Some Results on Secant Varieties Leading to a Geometric Flip Construction

PETER VERMEIRE

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, U.S.A.
e-mail:petvermi@math.okstate.edu

Abstract. We study the relationship between the equations defining a projective variety and properties of its secant varieties. In particular, we use information about the syzygies among the defining equations to derive smoothness and normality statements about $SecX$ and also to obtain information about linear systems on the blow up of projective space along a variety X. We use these results to geometrically construct, for varieties of arbitrary dimension, a flip first described in the case of curves by M. Thaddeus via Geometric Invariant Theory.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a projective variety, scheme theoretically defined by homogeneous polynomials $F_0, \ldots, F_s$ of degree d. The F_i induce a rational map $\varphi : \mathbb{P}^n -- \to \mathbb{P}^s$ defined off X. Equivalently, φ is determined by a linear system $V \subset \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ with base scheme X. One resolves φ to a morphism $\widetilde{\varphi} : \widetilde{\mathbb{P}^n} \to \mathbb{P}^s$ by blowing up $\mathbb{P}^n$ along X. φ and $\widetilde{\varphi}$ have been studied in a number of contexts, including Cremona transformations [10, 11, 13, 20], linear systems on the blow up of projective space [15], and in somewhat greater generality in the form of projections from subvarieties [3].

We will be most concerned with the case $d = 2$. It is well known that if $\mathcal{L}$ is an ample line bundle on a variety X, then X is ideal theoretically defined by quadrics for all embeddings induced by sufficiently large multiples of $\mathcal{L}$. Explicit examples include smooth curves embedded by line bundles of degree at least $2g + 2$ and canonical curves with Clifford index at least 2.

We begin with the observation that, in the case $d = 2$, if $L \subset \mathbb{P}^n$ is a secant line to X then φ collapses L to a point as the restriction of the F_i to L forms a system of quadrics with a base scheme of length two, determining a unique quadric. This raises the natural question: When is φ an embedding off $SecX$? Corollary 2.5 shows this is the case if a condition slightly weaker than Green's condition (N_2) is imposed. In fact, in Theorem 2.10 we prove a more general statement (for arbitrary d) about

when $\widetilde{\varphi}$ is an embedding off the proper transform of an appropriate secant variety. We also give in Corollary 2.12 an application to the vanishing of the cohomology of powers of ideal sheaves.

Section 3 is concerned with the structure of $SecX$ and its proper transform in the case $d = 2$. The main results in this section are Theorem 3.9 and its Corollary 3.10 where it is shown that the restriction of $\widetilde{\varphi}$ to the proper transform of the secant variety is a $\mathbb{P}^1$-bundle over the length two Hilbert scheme of X, extending a result in [4] to varieties of arbitrary dimension. We also give (Remark 3.12) a geometric criterion for determining the dimension of the secant variety to a variety satisfying Green's condition (N_2).

In Section 4, we obtain a partial answer to a question raised in [7, §1] by constructing a flip first described in the case of smooth curves by M. Thaddeus [28] via GIT. The construction proceeds in several stages, with the end result summarized in Theorem 4.12. The reader should note that these are not K_X-flips in the sense of the Minimal Model Program. It is shown in [7] that the flips constructed by Thaddeus are, in fact, log flips; however we do not address that question here. Again (Corollary 4.15) we give a simple application to the cohomology of ideal sheaves.

Aside from gaining an understanding of the geometry of secant varieties and how this geometry relates to syzygies, a practical goal of the flip construction is the derivation of vanishing theorems for the groups $H^i(\mathbb{P}^n, \mathcal{I}_X^a(b))$ (Cf. [8, 35]). Specifically, in [28], Thaddeus obtains vanishing theorems on the flipped spaces via Kodaira vanishing as he is able to identify the ample cone on each space. In a related direction, Bertram [6] uses a generalization of Kodaira vanishing to prove vanishing theorems directly on the space $\widetilde{\mathbb{P}^n}$, deduced from the existence of log canonical divisors. As discussed in [6], a combination of the two techniques should reveal the strongest results. The construction of further flips is taken up in [32], and the question of vanishing theorems in [33].

NOTATION

We will decorate a projective variety X as follows: X^d is the dth Cartesian product of X; $S^d X$ is $Sym^d X = X^d/S_d$, the dth symmetric product of X; and $\mathcal{H}^d X$ is $Hilb^d(X)$, the Hilbert Scheme of zero dimensional subschemes of X of length d. Recall (Cf. [16]) that if X is a smooth projective variety then $\mathcal{H}^d X$ is also projective, and is smooth if either $dim X \leqslant 2$ or $d \leqslant 3$.

If V is a k-vector space, $\mathbb{P}(V)$ is the space of one-dimensional *quotients* of V. We work throughout over the field $k = \mathbb{C}$ of complex numbers. We use the terms locally free sheaf (resp. invertible sheaf) and vector bundle (resp. line bundle) interchangeably. If $D \subset X$ is a Cartier divisor, then the associated invertible sheaf is denoted $\mathcal{O}_X(D)$. We conform to the convention that products of line bundles corresponding to explicit divisors are written additively, while other products are written multiplicatively, e.g. $(\mathcal{L} \otimes \mathcal{O}_X(D))^{\otimes n} \cong \mathcal{L}^{\otimes n} \otimes \mathcal{O}_X(nD)$. A line bundle $\mathcal{L}$ on X is *nef* if $\mathcal{L} \cdot C \geqslant 0$ for every irreducible curve $C \subset X$. A line bundle $\mathcal{L}$ is *big* if

$\mathcal{L}^{\otimes n}$ induces a birational map for all $n \gg 0$. The term *conic* is used to mean a quadric hypersurface in some projective space.

2. Condition (K_d)

We begin with the situation $d = 2$ from the Introduction and establish a useful technical result:

PROPOSITION 2.1. *Let $V \subseteq \Gamma(\mathbb{P}^n, \mathcal{O}(2))$ be a linear system with base scheme X, with induced map $\varphi : \mathbb{P}^n --\to \mathbb{P}(V)$. Then the following are equivalent:*

(1) *φ is an embedding off $SecX$.*
(2) *If $L \subset \mathbb{P}^n$ is a line not intersecting X, $L \not\subset SecX$, then the natural restriction map $r_L : V \to \Gamma(L, \mathcal{O}_L(2))$ is surjective.*

Proof. Assume φ is an embedding off $SecX$, and let $L \not\subset SecX$ be a line not intersecting X. Then the restriction of φ to L is base point free, hence $corank(r_L) \leqslant 1$. If $corank(r_L) = 1$, then $\varphi|_L$ is a ramified double cover of $\mathbb{P}^1$, contradiction the assumption that φ is an embedding off $SecX$.

Conversely, choose a length two subscheme $Z \subset \mathbb{P}^n \setminus SecX$; Z determines a unique line $L \not\subset SecX$. If L intersects X in a single point, then the restriction of φ to L resolves to a linear embedding of L. If L does not intersect X, then the surjectivity of the restriction map implies that φ is an embedding along L. Hence points and tangents are separated off $SecX$.   □

In other words, we need only avoid the case where, after a choice of coordinates on L, the restriction of φ to L is the system generated by x^2, y^2. We introduce a condition that guarantees this does not happen.

DEFINITION 2.2. A subscheme $X \subset \mathbb{P}^n$ satisfies condition (K_d) if X is scheme theoretically cut out by forms $F_0, \ldots, F_s$ of degree d such that the trivial (or Koszul) relations among the F_i are generated by linear syzygies.

More generally, let $V \subseteq H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ be a linear system of forms of degree d with (possibly empty) base scheme X. Then the pair (X, V) *satisfies condition* (K_d) if the trivial relations among the elements of V are generated by linear syzygies. Write (X, F_i) for the pair (X, V) if the set $\{F_i\}$ generates the linear system V. Perhaps the simplest example of varieties that do *not* satisfy (K_d) is that of scheme theoretic complete intersections of hypersurfaces of degree d.

Remark 2.3. For a projective variety $X \subset \mathbb{P}^n$, M. Green [17] defines *condition* (N_2) as: X projectively normal, ideal theoretically defined by quadrics F_i, and all of the syzygies among the F_i are generated by linear ones. Examples include a smooth curve embedded by a line bundle of degree at least $2g + 3$ [17], canonical curves with

Clifford index at least 3 [26, 34], Veronese embeddings of $\mathbb{P}^n$, e.g. [12], and all sufficiently large embeddings of any projective variety [17, 22].

Clearly, if X satisfies (N_2), then X satisfies the weaker condition (K_2). Though (K_2) is a technically simpler condition and will arise naturally, in practice most examples we consider that satisfy (K_2) will actually satisfy the stronger, and well studied, condition (N_2). As such, we have made no attempt to understand examples where X satisfies (K_2) but not (N_2). It is not difficult, however, to see where such examples might arise. Specifically, if $X \subset \mathbb{P}^n$ is a smooth surface satisfying (N_2) and if $h^1(\mathcal{O}_X(3)) < h^1(\mathcal{O}_X(1))$, then a general quadric section of X is not projectively normal, but is certainly scheme theoretically defined by quadrics.

