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SOME RESULTS ON THE LARGEST AND LEAST EIGENVALUES OF GRAPHS*

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Abstract. Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. In this paper, first some sharp upper and lower bounds on the largest and least eigenvalues of graphs are given when vertices are removed. Some conjectures in [M. Aouchiche. *Comparaison Automatisée d'Invariants en Théorie des Graphes.* Ph.D. Thesis, École Polytechnique de Montréal, February 2006.] and [M. Aouchiche, G. Caporossi, and P. Hansen. Variable neighborhood search for extremal graphs, 20. Automated comparison of graph invariants. *MATCH Commun. Math. Comput. Chem.*, 58:365–384, 2007.] involving the spectral radius, diameter and matching number are also proved. Furthermore, the extremal graph which attains the minimum least eigenvalue among all quasi-tree graphs is characterized.

Key words. Spectral radius, Diameter, Matching number, Least eigenvalue, Quasi-tree graph.

AMS subject classifications. 05C50.

1. Introduction. Throughout this paper, we consider only simple graphs, herein called just graphs. Let $G = (V, E)$ be a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The *distance* between vertices u and v is denoted by $d(u, v)$. The *diameter* of a graph is the maximal distance between any two vertices. The *adjacency matrix* of a graph G is denoted by $A(G)$ and defined as the $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise. Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The largest eigenvalue $\lambda_1(G)$ or $\rho(G)$ is called the *spectral radius* or the *index* of G. By Perron-Frobenius Theorem, $\lambda_1(G)$ is simple and has a unique positive unit eigenvector corresponding to it. We will refer to such an eigenvector as the *Perron vector* of G. It is known that $\lambda_n(G) = -\lambda_1(G)$ for a bipartite graph G (see [7]). A unit eigenvector corresponding to $\lambda_n(G)$ is called a *least vector* of G.

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A *kite* $Ki_{n,\omega}$ is the graph obtained from a complete graph K_ω and a path $P_{n-\omega}$ by adding an edge between a vertex of K_ω and an end point of $P_{n-\omega}$. A *matching* in a graph is a set of disjoint edges, and the maximum cardinality of a matching over all possible matching in a graph G is called the *matching number* of G, denoted by μ.

In a susceptible-infectious-susceptible (SIS) type of network infection, the steady-state infection of the network is determined by a phase transition at the epidemic threshold $\tau_c = \frac{1}{\lambda_1}$: When the effective infection rate $\tau > \tau_c$, the network is infected, whereas below τ_c, the network is virus free. Motivated by a $\frac{1}{\lambda_1(A)}$ threshold separating two different phases of a dynamic process on a network, we want to change the network in order to enlarge the network's epidemic threshold τ_c, or, equivalently, to lower $\lambda_1(A)$. We are searching for a strategy so that, after removing k vertices, $\lambda_1(A)$ is minimal. Recently, Li et al. [13] presented a lower bound for the spectral radius of a graph in which some vertices are removed, and Mieghem et al. [16] gave lower and upper bounds for the spectral radius of a graph when some edges are removed. Naturally, Xing and Zhou [23] established an upper bound for the least eigenvalue of a graph when some vertices are removed using the components of the least vector. Furthermore, the authors [23] also gave lower and upper bounds for the least eigenvalue of a graph when some edges are removed. In Section 2 of this paper, we consider the case of connected graphs, and present an incomparable sharp lower bound for the spectral radius of a graph and an incomparable sharp upper bound for the least eigenvalue of a graph when some vertices are removed.

In [1, 2], Aouchiche et al. gave the following conjectures involving index, diameter and matching number of G (see also [3]).

Conjecture 1.1 ([4]). Let G be a connected graph with diameter D. Then

$$\lambda_1(G) + D \le n - 1 + 2\cos\frac{\pi}{n+1},$$

and equality holds if and only if $G \cong P_n$.

Conjecture 1.2 ([1, 2, 3]). Let G be a connected graph with matching number μ. Then

$$\lambda_1(G) - \mu \le n - 1 - \lfloor n/2 \rfloor,$$

and equality holds if and only if $G \cong K_n$.

Conjecture 1.3 ([1, 2, 3]). Let G be a connected graph with matching number μ. Then

$$\frac{\lambda_1(G)}{\mu} \le \sqrt{n-1},$$

and equality holds if and only if $G \cong K_{1,n-1}$.

