using neighborhoods.

Proposition 1 (Anderson[2],[4p.115) *] Let G be a graph with an even number of vertices. If*

$$\Gamma_G(X) = V(G) \quad or \quad |\Gamma_G(X)| \geq \frac{4}{3}|X| - \frac{2}{3}$$

for all $X \subset V(G)$, then G has a 1-factor.

Our next main theorem is an extension of this proposition, and its proof is analogous to that of Proposition 1.

Theorem 1 *Let G be a graph with an even number of vertices, and let n be a positive integer. If*

$$\Gamma_G(X) = V(G) \ \ or \ \ |\Gamma_G(X)| > (1 + \frac{1}{3(2n-1)})|X| - \frac{1}{2n-1}$$

for all $X \subset V(G)$, then G has a $\{1, 3, \cdots, 2n-1\}$-factor.

This theorem is best possible in the sense that the condition in Theorem 1 cannot be replaced by the condition that

$$\Gamma_G(X) = V(G) \ \ or \ \ |\Gamma_G(X)| \geq (1 + \frac{1}{3(2n-1)})|X| - \frac{1}{2n-1}$$

for all $X \subset V(G)$. This fact will be shown in Theorem 2

We give some definitions before proving Theorem 1. For a subset S of $V(G)$, we denote by $G - S$ the subgraph of G obtained from G by deleting the vertices in S together with their incident edges. We write $o(G)$ for the number of odd components(components with odd order) of G. Our proof of Theorem 1 depends on the following theorem, which is a generalization of Tutte's 1-factor Theorem and will be extended in Theorem 3.

Proposition 2 (Amahashi[1]) *Let n be a positive integer. Then a graph G has a $\{1, 3, \cdots, 2n-1\}$-factor if and only if*

$$odd(G - X) \leq (2n-1)|X| \ \ for\ all\ X \subset V(G).$$

Proof of Theorem 1. Suppose that G satisfies the condition in the theorem but has no $\{1, 3, \cdots, 2n-1\}$-factor. Then there exist a subset $S \subset V(G)$ with $o(G - S) > (2n-1)|S|$ by Proposition 2. Let $|V(G) = p$. Since p is even, by parity, we may assume $o(G - S) \geq (2n-1)|S| + 2$. Let m denote the number of isolated vertices of $G - S$, and put $t = 1 + (1/3(2n-1))$ and $r = 1/(2n-1)$. We consider two case.

Case 1. $m > o$. Since $|\Gamma_G(V(G) - S)| \neq V(G)$, we have

$$|\Gamma_G(V(G) - S)| > t|V(G) - S| - r = tp - t|S| - r.$$

It is clear that $|\Gamma_G(V(G) - S)| \geq p - m$. From these inequalities, we obtain

$$p < \frac{t|S| + r - m}{t - 1}. \tag{1}$$

On the other hand, counting the vertices of the odd components of $G-S$, we have $m+3((2n-1)|S|+2-m) \le p-|s|$, and thus

$$(3(2n-1)+1)|S|+6-2m \le p. \tag{2}$$

Combining inequalities (1) and (2), we obtain

$$(3(2n-1)+1)|S|+6-2m < \frac{t|S|+r-m}{t-1}. \tag{3}$$

Substituting the values of t and r into (3), we can get $3+(6n-5)m<0$, a contradiction.

Case 2. $m=o$. In this case, every odd component has at least three vertices. Let X be the set of vertices of any $(2n-1)|S|+1$ odd components of $G-S$. Since $\Gamma_G(X) \ne V(G)$, we have $|\Gamma_G(X) > t|S|-r$ and hence

$$|X| < \frac{|S|+r}{t-1}. \tag{4}$$

On the other hand, $|X| \ge 3((2n-1)|S|+1)$ as well. So combining it with inequality (4), we obtain

$$3((2n-1)|S|+1) < \frac{|S|+r}{t-1}.$$

Substituting the values of t and r into the above inequality, we get $0<0$, a contradiction.