In light of Lemma 2.4 below, one may also expect to find examples by taking hyperplane sections of non arithmetically Cohen–Macaulay varieties.

An advantage of the weaker condition (K_d) is the following simple:

LEMMA 2.4. *Let V be a linear system on $\mathbb{P}^n$ that satisfies (K_d), and let $M \cong \mathbb{P}^k$ be a linear subspace of $\mathbb{P}^n$. Then the restriction of V to M satisfies (K_d).* □

This gives

COROLLARY 2.5. *Let $X \subset \mathbb{P}^n$ be scheme theoretically defined by quadrics $F_0, \ldots, F_s$ satisfying (K_2). Then the induced map φ is an embedding off $Sec X$.*

Proof. Let $L \subset \mathbb{P}^n$, $L \not\subset Sec X$, be a line not intersecting X. By Lemma 2.4, the restriction of the F_i to L must satisfy (K_2). However, it is easy to check that the only base point free system of quadrics on $\mathbb{P}^1$ satisfying (K_2) is the complete system of quadrics. Hence by Proposition 2.1, φ is an embedding off $Sec X$. □

Remark 2.6. A similar result was discovered independently by K. Hulek and W. Oxbury [21]. □

Recall that a base point free linear system W on X is said to be *k-very ample* if every zero-dimensional subscheme of X of length k spans a $\mathbb{P}^{k-1}$ in $\mathbb{P}(W)$. We record the following elementary:

LEMMA 2.7. *Assume $X \subset \mathbb{P}(W) = \mathbb{P}^n$ satisfies condition (K_2) and contains no lines or conics. Then W is a 4-very ample linear system on X.*

Proof. Assume to the contrary that there is a 2-plane M that intersects X in a scheme Z of length $k \geqslant 4$. Note that by hypothesis M cannot intersect X in a scheme of positive dimension.

Two conics in M intersect in a scheme of length 4 if and only if they have no common component. However, a pair of plane conics cannot satisfy (K_2) unless they share a linear factor, which would imply a positive dimensional base scheme. Therefore, there can be no such 2-plane. □

We return to the case of arbitrary d. Via the closure of the graph $\overline{\Gamma_\varphi} \subset \mathbb{P}^n \times \mathbb{P}^s$, we have a resolution of the rational map φ:

$$
\begin{array}{ccc}
\overline{\Gamma_\varphi} & & \\
\pi_1 \downarrow & \searrow{\tilde{\varphi}} & \\
\mathbb{P}^n & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^s
\end{array}
$$

where $\tilde{\varphi}$ is the restriction of the projection onto the second factor. Note that $\tilde{\varphi}$ can be identified with the morphism $\mathrm{Bl}_X(\mathbb{P}^n) \to \mathbb{P}^s$ induced by lifting the appropriate sections of $\Gamma(\mathbb{P}^n, \mathcal{I}_X(d))$ to sections of $\Gamma(\widetilde{\mathbb{P}^n}, \mathcal{O}_{\widetilde{\mathbb{P}^n}}(dH - E))$.

PROPOSITION 2.8. *Let (X, F_i) satisfy (K_d) and assume that X does not contain a line. If $a \in \mathrm{Im}\, \tilde{\varphi}$, then $\tilde{\varphi}^{-1}(a) = \mathbb{P}^k \times \{a\} \subset \mathbb{P}^n \times \mathbb{P}^s$, where either*

(1) $k = 0$ *or*
(2) $\pi_1(\mathbb{P}^k \times \{a\}) = \mathbb{P}^k \subset \mathbb{P}^n$ *intersects X in a hypersurface of degree d in $\mathbb{P}^k$.*

Proof. Take coordinates $[z_0, \ldots, z_n; t_0, \ldots, t_s]$ on $\mathbb{P}^n \times \mathbb{P}^s$ and let S be the scheme defined by the equations $\{F_i t_j - F_j t_i = 0\} \forall i, j$. Clearly, $\overline{\Gamma_\varphi} \subset S$ as schemes and it is easy to verify that $\overline{\Gamma_\varphi} = S$ off of E, the exceptional divisor of the blow up.

Now, considering a syzygy as a vector of forms, let

$$\{(a_{\ell 0}, \ldots, a_{\ell s}); 0 \leqslant \ell \leqslant r\}$$

generate the linear syzygies among the F_i. Let $T \subset \mathbb{P}^n \times \mathbb{P}^s$ be the subscheme defined by the equations $\{\sum_{k=0}^{s} a_{\ell k}(z)t_k; 0 \leqslant \ell \leqslant r\}$. Again it is clear that $\overline{\Gamma_\varphi} \subset T$ as schemes, (Cf. [20, §1]) but condition (K_d) implies that $T \subset S$, hence $\overline{\Gamma_\varphi} = T$ off of E.

$\tilde{\varphi}^{-1}(a)$ is contained as a scheme in T_a, where T_a is the fiber over a of the projection map restricted to T. Without loss of generality, assume $a = [1, 0, \ldots, 0] \in \mathbb{P}^s$. T_a is then scheme theoretically defined by the bihomogeneous equations

$$\left\{ \sum_{k=0}^{s} a_{0k} t_k, \ldots, \sum_{k=0}^{s} a_{rk} t_k, t_1, t_2, \ldots, t_s \right\}$$

and so more simply by

$$\{a_{00} t_0, \ldots, a_{r0} t_0, t_1, t_2, \ldots, t_s\}$$

giving:

$$
\begin{aligned}
T_a &= \mathbb{P}^k \times [1, 0, \ldots, 0] \subset \mathbb{P}^n \times \mathbb{P}^s \\
&\cong \mathbb{P}^k,
\end{aligned}
$$

where $\mathbb{P}^k$ is the linear subspace of $\mathbb{P}^n$ defined by the $\{a_{\ell 0}\}$. We have just seen $\tilde{\varphi}^{-1}(a) = T_a$ off of E; T_a is irreducible, however, so $\tilde{\varphi}^{-1}(a) = T_a \cong \mathbb{P}^k$ as long as either of the following is true:

(1) T_a is a reduced point (i.e. $k = 0$).

(2) T_a and $\tilde{\varphi}^{-1}(a)$ are not both contained in E.

To guarantee that the second possibility occurs if T_a has positive dimension, note that X does not contain a line. Therefore $\pi_1(T_a)$, which is a linear subspace of $\mathbb{P}^n$, cannot be isomorphic to a positive-dimensional reduced linear subscheme of X. Hence, T_a cannot be contained in E. $\square$

Remark 2.9. Rather than the hypothesis that X contain no lines, one could instead insist that X is not set theoretically cut out by any subsystem of the F_i. We choose the former hypothesis as it will be necessary below. $\square$

Denoting by $Sec_d^1 X$ the variety of lines intersecting X in a subscheme of length at least d, we have a natural extension of Corollary 2.5:

THEOREM 2.10. *Let (X, F_i) be a pair that satisfies (K_d), and assume that X does not contain a line. Then $\tilde{\varphi}$ is an embedding off the proper transform of $Sec_d^1 X$. Furthermore, if X is smooth then the image of $\tilde{\varphi}$ is a normal subvariety of $\mathbb{P}^s$.*

Proof. To prove the first claim, note that because the fibers of $\tilde{\varphi}$ are reduced, we need only show that points are separated.

Take $p, q \in \overline{\Gamma_\varphi}$, and assume $\tilde{\varphi}(p) = \tilde{\varphi}(q) = \{r\} \in \mathbb{P}^s$. Then by Proposition 2.8, $S = \tilde{\varphi}^{-1}(r)$ satisfies $\pi(S) = \mathbb{P}^k, k > 0$. There are then two possibilities:

(1) $\pi(S) \cap X$ is a d-ic hypersurface in $\pi(S)$ and, hence, every line in $\pi(S)$ is a d-secant line of X, which implies that $\pi(S) \subseteq Sec_d^1(X)$, and so p and q are in the proper transform of the variety of d-secant lines.

(2) $\pi(S) \subseteq X$. In this case, however, $\pi(S)$ is a positive dimensional linear subvariety of X, which is not allowed by hypothesis.

To see that the image is normal if X is smooth, notice that by identification with the blow up of $\mathbb{P}^n$ along X, $\overline{\Gamma_\varphi}$ is smooth, hence normal. We have just shown that the fibers are reduced and connected, hence the result follows from the fact that the image of a proper morphism from a normal variety with reduced, connected fibers is normal. $\square$

Remark 2.11. The proof of Theorem 2.10 implies that if $X \subset \mathbb{P}^n$ is scheme theoretically defined by forms of degree d that satisfy (K_d), then the map $\varphi : \mathbb{P}^n \setminus X \to \mathbb{P}^s$ is an embedding off of $Sec_d^1 X$, even if X *does* contain a line, extending Corollary 2.5. $\square$

Theorem 2.10 provides a simple extension of a vanishing result for powers of ideal sheaves of projective varieties in [8, 1.10]. The bound is only improved by one degree; this comes precisely from the fact that we know $\mathcal{O}(dH - E)$ is big.