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D. Stevanović [20] proved Conjectures 1.2 and 1.3. However, we observed that the extremal graphs in the statement of Conjecture 1.3 are not complete. Moreover, Stevanovic's theorem is also missing some extremal graphs. When $n = 5$, K_5 is also the extremal graph in Conjecture 1.3 which is not considered in [20]. In Section 3 of this paper, we show that Conjecture 1.1 is right, and Conjectures 1.2 and 1.3 still hold when removing the condition that G is connected.

Recently, researchers have paid much attention to the least eigenvalues of graphs with a given value of graph invariant, for instance: order and size [5, 6, 10, 19], unicyclic graphs with a given number of pendant vertices [15], matching number and independence number [21], number of cut vertices [22], connectivity [24], domination number [25]. A connected graph G is called a *quasi-tree graph* if there exists $v \in V(G)$ such that $G-v$ is a tree. H. Liu and M. Lu [14] determined the maximal and the second maximal spectral radii among all quasi-tree graphs. In section 4, we characterize the extremal graph which attains the minimum least eigenvalue among all quasi-tree graphs.

2. On the largest and least eigenvalues of graphs when vertices are removed.

THEOREM 2.1. *Let G be a connected graph with $V(G) = V_1 \cup V_2 = \{v_1, \ldots, v_n\}$ where $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_n\}$, and $G[V_i]$ be the induced subgraphs of G for $i = 1, 2$. Suppose that A and A_i are the adjacency matrices of G and $G[V_i]$ for $i = 1, 2$, respectively. Let $X = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^t$ be the Perron vector of G, where x_i corresponds to v_i for $i = 1, \ldots, n$. Then*

$$\lambda_1(A_1) \geq \lambda_1(A) - \frac{\sum_{v_iv_j \in E[V_1,V_2]} x_ix_j}{\sum_{v_i \in V_1} x_i^2},$$

and equality holds if and only if $X_1 = (x_1, \ldots, x_k)^t$ is an eigenvector of G_1 corresponding to $\lambda_1(A_1)$.

Proof. Let $A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ where $X_1 = (x_1, \ldots, x_k)^t$ and $X_2 = (x_{k+1}, \ldots, x_n)^t$. Since $AX = \lambda_1(A)X$, thus

$$\begin{cases} \lambda_1(A)X_1 = A_1X_1 + B_1X_2, \\ \lambda_1(A)X_2 = B_2X_1 + A_2X_2. \end{cases} \tag{2.1}$$

Note that

$$X_1^tB_1X_2 = \sum_{i=1}^{k}\sum_{j=k+1}^{n} x_ia_{ij}x_j = \sum_{v_iv_j \in E[V_1,V_2]} x_ix_j$$

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and by the second equation in (2.1),

$$X_2^t B_2 X_1 + X_2^t A_2 X_2 = \lambda_1(A) X_2^t X_2.$$

Thus,

$$\begin{aligned}\lambda_1(A) = X^t A X &= (X_1^t \ X_2^t) \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ &= X_1^t A_1 X_1 + X_2^t B_2 X_1 + X_1^t B_1 X_2 + X_2^t A_2 X_2 \\ &= X_1^t A_1 X_1 + \sum_{v_i v_j \in E[V_1, V_2]} x_i x_j + \lambda_1(A) X_2^t X_2.\end{aligned}$$

Note that $X_1^t X_1 + X_2^t X_2 = 1$, then

$$\begin{aligned}\lambda_1(A_1) \ \geq \ \frac{X_1^t A_1 X_1}{X_1^t X_1} &= \frac{\lambda_1(A) - \lambda_1(A) X_2^t X_2 - \sum_{v_i v_j \in E[V_1, V_2]} x_i x_j}{X_1^t X_1} \\ &= \lambda_1(A) - \frac{\sum_{v_i v_j \in E[V_1, V_2]} x_i x_j}{\sum_{v_i \in V_1} x_i^2}.\end{aligned}$$

Equality holds if and only if X_1 is an eigenvector of G_1 corresponding to $\lambda_1(A_1)$. $\square$

