

SOME RESULTS ON THE CENTER OF A RING WITH POLYNOMIAL IDENTITY

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Introduction. The purpose of this paper is to provide a fresh outlook to various questions on rings with polynomial identity by examining the centers of such rings. This approach yields the interesting result that any nonzero ideal of a semiprime ring with polynomial identity intersects the center nontrivially (Theorem 2).

There are at least two interesting consequences to Theorem 2: a generalization of Wedderburn's theorem (any semiprimitive ring with polynomial identity, whose center is a field, is simple) and a strengthening of Posner's theorem [1] (any prime ring with a polynomial identity has a simple ring of quotients whose center is the quotient field of the center of the prime ring).

The proofs are elementary modulo Jacobson [3]. Of course rings are not necessarily commutative and for the sake of simplicity we assume a unit 1.

The key argument in this paper is an application of Formanek's central polynomials for matrix algebras over a field, whose important properties are [2]: Let M_n be an $n \times n$ matrix algebra over an arbitrary field. Then there exists a polynomial $g_n(X_1, \dots, X_m)$ which has coefficients in \mathbf{Z} ; is homogeneous (degree > 0) in every variable and linear in all but the first variable; takes values in the center for every specialization in M_n ; and is nonvanishing for some specialization.

LEMMA 1. $g_n(X_1, \dots, X_m)$ is central, nonvanishing for any central simple algebra S of degree n over its center C .

PROOF. Let us first consider C finite. Since by Wedderburn's structure theorem S is a matrix algebra over a division ring D which is finite dimensional over C , which is finite, we have D is finite and thus a field (Wedderburn's theorem on finite division rings [3, p. 183]). Thus $D = C$ and S is in fact a matrix algebra over C , a field, and g_n is by hypothesis a central, nonvanishing polynomial for S , so that there is nothing to prove.

So we may assume C is infinite. Again let S be a matrix algebra over D , a division ring finite dimensional over C . Let F be a splitting subfield

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of D [3, p. 120]. Then $S \otimes_C F = M_n(F)$, for which g_n is central and nonvanishing, so $g_n X_{m+1} - X_{m+1} g_n$ is a polynomial identity for S (since C is a field, $S \hookrightarrow S \otimes_C F$ by $s \rightarrow s \otimes 1$, so $g_n X_{m+1} - X_{m+1} g_n$, which vanishes on $M_n(F)$, vanishes on the subalgebra S). Thus, g_n takes values in the center of S . Is g_n nonvanishing? If g_n were a polynomial identity for S , it would be a polynomial identity for $M_n(F) = S \otimes_C F$ since C is infinite [3, p. 231], so g_n would vanish for all specializations in $M_n(F)$, false. So g_n does not vanish for some specializations in S , and we are done.

THEOREM 1. *Let R be semiprimitive (= semisimple in the sense of Jacobson [3]) with center C and polynomial identity f . Let A be any ideal $\neq 0$ of R . Then $A \cap C \neq 0$.*

PROOF. As a semiprimitive ring, R is a subdirect product of primitive rings S_i , each of which is a homomorphic image of R . Thus, S_i satisfies the same polynomial identity f and is therefore simple of dimension $\leq [d/2]^2$ over its center, which we shall call C_i (by Kaplansky's Theorem, [3, p. 226]).

Since $\bigcap_i \ker p_i = 0$ where p_i is the canonical projection $p_i: R \rightarrow S_i$, we must have the restriction of p_i to A is nonzero for some i . Let $I = \{i: p_i \text{ restricted to } A \text{ is nonzero}\}$, i.e. for $i \notin I$ the i th component of any element of A is 0. Since $p_i: R \rightarrow S_i$ is surjective, $p_i(A)$ is a nonzero ideal of S_i for $i \in I$. But S_i is simple. Therefore, $\forall i \in I$, p_i restricted to A is surjective.