COROLLARY 2.12. *If X is smooth, irreducible and satisfies (K_d) then Theorem 2.10 shows that the linear system $\mathcal{O}(dH - E)$ on $\overline{\Gamma_\varphi} = \mathrm{Bl}_X(\mathbb{P}^n)$ is big and nef, and $\mathcal{O}(kH - mE)$ is very ample if $km > d$. A simple application of the Kawamata–Viehweg vanishing theorem gives*

$$H^i\big(\mathbb{P}^n, \mathcal{I}_X^a(k)\big) = 0, i > 0, k \geqslant d(e + a - 1) - (n + 1),$$

where e is the codimension of X in $\mathbb{P}^n$. □

3. Results on Secant Varieties

We describe a vector bundle on $\mathcal{H}^2 X = Hilb^2(X)$ and a morphism to projective space giving rise to $Sec X$. Our construction follows that of [4, §1], [2, VIII.2], and [27] where this is done for curves with the identification of $\mathcal{H}^2 X$ with $S^2 X$.

Let $V \subseteq \Gamma(X, L)$ be a very ample linear system and denote by $\mathcal{D}$ the universal subscheme of $X \times \mathcal{H}^2 X$ and note $\mathcal{D} \cong \mathrm{Bl}_\Delta(X \times X)$. Let $\pi : X \times \mathcal{H}^2 X \to X$ and $\pi_2 : X \times \mathcal{H}^2 X \to \mathcal{H}^2 X$ be the projections, and let L be any line bundle on X. Form the invertible sheaf $\mathcal{O}_\mathcal{D} \otimes \pi^* L$ on $\mathcal{D} \subset X \times \mathcal{H}^2 X$. Now $\pi_2|_\mathcal{D} : \mathcal{D} \to \mathcal{H}^2 X$ is flat of degree 2, hence $\mathcal{E}_L = (\pi_2)_*(\mathcal{O}_\mathcal{D} \otimes \pi^* L)$ is a locally free sheaf of rank 2 on $\mathcal{H}^2 X$. We define the *first secant bundle* of X with respect to L to be the $\mathbb{P}^1$-bundle $B^1(L) = \mathbb{P}_{\mathcal{H}^2 X}(\mathcal{E}_L)$.

To define the desired map, push the natural restriction $\pi^* L \to \mathcal{O}_\mathcal{D} \otimes \pi^* L$ down to $\mathcal{H}^2 X$ giving an evaluation map $H^0(X, L) \otimes \mathcal{O}_{\mathcal{H}^2 X} \to \mathcal{E}_L$ which in turn for any linear system $V \subseteq H^0(X, L)$ restricts to $V \otimes \mathcal{O}_{\mathcal{H}^2 X} \to \mathcal{E}_L$. Now a fiber of $\mathcal{E}_L$ over a point $Z \in \mathcal{H}^2 X$ is $H^0(X, L \otimes \mathcal{O}_Z)$, so if V is very ample then this map is surjective and we obtain a morphism:

$$\beta_1 : B^1(L) \to \mathbb{P}(V) \times \mathcal{H}^2 X \to \mathbb{P}(V).$$

The image of this morphism is the secant variety to X in $\mathbb{P}(V)$.

Remark 3.1. It will be useful to note that the above surjection also induces a morphism $\mathcal{H}^2 X \to \mathbb{G}(1, V)$ which is an embedding as long as V is 3-very ample [9]. □

Notation 3.2. To help simplify notational clutter, we denote $\Sigma = Sec_2^1 X$ and $\mathcal{T} = Tan X$. This should cause no confusion as we will be concerned with a fixed variety X, and will be primarily concerned only with the first secant variety. □

HYPOTHESIS 3.3. For the remainder of this section, $X \subset \mathbb{P}^n$ will denote a smooth, irreducible, non-degenerate variety, scheme theoretically defined by quadrics $(F_0, \ldots, F_s) = V \subset \Gamma(\mathbb{P}^n, \mathcal{O}(2))$ satisfying (K_2). Assume that *X contains no lines and no conics*. In particular, the embedding of X is 4-very ample. □

In this situation, Theorem 2.10 implies that the map $\tilde{\varphi}$ is an embedding off the proper transform of the secant variety to X. Here we study what the map does when restricted to the proper transform of the secant variety. Denote by $\tilde{\Sigma}$ the proper transform of the secant variety under the blowing up $\pi : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ of $\mathbb{P}^n$ along X. By a slight abuse of notation, write $\tilde{\varphi} : \tilde{\Sigma} \to \mathbb{P}^s$ for the restriction of $\tilde{\varphi} : \widetilde{\mathbb{P}^n} \to \mathbb{P}^s$. We show first that there is a map $\tilde{\Sigma} \to \mathcal{H}^2 X$, and then an embedding of $\mathcal{H}^2 X$ into the image variety of $\tilde{\varphi}$, such that the composition factors $\tilde{\varphi} : \tilde{\Sigma} \to \mathbb{P}^s$ and the map $\tilde{\Sigma} \to \mathcal{H}^2 X$ is a $\mathbb{P}^1$-bundle.

Remark 3.4. Because of the assumption that X contains no lines and no conics, each fiber of $\tilde{\varphi} : \tilde{\Sigma} \to \mathbb{P}^s$ is isomorphic to $\mathbb{P}^1$ by Proposition 2.8. In particular, given a point in $\tilde{\Sigma}$ or in $\Sigma \setminus X$, one can say on which secant or tangent line it lies. $\square$

LEMMA. 3.5. *There is a morphism $g : \tilde{\Sigma} \to \mathcal{H}^2 X$ taking a point p to the length 2 subscheme Z of X determining the secant line on which p lies.*

Proof. We construct a morphism $\tilde{\Sigma} \to \mathbb{G}(1, n)$ whose image is $\mathcal{H}^2 X$. Let $Y = Im\tilde{\varphi}$, and push the surjection

$$H^0(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}(H)) \otimes \mathcal{O}_{\tilde{\Sigma}} \to \mathcal{O}_{\tilde{\Sigma}}(H) \to 0$$

down to Y:

$$H^0(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}(H)) \otimes \mathcal{O}_Y \to \tilde{\varphi}_* \mathcal{O}_{\tilde{\Sigma}}(H)$$

The sheaf $\tilde{\varphi}_* \mathcal{O}_{\tilde{\Sigma}}(H)$ is locally free of rank 2, and the map is surjective as $\mathcal{O}(H)$ maps a fiber of $\tilde{\varphi}$ to a linearly embedded $\mathbb{P}^1 \subset \mathbb{P}^n$. Pulling this surjection back to $\tilde{\Sigma}$ gives a surjection from a free rank $n + 1$ sheaf to a rank 2 vector bundle, hence a morphism $\tilde{\Sigma} \to \mathbb{G}(1, n)$ taking a fiber of $\tilde{\varphi}$ to the point representing the associated secant line. The image of this morphism is clearly $\mathcal{H}^2 X \hookrightarrow \mathbb{G}(1, n)$ from Remark 3.1. $\square$

Having constructed a map $g : \tilde{\Sigma} \to \mathcal{H}^2 X$, we construct an embedding $f : \mathcal{H}^2 X \hookrightarrow \mathbb{P}^s$ so that $\tilde{\varphi} = (f \circ g) : \tilde{\Sigma} \to \mathbb{P}^s$.

Let $Z \in \mathcal{H}^2 X$ be a length 2 subscheme of X, and let $\ell_Z \subset \mathbb{P}^n$ be the line determined by Z. Note that by hypothesis ℓ_Z does not lie on X. There are homomorphisms:

$$r_Z : V \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \otimes \mathcal{O}_{\ell_Z}).$$

f is set theoretically given by associating to every $Z \in \mathcal{H}^2 X$ the one-dimensional quotient $V/ker(r_Z)$. This association is injective by Remark 3.4.