For any graph G (not necessarily connected) with $V(G) = V_1 \cup V_2$ and corresponding graphs $G[V_i]$ for $i = 1, 2$ are the induced subgraphs of G. Suppose that A and A_i are the adjacency matrices of G and $G[V_i]$ for $i = 1, 2$, respectively. C. Li et al. [13] gave a lower bound on $\lambda_1(A_1)$, that is,

$$\lambda_1(A_1) \ \geq \ \left(1 - 2 \sum_{v_i \in V_2} x_i^2\right) \lambda_1(A) \ + \sum_{v_i v_j \in E(G[V_1])} x_i x_j,$$

where $X = (x_1, \ldots, x_n)^t$ is the eigenvector of G corresponding to $\lambda_1(G)$. Let $G_i = G - v_i$, Then $\lambda_1(G_1) \geq (1 - 2x_i^2)\lambda_1(G)$. Nikiforov [17] improved the lower bound of $\lambda_1(G_1)$. When G is connected, we provide a necessary and sufficient condition for the lower bound is attained (see Theorem 2.2).

THEOREM 2.2 ([17]). *Let G be a connected graph with order n. Let $v_i \in V(G)$ and $G_i = G - v_i$. Suppose that $X = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_i)^t$ is the Perron vector of G corresponding to $\lambda_1(G)$. Then*

$$\lambda_1(G_i) \geq \lambda_1(G) \frac{1 - 2x_i^2}{1 - x_i^2}$$

and equality holds if and only if $X_1 = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)^t$ is an eigenvector of G_i corresponding to $\lambda_1(G_1)$.

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Proof. Let $A = \begin{pmatrix} A_1 & b \\ b^t & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 \\ x_i \end{pmatrix}$ where A_1 is the adjacency matrix of G_i. Since $AX = \lambda_1(A)X$, thus we have $\lambda_1(A)x_i = b^t X_1$. Therefore,

$$\begin{aligned}\lambda_1(A) = X^t AX &= (X_1^t \ x_i)\begin{pmatrix} A_1 & b \\ b^t & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ x_i \end{pmatrix}\\ &= X_1^t A_1 X_1 + x_i b^t X_1 + X_1^t b x_i\\ &= X_1^t A_1 X_1 + 2\lambda_1(A)x_i^2.\end{aligned}$$

Note that $X_1^t X_1 + x_i^2 = 1$, then

$$\lambda_1(A_1) \geq \frac{X_1^t A_1 X_1}{X_1^t X_1} = \lambda_1(A)\frac{1-2x_i^2}{1-x_i^2}.$$

Obviously, the equality holds if and only if $X_1 = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)^t$ is an eigenvector of G_i corresponding to $\lambda_1(G_i)$. ☐

COROLLARY 2.3. *Let G be a connected graph with order n. Suppose that $X = (x_1, x_2, \ldots, x_n)^t$ is the Perron vector of G, where $x_1 \geq x_2 \geq \cdots \geq x_n$. Then $\max_{i=1}^n \lambda_1(G_i) \geq \lambda_1(G)\frac{n-2}{n-1}$, and the equality holds if and only if $G \cong K_n$. Meanwhile, $\min_{i=1}^n \lambda_1(G_i) \geq 0$, and the equality holds if and only if $G \cong K_{1,n-1}$.*

Proof. Let $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of G where x_i corresponds to v_i for $i = 1, \ldots, n$. It is easy to see that $f(x) = \frac{1-2x^2}{1-x^2}$ is a decreasing function when $0 < x < 1$. So $\lambda_1(G_i)$ attains the maximum if $v_i = v_n$ and the minimum if $v_i = v_1$. Note that $\frac{1}{\sqrt{n}} \leq x_1 \leq \frac{\sqrt{2}}{2}$ and $0 < x_n \leq \frac{1}{\sqrt{n}}$. Therefore $\lambda_1(G_1) \geq 0$ and the equality holds if and only if $x_1 = \frac{\sqrt{2}}{2}$, that is $G \cong K_{1,n-1}$. On the other hand, $\lambda_1(G_n) \geq \lambda_1(G)\frac{n-2}{n-1}$ and the equality holds if and only if $x_n = \frac{1}{\sqrt{n}}$. Then $x_i = \frac{1}{\sqrt{n}}$ for $i = 1, \ldots, n$ and then G is a regular graph. By Theorem 2.2, the equality holds if and only if $X_1 = (x_1, \ldots, x_{n-1})^t = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)^t$ is the eigenvector of G_n corresponding to $\lambda_1(G_n)$, that is G_n is also a regular graph. Therefore, it is easy to see that $G \cong K_n$. ☐