Now let us choose $i_0 \in I$ such that S_{i_0} has maximal dimension n_0^2 over its center $\forall i \in I$ (the dimension is bounded by $[d/2]^2$ by Kaplansky's theorem [3, p. 226]). Then let g_{n_0} be the nonvanishing central polynomial of Formanek, described in Lemma 1. Since g_{n_0} is central for M_{n_0} , g_{n_0} is central (Procesi has shown indeed vanishing) for smaller-degree matrix algebras, hence for all smaller-degree central simple algebras, by the argument of Lemma 1. So g_{n_0} is central for S_i , $\forall i \in I$.

Let $s_1, \dots, s_m \in S_{i_0}$ such that $g_{n_0}(s_1, \dots, s_m) \neq 0$. Let $a_1, \dots, a_m \in A$ such that $p_{i_0}(a_j) = s_j$, $1 \leq j \leq m$. Since g_{n_0} is homogeneous of degree > 0 , its constant term is 0, and

$$\forall i \notin I, \quad p_i(g_{n_0}(a_1, \dots, a_m)) = g_{n_0}(p_i(a_1), \dots, p_i(a_m)) = g_{n_0}(0, \dots, 0) = 0.$$

$$\forall i \in I, \quad p_i(g_{n_0}(a_1, \dots, a_m)) = g_{n_0}(p_i(a_1), \dots, p_i(a_m)) \in C_i$$

and in particular

$$\begin{aligned} p_{i_0}(g_{n_0}(a_1, \dots, a_m)) &= g_{n_0}(p_{i_0}(a_1), \dots, p_{i_0}(a_m)) \\ &= g_{n_0}(s_1, \dots, s_m) \neq 0 \quad \text{but } \in C_{i_0}. \end{aligned}$$

So $g_{n_0}(a_1, \dots, a_m) \in C$ and is nonzero. But $g_{n_0}(a_1, \dots, a_m) \in A$, so $A \cap C \neq 0$. Q.E.D.

COROLLARY. *Let R be semiprimitive with center C , a field, and with polynomial identity f . Then R is simple.*

PROOF. If A is a nonzero ideal of R , $A \cap C \neq 0$ by Theorem 1. But C is a field, so $A = R$. Q.E.D.

It is clear that the conditions of the corollary are necessary; appropriate examples of their necessity are the nonsimple rings \mathbf{Z} , the ring of integers (center is not a field), and the ring of linear transformations over an infinite-dimensional vector space over a division ring (primitive, center is a field, but has no polynomial identity).

We now generalize Theorem 1:

THEOREM 2. *Let R be semiprime with center C and polynomial identity f . Let A be any ideal $\neq 0$ of R . Then $A \cap C \neq 0$.*

PROOF. Since a semiprime ring with polynomial identity has no nil ideals (an immediate consequence of [3, p. 232, Theorem 1]), $R[\lambda]$ is semiprimitive [3, p. 12], where $R[\lambda]$ is the ring of polynomials in the commuting indeterminate λ , with coefficients in R (these reductions have been made standard by Amitsur).

We note that the center of $R[\lambda]$ is actually $C[\lambda]$, since if $c(\lambda) = \sum r_i \lambda^i$, $r_i \in R$, then $\forall r \in R$, $c(\lambda)r = rc(\lambda) \Rightarrow \sum (r_i r - r r_i) \lambda^i = 0 \Rightarrow r_i \in C$, so $c(\lambda) \in C[\lambda]$.

Now A is an ideal of R , so $A[\lambda]$ is an ideal of $R[\lambda]$. $R[\lambda]$ is semiprimitive, so by Theorem 1 $A[\lambda] \cap C[\lambda] \neq 0$. Comparing coefficients we get $A \cap C \neq 0$. Q.E.D.