LEMMA. 3.6. *$f : \mathcal{H}^2 X \to \mathbb{P}(V) = \mathbb{P}^s$ is a morphism.*

Proof. Let $L = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(2H)$ and form $\pi_1^* L$ on $\widetilde{\mathbb{P}^n} \times \mathcal{H}^2 X$. Embed:

$$\tilde{\Sigma} \hookrightarrow \widetilde{\mathbb{P}^n} \times \mathcal{H}^2 X,$$
$$p \mapsto (i(p), g(p)),$$

where $i : \tilde{\Sigma} \hookrightarrow \widetilde{\mathbb{P}^n}$ is the inclusion. Applying π_{2*} to the surjection $\pi_1^* L \to \pi_1^* L \otimes \mathcal{O}_{\tilde{\Sigma}} \to 0$ gives a map:

$$H^0(\widetilde{\mathbb{P}^n}, L) \otimes \mathcal{O}_{\mathcal{H}^2 X} \to \pi_{2*}\big(\pi_1^* L \otimes \mathcal{O}_{\tilde{\Sigma}}\big).$$

Recalling $V \subseteq H^0(\widetilde{\mathbb{P}^n}, L \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-E))$, there is a map:

$$V \otimes \mathcal{O}_{\mathcal{H}^2 X} \to \pi_{2*}\big(\pi_1^* L \otimes \mathcal{O}_{\tilde{\Sigma}}\big),$$

where a fiber of the coherent sheaf $\pi_{2*}(\pi_1^* L \otimes \mathcal{O}_{\tilde{\Sigma}})$ over a point $Z \in \mathcal{H}^2 X$ is isomorphic to $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \otimes \mathcal{O}_{\ell_Z})$. By the above remarks, this map has rank 1, hence gives a surjection to a line bundle on $\mathcal{H}^2 X$ with fiber over Z isomorphic to $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \otimes \mathcal{I}_X \otimes \mathcal{O}_{\ell_Z})$. f is the morphism induced by this surjection. $\square$

The diagram

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\ g\ } & \mathcal{H}^2 X \\ & \searrow{\scriptstyle \tilde{\varphi}} & \downarrow{\scriptstyle f} \\ & & \mathbb{P}^s \end{array}$$

then commutes, and because the fibers of $\tilde{\varphi}$ are reduced, those of f are as well:

PROPOSITION 3.7. *Let (X, V) be a pair satisfying (K_2), and assume $X \subset \mathbb{P}^n$ is smooth, irreducible, and contains no lines or conics. Then the morphism $f : \mathcal{H}^2 X \hookrightarrow \mathbb{P}(V)$ above is an embedding* $\square$

LEMMA 3.8. *With hypotheses as in Proposition 3.7, the exceptional divisor of the blow up $\tilde{\Sigma} \to \Sigma$ is isomorphic to $\mathrm{Bl}_\Delta(X \times X)$.*

Proof. Let $F \subset \tilde{\Sigma}$ be the exceptional divisor of this blow up. Let Y be the image of:

$$F \to X \times \mathcal{H}^2 X$$
$$p \mapsto (\pi(p), \tilde{\varphi}(p)).$$

Y is flat of degree 2 over $\mathcal{H}^2 X$. Indeed, by the structure of $\tilde{\varphi}$ the fiber over a point in $\mathcal{H}^2 X$ is exactly the corresponding length 2 subscheme of X, hence Y induces the identity morphism $id : \mathcal{H}^2 X \to \mathcal{H}^2 X$. By the universal property of $\mathcal{D} \cong \mathrm{Bl}_\Delta(X \times X)$, the universal subscheme of $X \times \mathcal{H}^2 X$, we have:

$$Y \cong (id_X \times id_{\mathcal{H}^2 X})^{-1}(\mathcal{D}).$$

The map from F to Y is a finite birational morphism to a smooth variety, so is an isomorphism, hence $F \cong \mathrm{Bl}_\Delta(X \times X)$. $\square$

This allows another construction of the secant bundle $B^1(L)$: Writing $Pic\,\widetilde{\mathbb{P}^n} \cong \mathbb{Z}H + \mathbb{Z}E$, form the line bundle $H \otimes \mathcal{O}_F$ on the exceptional divisor $F \subset \tilde{\Sigma}$. The restriction of $\tilde{\varphi}$ to F is a degree two map to $\mathcal{H}^2 X$. Let

$\mathcal{E} = \tilde{\varphi}_*(H \otimes \mathcal{O}_F)$. By the identification of F with $\mathcal{D}$ and of $H \cong \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ with L on X, we see $B^1(L) \cong \mathbb{P}_{\mathcal{H}^2 X}(\mathcal{E})$.

THEOREM 3.9. *Let $X \subset \mathbb{P}^n$ be smooth, irreducible and satisfy (K_2). If X contains no lines and no conics then $\tilde{\varphi} : \tilde{\Sigma} \to \mathcal{H}^2 X$ is the $\mathbb{P}^1$-bundle $\mathbb{P}_{\mathcal{H}^2 X}(\mathcal{E}) \to \mathcal{H}^2 X$.*

Proof. Note that $\mathcal{E} = \tilde{\varphi}_*(H \otimes \mathcal{O}_F)$ and $\tilde{\varphi}_*(H \otimes \mathcal{O}_{\tilde{\Sigma}})$ are isomorphic rank two vector bundles on $\mathcal{H}^2 X$ and that $H \otimes \mathcal{O}_{\tilde{\Sigma}}$ is generated by its global sections. Hence there is a surjection $\tilde{\varphi}^*\mathcal{E} \to H \otimes \mathcal{O}_{\tilde{\Sigma}} \to 0$ which induces a morphism $\kappa : \tilde{\Sigma} \to \mathbb{P}_{\mathcal{H}^2 X}(\mathcal{E})$.

This gives the diagrams:

$$
\begin{array}{ccc}
\tilde{\Sigma} & \overset{\kappa}{\longrightarrow} & \mathbb{P}_{\mathcal{H}^2 X}(\mathcal{E}) \\
\downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \beta_1} & \\
\Sigma & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\tilde{\Sigma} & \overset{\kappa}{\longrightarrow} & \mathbb{P}_{\mathcal{H}^2 X}(\mathcal{E}) \\
& \searrow{\scriptstyle \tilde{\varphi}} & \downarrow{\scriptstyle p} \\
& & \mathcal{H}^2 X
\end{array}
$$

where p is the natural projection map. κ makes both triangles commute, and so is a finite (by the second diagram) birational (by the first) morphism to a smooth variety, hence an isomorphism. $\square$

COROLLARY 3.10. *Under the hypotheses of Theorem 3.9:*

(1) $\tilde{\Sigma}$ is smooth and Σ is smooth off X.
(2) $\tilde{\mathcal{T}}$ is smooth and $\mathcal{T}$ is smooth off X.
(3) Σ is normal.

Proof. The smoothness of $\tilde{\Sigma}$ is immediate from Theorem 3.9.

For the second claim, note that $\tilde{\varphi}$ maps $\tilde{\mathcal{T}}$ to the diagonal in $\mathcal{H}^2 X$, which is the projectivized tangent bundle to X, hence smooth (by the *diagonal* in $\mathcal{H}^2 X$, we mean the proper transform on the diagonal under the birational morphism $\mathcal{H}^2 X \to S^2 X$). Therefore, $\tilde{\mathcal{T}}$ is smooth as above.

To show Σ is normal, we use the (just proven) fact that $\tilde{\Sigma}$ is smooth, hence normal. It suffices to show that for $p \in X$, $\pi^{-1}(p)$ is reduced and connected where $\pi : \tilde{\Sigma} \to \Sigma$ is the blow up of Σ along X. But by Lemma 3.8, $\pi^{-1}(p) \cong \mathrm{Bl}_p(X)$. $\square$

Remark 3.11. In [4], A. Bertram shows directly that $\tilde{\Sigma}$ is isomorphic to $B^1(L)$ if X is a smooth curve embedded by the complete linear system associated to a line bundle L that is 4-very ample. $\square$

Remark 3.12. A simple consequence of the above results is that a smooth, irreducible variety $X \subset \mathbb{P}^n$ satisfying (K_2) with no lines and no conics has a non-deficient secant variety. It can be shown [31, 3.6.1] that if $\delta = 2r + 1 - dim(\Sigma)$ is the deficiency of the secant variety to X, then $\delta = (2r - dim Y)/2$ where Y is the image variety of $\tilde{\varphi} : \tilde{\Sigma} \to \mathbb{P}^s$.

In particular, this shows that the dimension of Σ is determined by the dimension of the fibers of $\tilde{\varphi}$: The generic pair of points of X lies on a quadric hypersurface of (maximal) dimension d if and only if $dim(\Sigma) = 2r + 1 - d$.

4. Geometric Flip Construction

In this section we are motivated by [28] to construct a flip centered about $\mathcal{H}^2 X$. We recall his construction.

4.1. WORK OF THADDEUS

In [28] Thaddeus considers the moduli problem of semi-stable pairs (E, s) consisting of a rank two bundle E with $\wedge^2 E = \Lambda$, and a section $s \in \Gamma(X, E) - \{0\}$. This is interpreted as a GIT problem, and by varying the linearization of the group action, a collection of (smooth) moduli spaces $M_1, M_2, \dots, M_k$, $k = [(d - 1)/2]$, is constructed. As stability is an open condition, these spaces are birational. In fact, they are isomorphic in codimension one, and may be linked via a diagram
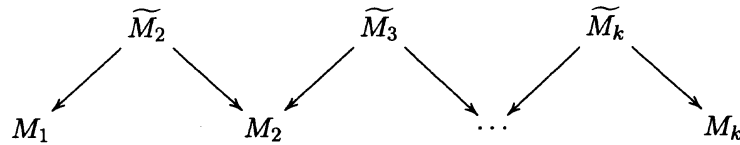
where there is a morphism $M_k \to M(2, \Lambda)$. The relevant observations are first that this is a diagram of flips (in fact it is shown in [7] that it is a sequence of log flips) where the ample cone of each M_i is known, second that M_1 is the blow up of $\mathbb{P}(\Gamma(X, K_X \otimes \Lambda)^*)$ along X, and finally that $\tilde{M}_2$ is the blow up of M_1 along the proper transform of the secant variety and that all of the flips can be seen as blowing up and down various higher secant varieties.