THEOREM 2.4. *Let G be a connected graph with $V(G) = V_1 \cup V_2 = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ where $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_n\}$ and $G_i = G[V_i]$ be the subgraphs of G for $i = 1, 2$. Suppose that A and A_i are the adjacency matrices of G and G_i for $i = 1, 2$, respectively. Let $X = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^t$ be a least vector of G where x_i corresponds to v_i for $i = 1, \ldots, n$. Then*

$$\lambda_n(A_1) \leq \frac{\lambda_n(A)\left(1 - 2\sum_{v_i \in V_2} x_i^2\right) + \sum_{v_i v_j \in E(G_2)} x_i x_j}{1 - \sum_{v_i \in V_2} x_i^2},$$

and equality holds if and only if $X_1 = (x_1, \ldots, x_k)^t$ is a least vector of G_1.

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Proof. Let $A = \begin{pmatrix} A_1 & B \\ B^t & A_2 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ where $X_1 = (x_1, \ldots, x_k)^t$ and $X_2 = (x_{k+1}, \ldots, x_n)^t$. Since $AX = \lambda_n(A)X$, thus

$$\begin{cases} \lambda_n(A)X_1 = A_1X_1 + BX_2, \\ \lambda_n(A)X_2 = B^tX_1 + A_2X_2. \end{cases}$$

Then

$$\begin{aligned} \lambda_n(A) = X^tAX &= (X_1^t\ X_2^t)\begin{pmatrix} A_1 & B \\ B^t & A_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ &= X_1^tA_1X_1 + X_2^tB^tX_1 + X_1^tBX_2 + X_2^tA_2X_2 \\ &= X_1^tA_1X_1 + 2X_2^t(B^tX_1 + A_2X_2) - X_2^tA_2X_2 \\ &= X_1^tA_1X_1 + 2\lambda_n(A)X_2^tX_2 - X_2^tA_2X_2. \end{aligned}$$

Note that $X_1^tX_1 + X_2^tX_2 = 1$, then

$$\begin{aligned} \lambda_n(A_1) \le \frac{X_1^tA_1X_1}{X_1^tX_1} &= \frac{\lambda_n(A) - 2\lambda_n(A)X_2^tX_2 + \sum_{v_iv_j \in E(G_2)} x_ix_j}{X_1^tX_1} \\ &= \frac{\lambda_n(A)\left(1 - 2\sum_{v_i \in V_2} x_i^2\right) + \sum_{v_iv_j \in E(G_2)} x_ix_j}{1 - \sum_{v_i \in V_2} x_i^2}. \end{aligned}$$

Equality holds if and only X_1 is a least vector of A_1 corresponding to $\lambda_n(A_1)$. ☐

COROLLARY 2.5. *Let G be a connected graph with $V(G) = V_1 \cup V_2 = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ where $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_n\}$ and $G_i = G[V_i]$ be the subgraphs of G for $i = 1, 2$. Suppose that A and A_i are the adjacency matrices of G and G_i for $i = 1, 2$, respectively. Let $X = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^t$ be a least vector of G where x_i corresponds to v_i for $i = 1, \ldots, n$. If $x_i = 0$ for $i = k+1, \ldots, n$, then $\lambda_n(A) = \lambda_n(A_1)$.*

Proof. By Theorem 2.4, $\lambda_n(A_1) \le \lambda_n(A)$. On the other hand, by Cauchy interlacing theorem, $\lambda_n(A_1) \ge \lambda_n(A)$. Thus, $\lambda_n(A_1) = \lambda_n(A)$. ☐

3. On conjectures involving the spectral radius of graphs. The following inequalities are well-known Courant-Weyl inequalities.

LEMMA 3.1. *Let A and B be $n \times n$ Hermitian matrices and $C = A + B$. Then*

$$\lambda_i(C) \le \lambda_j(A) + \lambda_{i-j+1}(B)\ (n \ge i \ge j \ge 1),$$

$$\lambda_i(C) \ge \lambda_j(A) + \lambda_{i-j+n}(B)\ (1 \le i \le j \le n).$$

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Similar to the Courant-Weyl inequalities, we have the following result.