COROLLARY 1. (*Strengthening of Posner's theorem*). *Let R be a prime ring with center C and polynomial identity. Then there exists a simple ring $S = R\tilde{C}$, where \tilde{C} is the quotient field of C , which is a ring of quotients of R and is finite dimensional over \tilde{C} (this is a statement of Posner's theorem [1]). Moreover, the center of S is \tilde{C} (not part of the previous formulations of Posner's theorem).*

PROOF. Since R is a prime, its center C has no zero-divisors. So we define S formally as $\{rc^{-1}, r \in R, c \neq 0 \in C \mid r_1 c_1^{-1} = r_2 c_2^{-1} \text{ if and only if } r_1 c_2 = r_2 c_1\}$. Then multiplication and addition are set in the obvious ways:

$$(r_1 c_1^{-1})(r_2 c_2^{-1}) = (r_1 r_2)(c_1 c_2)^{-1}$$

and

$$r_1c_1^{-1} + r_2c_2^{-1} = (r_1c_1 + r_2c_1)(c_1c_2)^{-1}.$$

All the necessary properties of multiplication and addition in S are natural and immediate. Obviously $S = R\tilde{C}$, and we claim $\tilde{C} = \text{cent } S$:

It is obvious that $\tilde{C} \subseteq \text{cent } S$. To go the other way, let $rc^{-1} \in \text{cent } S$. Then $(rc^{-1})c \in \text{cent } S$, so $r \in R \cap \text{cent } S \subseteq C$. So $rc^{-1} \in \tilde{C}$ and $\text{cent } S = \tilde{C}$, a field.

Clearly S is a prime: Suppose $s_1s_2 = 0$, $s_1, s_2 \in S$. Then $s_1(rc^{-1})s = 0$, $\forall r \in R, c \neq 0 \in C$. Let $s_1 = r_1c_1^{-1}$, $s_2 = r_2c_2^{-1}$, $r_i \in R$, $i = 1, 2$, $c_i \in C$, $i = 1, 2$. Then $0 = r_1c_1^{-1}rc^{-1}r_2c_2^{-1} = (r_1rr_2)(c_2cc_1)^{-1}$, $\forall r \in R, c \in C$. So $r_1rr_2 = 0$, $\forall r \in R$. But R is prime, so either r_1 or r_2 is 0, so either s_1 or s_2 is 0.

So S is prime and $S = R\tilde{C}$ satisfies a polynomial identity. Therefore by Theorem 2, if A is a nonzero ideal of S , $A \cap \tilde{C} \neq 0$. But \tilde{C} is a field, so $A = S$ and S is simple. S is finite dimensional over \tilde{C} by Kaplansky's theorem. Q.E.D.

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ADDENDUM. Subsequent to the submission of this note for publication, several facts have come to my attention. The most significant fact is that the statement of Corollary 1 of Theorem 2 (the strengthening of Posner's theorem) has been discovered also by Procesi, Formanek, Martindale, and Small, so this result cannot be considered new. What is new is the straightforward and direct proof, relying only on the existence of central polynomials for matrix rings. Using the stronger version of Posner's theorem, Small (and perhaps others) proved the special case of Theorem 2 for R prime.

Martindale has observed to me that the results of this paper remain true even if one does not assume R contains unity; this assumption is not essential to any of the proofs. At any rate, one could always adjoin unity to a semiprime ring with polynomial identity to obtain a semiprime ring with polynomial identity and with unity, thereby extending Theorem 2 to the analogous result for rings without unity. It follows clearly that any semiprime ring with polynomial identity has a nontrivial center. That semisimple rings with polynomial identity have nontrivial centers, was first shown by Procesi (remark 5 of [2]). Using standard methods of embedding semiprime rings with polynomial identity into semisimple rings with polynomial identity (as indicated in the proof of Theorem 2), can one extend Procesi's result to semiprime rings with polynomial identity, as was observed by Small. This result is much weaker than Theorem 2, however, because Theorem 2 fails for rings which satisfy precisely the same polynomial identities as $n \times n$ matrix rings; it is easy to see that these rings have nontrivial centers.

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