Our inspiration can be stated as follows: The sequence of flips in Thaddeus' construction, constructed via Geometric Invariant Theory, can be realized as a sequence of natural geometric constructions depending only on the original embedding of $X \subset \mathbb{P}^n$. An advantage of this approach is that the smooth curve X can be replaced by any smooth variety. Even in the curve case, ours applies to situations where the original construction does not hold (e.g. for canonical curves with Clifford index at least 3).

Thaddeus goes on [28, 7.8] to compute the dimension of the spaces $H^0(\mathbb{P}H^0(K_X \otimes \Lambda), \mathcal{I}_X^a(k))$ for certain values of d, g, a, k. In particular, this computation is used to verify the rank-two Verlinde formula. A part of our motivation is to try to extend this computation to a much larger class of varieties.

4.2. OUTLINE OF OUR CONSTRUCTION

With notation as above assume $X \subset \mathbb{P}^n$ is smooth, irreducible, satisfies (K_2), and contains no lines or conics. Let $r = dim X$ and assume that $n - 2r - 1 \geqslant 2$, i.e. that

Σ is not a hypersurface in $\mathbb{P}^n$. Write $\mathbb{P}(\mathcal{E})$ for the secant bundle $\mathbb{P}_{\mathcal{H}^2 X}(\mathcal{E})$ and identify $\tilde{\Sigma}$ with $\mathbb{P}(\mathcal{E})$ (Theorem 3.9). Simply write $\tilde{\varphi} : \mathbb{P}(\mathcal{E}) \to \mathcal{H}^2 X$ for the restriction of $\tilde{\varphi} : \widetilde{\mathbb{P}^n} \to \mathbb{P}^s$.

We begin with a general construction: Let $f : X -- \to Y$ and $g : X -- \to Z$ be rational maps of irreducible varieties, f birational. Form $\overline{\Gamma_{f,g}} \subset X \times Y \times Z$, the closure of the graph of $(f, g) : X -- \to Y \times Z$, and let $M_2 \subset Y \times Z$ be the image of $\overline{\Gamma_{f,g}}$ under the obvious projection. We have the following diagrams where all maps are projections, the left included in the right by restriction:
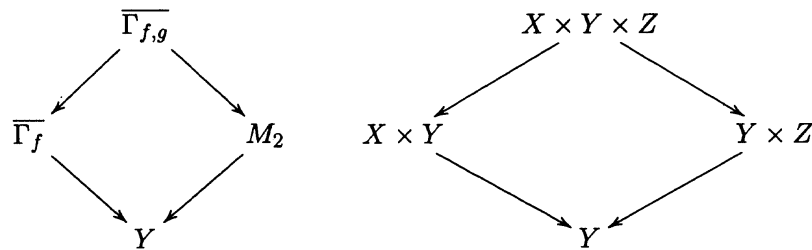
Note that $X, Y, \overline{\Gamma_f}, \overline{\Gamma_{f,g}}$, and M_2 are all birational.

Of particular interest is the case where $\overline{\Gamma_f} \to Y$ is a small morphism with exceptional locus W. Assume there exists a line bundle L on $\overline{\Gamma_f}$ with base scheme W and take $Z = \mathbb{P} H^0(\overline{\Gamma_f}, L)$. Then $\overline{\Gamma_f}$ and M_2 are isomorphic in codimension one. Furthermore, if $\overline{\Gamma_f}$ and M_2 are factorial varieties, then the image of L under the isomorphism $Pic\overline{\Gamma_f} \cong PicM_2$ is a globally generated line bundle on M_2.

We give an explicit construction of this birational transformation: take $\overline{\Gamma_f} \to Y$ above to be $\tilde{\varphi} : \widetilde{\mathbb{P}^n} \to \mathbb{P}^s$. As the exceptional locus of $\tilde{\varphi}$ is $\tilde{\Sigma}$, we find a line bundle on $\widetilde{\mathbb{P}^n}$ whose base scheme is $\tilde{\Sigma}$, identify explicitly the exceptional loci in the diagram, give explicit linear systems defining the morphisms and finally show that the space M_2 is smooth.

4.3 THE DIAGRAM OF EXCEPTIONAL LOCI

An examination of Thaddeus' construction [28, 3.11] suggests we identify a vector bundle $\mathcal{F}$ on $\mathcal{H}^2 X$ of rank $n - 2r - 1 = codim(\mathbb{P}(\mathcal{E}), \widetilde{\mathbb{P}^n})$ such that

$$\tilde{\varphi}^*(\mathcal{F}) \cong N^*_{\mathbb{P}(\mathcal{E})/\widetilde{\mathbb{P}^n}} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \tag{1}$$

and then construct one of the exceptional loci as $\mathbb{P}_{\mathcal{H}^2 X}(\mathcal{F})$. Writing $N_{\mathbb{P}(\mathcal{E})}(k) = N_{\mathbb{P}(\mathcal{E})/\widetilde{\mathbb{P}^n}} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(k)$, we verify $\mathcal{F} = \tilde{\varphi}_* N^*_{\mathbb{P}(\mathcal{E})}(-1)$ satisfies (1).

PROPOSITION 4.1. *Let $\tilde{Y} \cong \mathbb{P}^1$ be a fiber of $\tilde{\varphi} : \mathbb{P}(\mathcal{E}) \to \mathcal{H}^2 X$. Then $N_{\mathbb{P}(\mathcal{E})}(k) \otimes \mathcal{O}_{\tilde{Y}} \cong \oplus \mathcal{O}_{\tilde{Y}}(k-1)$ and it follows that*

$$\tilde{\varphi}^* \tilde{\varphi}_* N^*_{\mathbb{P}(\mathcal{E})}(-1) \cong N^*_{\mathbb{P}(\mathcal{E})}(-1).$$

Proof. Denoting tangent sheaves by Θ, it is easy to see that

$$(\pi^*\Theta_{\mathbb{P}^n}) \otimes \mathcal{O}_{\tilde{Y}} \cong \Theta_{\mathbb{P}^n} \otimes \mathcal{O}_Y \cong \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1).$$

Let F be the universal quotient bundle on the exceptional divisor E and $j : E \hookrightarrow \widetilde{\mathbb{P}^n}$ the inclusion. Because the intersection of $\tilde{Y}$ with E is a scheme of length two, pulling the exact sequence [14, 15.4] $0 \to \Theta_{\widetilde{\mathbb{P}^n}} \to \pi^*\Theta_{\mathbb{P}^n} \to j_*F \to 0$ back to $\tilde{Y}$ gives $\Theta_{\widetilde{\mathbb{P}^n}} \otimes \mathcal{O}_{\tilde{Y}}(k)$ ample for $k \geqslant 2$. The sequence

$$0 \to \Theta_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}_{\tilde{Y}} \to \Theta_{\widetilde{\mathbb{P}^n}} \otimes \mathcal{O}_{\tilde{Y}} \to N_{\mathbb{P}(\mathcal{E})/\widetilde{\mathbb{P}^n}} \otimes \mathcal{O}_{\tilde{Y}} \to 0$$

then gives $N_{\mathbb{P}(\mathcal{E})}(k) \otimes \mathcal{O}_{\tilde{Y}}$ ample for $k \geqslant 2$ because it is the quotient of an ample bundle [18, III.1.7]; hence $N_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{O}_{\tilde{Y}} \cong \oplus \mathcal{O}_{\tilde{Y}}(a_i)$ where the $a_i \geqslant -1$. A straightforward computation of the determinant via the isomorphism $\omega_{\mathbb{P}(\mathcal{E})} \cong \omega_{\widetilde{\mathbb{P}^n}} \otimes \Lambda^{n-2r-1} N_{\mathbb{P}(\mathcal{E})}$ shows $a_i = -1$, and the first part of the statement holds. The second follows immediately. $\qquad\square$