Consequently, the proof is complete. ■

If a graph G consists of $n(n \ge 2)$ disjoint copies of a graph H, then we write $G=nH$. The join $G=A+B$ has $V(G)=V(A)\cup V(B)$ and $E(G)=E(A)\cup E(B)\cup\{xy|x\in V(A) \text{ and } y \in V(B)\}$.

Theorem 2 *For every position integer n, there exists infinitely many graphs G that have no $\{1,3,\cdots,2n-1\}$-factor and satisfy*

$$\Gamma_G(X)=V(G) \quad or \quad |\Gamma_G(X)| \ge (1+\frac{1}{3(2n-1)})|X|-\frac{1}{2n-1}$$

for all $X \subset V(G)$.

Proof Let m be a positive integer. We define a graph G by $G=K_m+((2n-1)m+2)K_3$, where K_m and K_3 denote the complete graphs of order m and 3, respectively. It is trivial that G is of even order. Put $S=V(K_m)$. Then $o(G-S)$ has $(2n-1)m+2$ odd components, and so G has no

$\{1, 3, \cdots, 2n-1\}$-factor by Proposition 2. Let $K_3(i)$, $1 \le i \le (2n-1)m+2$, denote the disjoint copies of K_3 in $G - V(K_m)$. Let X be any subset of $V(G)$. We now prove that X satisfies the condition in this theorem. If $|X| = 1$, then

$$|\Gamma_G(X)| > 2 > (1 + \frac{1}{3(2n-1)}|X| - \frac{1}{2n-1}.$$

Hence we may assume that $|X| \ge 2$. It is clear that if $X \cap V(K_m) \ne \emptyset$, then $\Gamma_G(X) = V(G)$. Thus we may assume $X \cap V(K_m) = \emptyset$. Let $a = |\{i | X \cap V(K_3(i))| = 1\}$, $b = |\{i | X \cap V(K_3(i))| = 2\}$ and $c = |\{i | X \cap V(K_3(i))| = 3\}$. Then $|X| = a + 2b + 3c$ and $|\Gamma_G(X) = m + 2a + 3(b + c)$. If $3(2n-1) \times (m + a + b) \ge |X| - 3$, then we have

$$|\Gamma_G(X)| = m + |X| + a + b \ge (1 + \frac{1}{3(2n-1)})|X| - \frac{1}{2n-1}$$

Therefore, we may assume that $|X| > 3(2n-1)(m + a + b) + 3$. Since $|X| \le |V(G)| - |V(K_m)| = 3(2n-1)m + 6$, we obtain $a + b = 0$, which implies $|X| = 3c$, and $|X| = 3(2n-1)m + 6$. Namely, $X = V(G) - V(K_m)$. In this case, we have $|\Gamma_G(X)| = |G|$. Consequently, the theorem is proved.■

2 The extension of proposition 2

In this section, we give an extension of Amahashi's Theorem(proposition 2), which was mentioned before by Knao. let G be a graph and f be a function defined on $V(G)$ such that $f(x)$ is a position odd integer for every $x \in V(G)$. We denote such a function by $f : V(G) \to \{1, 3, 5, \cdots\}$. Then a spanning subgraph F of G is called an $(1, f)$-odd-factor if $deg_F(x) \in \{1, 3, 5, \cdots\}$ for all $x \in V(G)$. Ir is obvious that if $f(x) = 2n - 1$ for all $x \in V(G)$, then a $(1, f)$-odd-factor and a $\{1, 3, \cdots, 2n-1\}$-factor are the same. We prove the following theorem.

Theorem 3 *Let G be a graph and $f : V(G) \to \{1, 3, 5, \cdots\}$. Then G has a $(1, f)$-odd-factor if and only if*

$$o(G - S) \le \sum_{x \in S} f(x) \tag{5}$$

for all $S \subset V(G)$.

In order to prove Theorem 3, we need the following two lemmas.

Lemma 4 *Let G be a tree of even order and $f : V(G) \to \{1, 3, 5, \cdots\}$. Then G has a $(1, f)$-odd-factor if and only if*

$$o(G - x) \leq f(x) \;\; \textit{for all} \;\; x \in V(G).$$

Proof The proof is similar to that of Theorem 1 of [1].