We construct the diagram of exceptional loci for the flip: Let

$$E_2' = \mathbb{P}_{\mathbb{P}(\mathcal{E})}\left(N^*_{\mathbb{P}(\mathcal{E})}(-1)\right) = \mathbb{P}_{\mathbb{P}(\mathcal{E})}(\tilde{\varphi}^*\mathcal{F})$$

and

$$E_2 = \mathbb{P}_{\mathbb{P}(\mathcal{E})}(N^*_{\mathbb{P}(\mathcal{E})}).$$

Hence, E_2 is the exceptional divisor of the blow up $\mathrm{Bl}_{\mathbb{P}(\mathcal{E})}(\widetilde{\mathbb{P}^n})$. As the vector bundles defining E_2 and E_2' differ by the twist of a line bundle, there is an isomorphism γ [19, II.7.9]

$$
\begin{array}{ccc}
E_2 & \xrightarrow{\ \gamma\ } & E_2' \\
& \searrow{\scriptstyle \pi} & \downarrow{\scriptstyle \pi'} \\
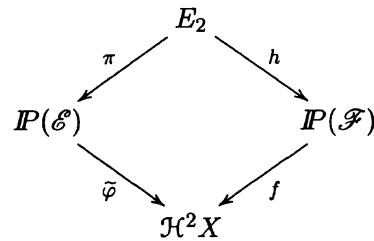& & \mathbb{P}(\mathcal{E})
\end{array}
$$

with the property that

$$\gamma^*(\mathcal{O}_{E_2'}(1)) \cong \mathcal{O}_{E_2}(1) \otimes \pi^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)). \tag{2}$$

Writing $\mathbb{P}(\mathcal{F}) = \mathbb{P}_{\mathcal{H}^2 X}(\mathcal{F})$, there is a morphism $E_2' \to \mathbb{P}(\mathcal{F})$ induced by the natural surjection $(\tilde{\varphi} \circ \pi')^*(\mathcal{F}) \to \mathcal{O}_{E_2'}(1) \to 0$. Via γ we get a morphism $h : E_2 \to \mathbb{P}(\mathcal{F})$ induced by the surjection (note (2))

$$(\tilde{\varphi} \circ \pi)^*(\mathcal{F}) \to \mathcal{O}_{E_2}(1) \otimes \pi^*\left(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)\right) \to 0$$

and, hence, a diagram

The isomorphism γ gives $E_2 \cong \mathbb{P}_{\mathbb{P}(\mathcal{E})}(\tilde{\varphi}^*\mathcal{F})$. Note the following symmetry property:

LEMMA 4.2. $E_2 \cong \mathbb{P}_{\mathbb{P}(\mathcal{F})}(f^*\mathcal{E})$.

Proof. To give a map $E_2 \rightarrow \mathbb{P}_{\mathbb{P}(\mathcal{F})}(f^*\mathcal{E})$ it is equivalent to give a surjection $h^*f^*\mathcal{E} \rightarrow \mathcal{K} \rightarrow 0$ for some line bundle $\mathcal{K}$ on E_2. By the above diagram, this is equivalent to a surjection $\pi^*\tilde{\varphi}^*\mathcal{E} \rightarrow \mathcal{K} \rightarrow 0$, which we obtain from the natural surjection $\tilde{\varphi}^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \rightarrow 0$ on $\mathbb{P}(\mathcal{E})$. As the fibers of h are isomorphic to $\mathbb{P}^1$, it is clear that the induced map is an isomorphism.   □

Let the very ample invertible sheaf M on $\mathcal{H}^2X \subset \mathbb{P}^s$ be the restriction of $\mathcal{O}_{\mathbb{P}^s}(1)$. Then for every k sufficiently large, $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes f^*M^k$ is very ample on $\mathbb{P}(\mathcal{F})$, [19], Ex. II.7.14], and so gives an embedding $i : \mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}^r$. The induced morphism $i \circ h : E_2 \rightarrow \mathbb{P}^r$ is given by a linear system associated to the line bundle:

$$\begin{aligned}
(i \circ h)^*(\mathcal{O}_{\mathbb{P}^r}(1)) &\cong h^*\big(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes f^*M^k\big) \\
&\cong h^*f^*M^k \otimes \mathcal{O}_{E_2}(1) \otimes \pi^*\big(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)\big).
\end{aligned} \tag{3}$$

Since $h_*\mathcal{O}_{E_2} = \mathcal{O}_{\mathbb{P}(\mathcal{F})}$ by Lemma 4.2, the projection formula yields:

$$\Gamma\big(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes f^*M^k\big) = \Gamma\big(E_2, h^*\big(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes f^*M^k\big)\big),$$

hence:

LEMMA 4.3. *The complete linear system* $|\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes f^*M^k|$ *on* $\mathbb{P}(\mathcal{F})$ *pulls back to the* complete *linear system* $|h^*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes f^*M^k)|$ *on* E_2.   □

4.4. THE TOTAL SPACES

We build the total spaces of the flip containing the diagram of exceptional loci, with those maps given by restriction. Three of the four spaces have been constructed already: $\widetilde{\mathbb{P}^n}$, $Im\tilde{\varphi}$, and $\text{Bl}_{\mathbb{P}(\mathcal{E})}(\widetilde{\mathbb{P}^n})$. We construct the fourth (and most interesting!) as the image of a linear system on $\text{Bl}_{\mathbb{P}(\mathcal{E})}(\widetilde{\mathbb{P}^n})$. This construction proceeds in several steps: First, we identify (4) an invertible sheaf on $\text{Bl}_{\mathbb{P}(\mathcal{E})}(\widetilde{\mathbb{P}^n})$ that restricts to $h^*\big(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes f^*M^k\big)$ on E_2 (Cf. Lemma 4.3). We then show that the associated complete linear system gives a birational morphism which is an embedding off E_2, and

that its restriction to E_2 is the complete linear system associated to $h^*\big(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \otimes f^* M^k\big)$.

Following the notation of [28], denote $\tilde{M}_2 = \mathrm{Bl}_{\mathbb{P}(\mathcal{E})}(\widetilde{\mathbb{P}^n})$. Writing

$$Pic\tilde{M}_2 = \mathbb{Z}H + \mathbb{Z}E_1 + \mathbb{Z}E_2 = \mathbb{Z}(\pi^*H) + \mathbb{Z}(\pi^*E) + \mathbb{Z}E_2$$

and noting $\mathcal{O}_{E_2}(-E_2) = \mathcal{O}_{E_2}(1)$, we have

$$\begin{aligned}
\mathcal{O}_{E_2}((2k-1)H - kE_1 - E_2) &\cong (f \circ h)^* M^k \otimes \mathcal{O}_{E_2}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \\
&\cong (i \circ h)^*(\mathcal{O}_{\mathbb{P}^r}(1)) \qquad\qquad (4)
\end{aligned}$$

by Equation (3).

PROPOSITION 4.4. *Let $\mathcal{L}$ be an invertible sheaf on a complete variety X, and let $\mathcal{B}$ be any locally free sheaf. Assume that the map $\lambda : X \to Y$ induced by $\mathcal{L}$ is a birational morphism and that λ is an isomorphism in a neighborhood of $p \in X$. Then for all n sufficiently large, the map*

$$H^0(X, \mathcal{B} \otimes \mathcal{L}^n) \to H^0(X, \mathcal{B} \otimes \mathcal{L}^n \otimes \mathcal{O}_p)$$

is surjective.

Proof. Push the exact sequence

$$0 \to \mathcal{B} \otimes \mathcal{L}^n \otimes \mathcal{I}_p \to \mathcal{B} \otimes \mathcal{L}^n \to \mathcal{B} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \to 0$$

down to Y. Because λ is an isomorphism in a neighborhood of p, the map

$$\lambda_*(\mathcal{B} \otimes \mathcal{L}^n \otimes \mathcal{O}_p) \to R^1 \lambda_*(\mathcal{B} \otimes \mathcal{L}^n \otimes \mathcal{I}_p)$$

is the zero map, hence there is an exact sequence on Y:

$$0 \to \lambda_*(\mathcal{B} \otimes \mathcal{I}_p)(n) \to \lambda_*(\mathcal{B})(n) \to \lambda_*(\mathcal{B})(n) \otimes \mathcal{O}_p \to 0,$$

where $\mathcal{L} \simeq \lambda^* \mathcal{O}_Y(1)$. Since $\mathcal{O}_Y(1)$ is (very) ample, $H^1(Y, \lambda_*(\mathcal{B} \otimes \mathcal{I}_p)(n)) = 0$ for all n sufficiently large. Therefore there is a section of $\lambda_*(\mathcal{B})(n)$ that does not vanish at $p \in Y$ which can be pulled back to a section of $\mathcal{B} \otimes \mathcal{L}^n$ that does not vanish at $p \in X$. □

Remark 4.5. Proposition 4.4 should be thought of as an analogue of the statement that if $\mathcal{L}$ is an ample line bundle, then $\mathcal{L}^n \otimes \mathcal{B}$ is globally generated for all $n \gg 0$. □

COROLLARY 4.6. *For k sufficiently large, the set theoretic base locus of the linear system $|(2k-1)H - kE|$ on $\widetilde{\mathbb{P}^n}$ is $\mathbb{P}(\mathcal{E})$.*

Proof. Clearly, the base locus of $|(2k-1)H - kE|$ contains $\mathbb{P}(\mathcal{E})$. Note, however, that as $|2H - E|$ is base point free, the base locus of $|(2k-1)H - kE|$ will stabilize for k sufficiently large. Hence it suffices to show that if p is a point not in $\mathbb{P}(\mathcal{E})$, then $|(2k-1)H - kE|$ is free at p for all $k \gg 0$. Now take $\mathcal{B} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(-H)$ and $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}^n}}(2H - E)$ in Proposition 4.4, and use Theorem 2.10. □

Notation 4.7. For the rest of this section, write $\mathcal{O}(a, b, c)$ for $\mathcal{O}_{\tilde{M}_2}(aH + bE_1 + cE_2)$, and write $\mathcal{L}_k = \mathcal{O}_{\tilde{M}_2}((2k-1)H - kE_1 - E_2)$, $k \in \mathbb{Q}$.

LEMMA 4.8. *$\mathcal{L}_k$ is nef for all k sufficiently large.*

Proof. Letting $C \subset \tilde{M}_2$ be an irreducible curve not contained in E_2, we have $\mathcal{L}_k \cdot C \geqslant 0$ for $k \gg 0$ by Corollary 4.6. Letting $C' \subset E_2$, $\mathcal{L}_k \otimes \mathcal{O}_{E_2}$ is globally generated on E_2 for $k \gg 0$ by (4), hence $\mathcal{L}_k \cdot C' \geqslant 0$ and $\mathcal{L}_k$ is nef. ☐

PROPOSITION 4.9. *With hypotheses as above and for k sufficiently large, the rational map on $\tilde{M}_2$ induced by the linear system $|\mathcal{L}_k|$ is a morphism, is an embedding off of E_2, and its restriction to E_2 is the morphism $h : E_2 \to \mathbb{P}(\mathcal{F})$.*

Proof. We first show that for k sufficiently large $|\mathcal{L}_k|$ restricts to the complete linear system on E_2 associated to the invertible sheaf $h^*f^*M^k \otimes \mathcal{O}_{E_2}(1) \otimes \pi^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))$ (Cf. Lemma 4.3). For this it suffices to prove

$$H^1(\tilde{M}_2, \mathcal{O}(2k-1, -k, -2)) = 0.$$

Writing $\mathcal{B} = \mathcal{O}(2k-1, -k, -2)$ and noting

$$K_{\tilde{M}_2} = \mathcal{O}(-n-1, n-r-1, n-2r-2):$$
$$\mathcal{B} \otimes K_{\tilde{M}_2}^{-1} \cong \mathcal{O}(2k+n, -k-n+r+1, -n+2r).$$

Let $\alpha = (k+n-r-1)/(n-2r)$ and rewrite the right side as $\mathcal{L}_\alpha^{n-2r} \otimes \mathcal{O}(2, 0, 0)$. For $k \gg 0$, $\mathcal{L}_\alpha$ is a nef $\mathbb{Q}$-divisor by Lemma 4.8, hence $\mathcal{B} \otimes K_{\tilde{M}_2}^{-1}$ is big and nef and the vanishing holds by the Kawamata–Viehweg vanishing theorem (note that by [25, 1.9], a nef line bundle tensored with a big and nef line bundle is again big and nef).

To see $|\mathcal{L}_k|$ is a morphism, note that by Corollary 4.6, the support of the base scheme is contained in E_2. By what has just been proven, however, $|\mathcal{L}_k|$ has no base points since the complete linear system on E_2 associated to $h^*f^*M^k \otimes \mathcal{O}_{E_2}(1) \otimes \pi^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))$ induces a morphism. Further, as $\mathcal{O}(2, -1, 0)$ induces an embedding off of E_2, $|\mathcal{L}_k|$ does as well for $k \gg 0$. ☐

Remark 4.10. It is unfortunate that this proof gives no bound on k; however, there is an important case when the value of k can be determined. Specifically, if $\Sigma \subset \mathbb{P}^n$ is scheme theoretically defined by cubics, then the line bundle $\mathcal{L}_2$ will be base point free and $k = 3$ suffices for Proposition 4.9. ☐

We simply write the morphism from Proposition 4.9 as $h : \tilde{M}_2 \to \mathbb{P}(|\mathcal{L}_k|)$. Denote by M_2 the image variety of h. Then $M_2 \setminus \mathbb{P}(\mathcal{F}) \cong \widetilde{\mathbb{P}^n} \setminus \mathbb{P}(\mathcal{E})$ by Proposition 4.9.

PROPOSITION 4.11. *M_2 is smooth.*

Proof. Note first that M_2 is normal since it is the image under a morphism of a normal variety with reduced, connected fibers.

Let $Z = h^{-1}(p) \cong \mathbb{P}^1$, where $p \in \mathbb{P}_{\mathcal{H}^2 X}(\mathcal{F})$. We have the normal bundle sequence:

$$0 \to N_{Z/E_2} \to N_{Z/\widetilde{M}_2} \to \mathcal{O}_{E_2}(E_2) \otimes \mathcal{O}_Z \to 0.$$

Because Z is a fiber of the $\mathbb{P}^1 \times \mathbb{P}^{n-2r-1}$-bundle E_2 over $\mathcal{H}^2 X$ by Proposition 4.9, this sequence becomes

$$0 \to \bigoplus_1^{n-2} \mathcal{O}_{\mathbb{P}^1} \to N_{Z/\widetilde{M}_2} \to \mathcal{O}_{\mathbb{P}^1}(-1) \to 0.$$

This sequence clearly splits and we apply a natural extension of the smoothness portion of Castelnuovo's contractibility criterion for surfaces (Cf. [1, 2.4]). □

Because $\widetilde{\mathbb{P}^n}$ and M_2 are smooth and isomorphic in codimension one, we have [19, II.6.5]

$$Pic\,\widetilde{\mathbb{P}^n} \cong Pic\,M_2 \quad \text{and} \quad H^0\left(\widetilde{\mathbb{P}^n}, \mathcal{O}_{\widetilde{\mathbb{P}^n}}(2H - E)\right) \cong H^0\left(M_2, \mathcal{O}_{M_2}(2H - E)\right).$$

Therefore, the line bundle $\mathcal{O}_{M_2}(2H - E)$ induces a morphism $f: M_2 \to \mathbb{P}^s$ which is an embedding off of $\mathbb{P}(\mathcal{F})$. Furthermore, because M_2 and $\mathbb{P}(\mathcal{F})$ are smooth, [13, 1.1] implies that $h : \tilde{M}_2 \to M_2$ is the blow up of M_2 along $\mathbb{P}(\mathcal{F})$.

Collecting these results:

THEOREM 4.12. *Let (X, V) satisfy (K_2) and assume $X \subset \mathbb{P}^n$ is smooth, irreducible, and contains no lines or conics. Then there is a flip as pictured below with:*

(1) $\widetilde{\mathbb{P}^n}$, $\tilde{M}_2$, *and* M_2 *smooth.*
(2) $\widetilde{\mathbb{P}^n} \setminus \mathbb{P}(\mathcal{E}) \cong M_2 \setminus \mathbb{P}(\mathcal{F})$.
(3) *h is the blow up of M_2 along $\mathbb{P}(\mathcal{F})$.*
(4) *π is the blow up of $\widetilde{\mathbb{P}^n}$ along $\mathbb{P}(\mathcal{E})$.*
(5) *f, induced by $\mathcal{O}_{M_2}(2H - E)$, is an embedding off of $\mathbb{P}(\mathcal{F})$, and the restriction of f is the projection $\mathbb{P}(\mathcal{F}) \to \mathcal{H}^2 X$.*
(6) *$\tilde{\varphi}$, induced by $\mathcal{O}_{\widetilde{\mathbb{P}^n}}(2H - E)$, is an embedding off of $\mathbb{P}(\mathcal{E})$, and the restriction of $\tilde{\varphi}$ is the projection $\mathbb{P}(\mathcal{E}) \to \mathcal{H}^2 X$:*

□
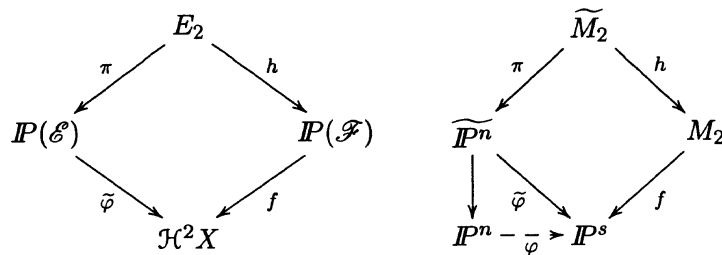
Note that the ample cone of the space M_2 is bounded on one side by $\mathcal{O}_{M_2}(2H - E)$ and on the other by a line bundle of the form $\mathcal{O}_{M_2}((2k - 1)H - kE)$, $k \geqslant 2$, hence the assertion that this construction yields a flip. In fact, as $\mathcal{O}_{M_2}(3H - 2E)$ is f-ample, this is a $(K + D)$-flip (in the sense of [24, 3.33]) where $D = (n + 4)H - (n - r + 1)E$. When $r = 1$ it is shown in [7] that D is log canonical, i.e. that this is a log flip.