Lemma 5 *Let G be a bipartite graph with partite sets X and Y, and let g be an integer valued function defined on X. Then G has a spanning subgraph H such that*

$$deg_H(x) = g(x) \;\; \textit{for all} \;\; x \in X \;\; \textit{and} \;\; deg_H(y) = 1 \;\; \textit{for all} \;\; y \in Y$$

if and only if

$$|Y| = \sum_{x \in X} g(x) \;\; \textit{and} \;\; |\Gamma_G(S)| \geq \sum_{x \in S} g(x) \;\; \textit{for all} \;\; S \subset X.$$

Proof The lemma is an immediate consequence of Hall's Marriage Theorem[3]. ■

Proof of Theorem 3. This theorem can be proved similar as proposition 2. Assume that G has a $(1, f)$-factor F. Then we have

$$o(G - S) \leq \sum_{x \in S} deg_F(x) \leq \sum_{x \in S} f(x)$$

since there exists at least one edge of F between every odd component of $G - S$ and S.

We next prove the sufficiency by induction on $|V(G)| + |E(G)|$. Without loss of generality, we may assume that G is connected. Moreover, we have that $|V(G)|$ is even by setting $S = \emptyset$ in (5). It is immediate that

$$o(G - S) \equiv |S| \equiv \sum_{x \in S} f(x) (mod\ 2). \tag{6}$$

By Lemma 1, if G is a tree, then G has a $(1, f)$-odd-factor. Hence we may assume that G is not a tree. We consider two cases.

Case 1. $o(G - S) < \sum_{x \in S} f(x)$ *whenever* $\emptyset \neq S \subset V(G)$.

There exists an edge e such that $G - e$ is connected, where $G - e$ denotes the subgraph of G obtained from G by deleting only the edge e. For every $S \subset V(G)$, it follows from (6) that

$$o((G - e) - S) \leq o(G - S) + 2 \leq \sum_{x \in S} f(x).$$

Thus $G - e$ has a $(1, f)$-odd-factor by the induction hypothesis, and hence G has a $(1, f)$-odd-factor.

case 2. $o(G - S) = \sum_{x \in S} f(x)$ for some nonempty $S \subset V(G)$.

Choose such a subset S_0 so that $|S_0|$ is maximum. Then every even component D of $G - S_0$ has a $(1, f)$-odd-factor $F(D)$ since D satisfies condition (5). Let X be the set of all odd components of $G - S_0$ and let B be a bipartite graph with partite sets X and S_0, in which $C \in X$ and $s \in S_0$ are joined by an edge if and only if G contains an edge joining s to a vertex of C. Then we can show that B has a spanning subgraph H such that

$$d_H(C) = 1 \quad \text{for all} \quad C \in X \quad \text{and } d_H(s) = f(s) \quad \text{for all} \quad s \in S_0$$

by Lemma 2 and by the choice of S_0. For every edge $e^{'} = Cs$ of H, there exists an edge e of G such that e joins a vertex of C to s. We can show that the subgraph $C + e$ of G, which is obtained from C by adding an edge e together with its end vertex s, has a $(1, f^{'})$-odd-factor $F^{'}(C + e)$ by the induction hypothesis, where $f^{'}(x) = f(x)$ if $x \neq s$ and $f^{'}(s) = 1$. Consequently, we obtain a desired $(1, f)$-odd-factor F of G given by

$$\begin{aligned} F = &\{F(D) | D \text{ are even components of } G - S_0\} \\ &\cup \{F^{'}(C + e) | C \text{ are odd components of } G - S_0 \text{ and } e^{'} \in E(H)\}. \end{aligned}$$ ■

Note that it seems to be difficult to give a sufficient condition for a graph to have a $(1, f)$-odd-factor by using neighborhoods. The following natural question is open: Is it possible to characterize graphs G that satisfy

$$odd(G - X) \leq 2n|X| \quad \text{for all} \quad X \subset V(G)$$

in terms of factors?

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