Remark 4.13 (on Conics). The hypothesis that X contain no conics is not always needed in order to construct this flip. The condition is imposed simply to keep the fibers of $\tilde{\varphi} : \tilde{\Sigma} \to \mathbb{P}^s$ equidimensional; problems occur when a variety has only a few conics. In the case of quadratic Veronese embeddings where there is a unique plane quadric through any two points, it is not difficult to modify the results of the previous sections to achieve similar results. We do not do this, however, as this case is already understood from the point of view of complete quadrics [29, 30].□

EXAMPLE 4.14. It is interesting to examine Theorem 4.12 in cases where $\mathcal{H}^2 X$ is well understood. For example, in the case $X = \mathbb{P}^r$, it is easy to see that $\mathcal{H}^2 X$ is itself a $\mathbb{P}^2$-bundle over the Grassmannian $\mathbb{G}(1, r)$ of lines in $\mathbb{P}^r$. If we embed $\mathbb{P}^r$ via $\mathcal{O}(d)$, $d \geqslant 3$, $\tilde{\Sigma}$ has the particularly nice structure of a $\mathbb{P}^1 \times \mathbb{P}^2$-bundle over $\mathbb{G}(1, r)$ (in the case $d = 2$, it is simply a $\mathbb{P}^2$-bundle; the missing factor of $\mathbb{P}^1$ is due precisely to the deficiency of Σ).

Furthermore, as the ideal of Σ is generated by cubics [23], the line bundle $\mathcal{O}_{M_2}(3H - 2E)$ is globally generated (see the discussion below). Applying Kawamata-Viehweg vanishing yields

$$H^i(M_2, \mathcal{O}(kH - aE)) = 0, i > 0, k > \tfrac{3}{2}(e + a - 1) - (n + 1),$$

where $n + 1 = \binom{r+d}{r}$ (compare Corollary 2.12). In the special case $k = 2a - 1$, this vanishing can be pulled directly back to $\widetilde{\mathbb{P}^n}$. More generally:

COROLLARY 4.15. *With hypotheses as in Theorem 4.12, assume further that* $\Sigma \subset \mathbb{P}^n$ *is not a hypersurface and is cut out as a scheme by cubics. Then* $H^i(\mathbb{P}^n, \mathcal{I}_X^a(2a - 1)) = 0, i > 0, a > n - 3r - 1.$ □

The continuation of this process following [28] is taken up in [32]. We need to construct a birational morphism $\tilde{\varphi}_2 : M_2 \to \mathbb{P}^{s_2}$ which contracts the image of 3-secant 2-planes to points, and is an embedding off their union. The natural candidate for this morphism is the linear system associated to $\mathcal{O}_{M_2}(3H - 2E)$, where we identify $Pic\widetilde{\mathbb{P}^n} \cong PicM_2$. Noting the fact that $h^*\mathcal{O}_{M_2}(3H - 2E) = \mathcal{O}_{\widetilde{M_2}}(3H - 2E_1 - E_2)$, it is not difficult to see (using Zariski's Main Theorem) that this system will be globally generated if $\Sigma \subset \mathbb{P}^n$ is scheme theoretically defined by cubics.

There are not yet theorems analogous to those for the quadric generation of varieties. However, there is evidence that such statements should exist (Cf. [23] where it is shown that $Sec(v_d(\mathbb{P}^n))$ is ideal theoretically defined by cubics for all d, n).

Furthermore, the author proves set theoretic statements for arbitrary smooth varieties in [32]. These statements also contain information about the syzygies among the generators that makes it possible to study the map $\tilde{\varphi}_2$ in much the same way as $\tilde{\varphi}$ was studied above.

Acknowledgements

References

1. Andreatta, M. and Wiśniewski, J. A.: A view on contractions of higher-dimensional varieties, In: *Algebraic Geometry (Santa Cruz 1995)*, Proc. Sympos. Pure Math. 62, Part I, Amer. Math. Soc., Providence, RI, 1997, pp. 153–183.
2. Arbarello, E., Cornalba, M., Griffiths, P. A. and Harris, J.: *Geomtry of Algebraic Curves*, Vol. I, Springer-Verlag, New York, 1985.
3. Beltrametti, M., Howard, A., Schneider, M. and Sommese, A.: Projections from Subvarieties, Preprint, math.AG/9804048.
4. Bertram, A.: Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective space, *J. Differential Geom.* **35** (1992), 429–469.
5. Bertram, A.: Complete extensions and their map to moduli space, In: *Complex Projective Geometry*, London Math. Soc. Lecture Note Ser. 179, Cambridge Univ. Press, Cambridge, 1992, pp. 81–91.
6. Bertram, A.: An application of general Kodaira vanishing to embedded projective varieties, Preprint, alg-geom/9707001.
7. Bertram, A: Stable pairs and log flips, In: *Algebraic Geometry (Santa Cruz 1995)*, Proc. Sympos. Pure Math. 62, Part 1, Amer. Math. Soc., Providence, RI, 1997, pp. 185–201.
8. Bertram, A., Ein, L. and Lazarsfeld, R.: Vanishing theorems, a theorem of Severi, and the equations defining projective varieties, *J. Amer. Math. Soc.* **4**(3) (1991), 587–602.
9. Catanese, F. and Göttsche, L.: *d*-very-ample line bundles and embeddings of Hilbert schemes of 0-cycles, *Manuscripta Math.* **68** (1990), 337–341.
10. Crauder, B. and Katz, S.: Cremona transformations with smooth irreducible fundamental locus, *Amer. J. Math.* **111** (1989), 289–307.
11. Crauder, B. and Katz, S.: Cremona transformations and Hartshorne's conjecture, *Amer. J. Math.* **113** (1991), 269–285.
12. Ein, L. and Lazarsfeld, R.: Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimensions, *Invent. Math.* **111** (1993), 51–67.
13. Ein, L. and Shepherd-Barron, N.: Some special Cremona transformations, *Amer. J. Math.* **111** (1989), 783–800.

14. Fulton, W.: *Intersection Theory*, Ergeb. Math. Grenzgeb. (3) 2, Springer, Berlin, 1984, 1998.
15. Gimigliano, A. and Lorenzini, A.: Blowing up determinantal space curves and varieties of codimension 2, *Comm. Algebra* **27**(3) (1999), 1141–1164.
16. Göttsche, L.: *Hilbert Schemes of Zero-Dimensional Subschemes of Smooth Varieties*, Lecture Notes in Math. 1572, Springer, New York, 1994.
17. Green, M.: Koszul cohomology and the geometry of projective varieties, *J. Differential Geom.* **19** (1984), 125–171.
18. Hartshorne, R.: *Ample Subvarieties of Algebraic Varietes*, Lecture Notes in Math. 156, Springer, New York, 1970.
19. Hartshorne, R.: *Algebraic Geometry*, Springer, New York, 1977.
20. Hulek, K. Katz, S. and Schreyer, F. O.: Cremona transformations and syzygies, *Math. Z.* **209**(1) (1992), 419–443.
21. Hulek, K. and Oxbury, W.: personal communication.
22. Inamdar, S. P.: On syzgies of projective varieties, *Pacific J. Math.* **177**(1) (1997), 71–76.
23. Kanev, V.: Chordal varieties of Veronese varieties and catalecticant matrices, algebraic geometry, 9, *J. Math. Sci. (New York)* **94**(1) (1999), 1114–1125.
24. Kollár, J. and Mori, S.: *Birational Geometry of Algebraic Varieties* Cambridge Univ. Press, Cambridge, 1998.
25. Mori, S.: *Classification of Higher Dimensional Varieties*, Proc. Sympos. Pure Math. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331.
26. Schreyer, F. O.: A standard basis approach to syzygies of canonical curves, *J. Reine Angew. Math.* **421** (1991), 83–123.
27. Schwarzenberger, R. L. E.: The secant bundle of a projective variety, *Proc. London Math. Soc.* **14** (1964), 369–384.
28. Thaddeus, M.: Stable pairs, linear systems, and Verlinde formula, *Invent. Math.* **117** (1994), 317–353.
29. Thaddeus, M.:Complete collineations revisited, Preprint, math. AG/9808114.
30. Vainsencher, I.: Schubert calculus for complete quadrics, In: P. Le Barz and Y. Hervier (eds), *Geomtry and Classical Algebraic Geometry*, Birkhäuser, Basel, 1982, pp. 199–236.
31. Vermeire, P.: PhD thesis, Univ. of North Carolina, 1998.
32. Vermeire, P.: Secant varieties and birational geometry, Preprint, math.Ag/9911078.
33. Vermeire, P.: On the regularity of powers of ideal sheaves, Preprint, math.AG/9911171.
34. Voisin, C.: Courbes titragonales et cohomologie de Koszul, *J. Reine angew. Math.* **387** (1988), 111–121.
35. Wahl, J.: On cohomology of the square of an ideal sheaf, *J. Algebra Geom.* **6** (1977), 481–511.