LEMMA 3.2. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Suppose G_1 and G_2 are two subgraphs of G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ with $E(G_1), E(G_2) \neq \emptyset$. Then $\lambda_1(G) \leq \lambda_1(G_1) + \lambda_1(G_2)$, equality holds if and only if G, G_1 and G_2 have a common eigenvector corresponding to $\lambda_1(G)$, $\lambda_1(G_1)$ and $\lambda_1(G_2)$.*

Proof. Let A, A_1 and A_2 be the adjacent matrices of G, G_1 and G_2, respectively. Obviously, $A = A_1 + A_2$. Suppose that X is a eigenvector of A corresponding to $\lambda_1(A)$, then

$$\lambda_1(A) = X^tAX = X^tA_1X + X^tA_2X \leq \lambda_1(A_1) + \lambda_1(A_2).$$

If $\lambda_1(A) = \lambda_1(A_1)+\lambda_1(A_2)$, then $\lambda_1(A_1) = X^tA_1X$ and $\lambda_1(A_2) = X^tA_2X$, that is, X is a common eigenvector of A_1 and A_2 corresponding to $\lambda_1(A_1)$ and $\lambda_1(A_2)$. For the converse, suppose that X is a common eigenvector of A, A_1 and A_2 corresponding to $\lambda_1(G)$, $\lambda_1(G_1)$ and $\lambda_1(G_2)$, then it is easy to see that $\lambda_1(A_1) = X^tA_1X$ and $\lambda_1(A_2) = X^tA_2X$. Therefore $\lambda_1(A) = \lambda_1(A_1) + \lambda_1(A_2)$. ∎

By the above lemma, we get the following corollary.

COROLLARY 3.3. *Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Suppose G_1 and G_2 be two subgraphs of G such that $V(G_1) = V(G_2) = V(G)$ and $E(G) = E(G_1) \cup E(G_2)$ with $E(G_1), E(G_2) \neq \emptyset$. If one of G_1 and G_2 has an isolated vertex, and the other is connected, then $\lambda_1(G) < \lambda_1(G_1) + \lambda_1(G_2)$.*

Proof. Without loss of generality, we may assume that G_1 is connected and G_2 contains an isolated vertex, say u. Since G_1 is connected, by Perron-Frobenius Theorem, the eigenvector, say $X = (x_1, \ldots, x_n)$ of G_1 corresponding to $\lambda_1(G_1)$ is positive, that is $x_i > 0$ for $i = 1, \ldots, n$. By Lemma 3.2, $\lambda_1(G) \leq \lambda_1(G_1) + \lambda_1(G_2)$. If $\lambda_1(G) = \lambda_1(G_1) + \lambda_1(G_2)$, then X is also the eigenvector of G_2 corresponding to $\lambda_1(G_2)$. Then $A(G_2)X = \lambda_1(G_2)X$. Since u is an isolated vertex of $V(G_2)$, we have $(\lambda_1(G_2)X)_u = \lambda_1(G_2)x_u > 0$. But on the other hand, $(\lambda_1(G_2)X)_u = (A(G_2)X)_u = 0$, a contradiction. Therefore $\lambda_1(G) < \lambda_1(G_1) + \lambda_1(G_2)$. ∎

Let $M(n, D)$ be the graph obtained from a complete graph on $n - D + 2$ vertices by removing an edge, adding a pendant path of $\lceil D/2 \rceil - 1$ edges to one end vertex of the removed edge, and adding a pendant path of $\lfloor D/2 \rfloor - 1$ edges to its other end vertex as shown in Fig. 1.

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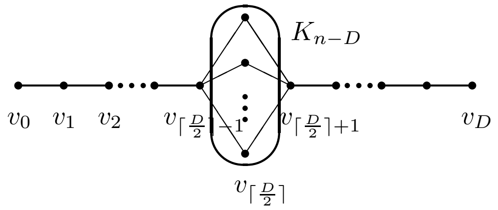

Fig. 1. The graph $M(n, D)$.

E.R. van Dam [8] and P. Hansen and D. Stevanović [12] independently determined that the graph $M(n, D)$ attains the maximal spectral radius among all graphs on n vertices with diameter D.

Lemma 3.4 ([8], [12]). *Let n and D be integers with $1 < D < n$. Then the graph $M(n, D)$ is the unique graph with maximal spectral radius among all graphs on n vertices with diameter D.*

Theorem 3.5 (Conjecture 4.2, [4]). *Let G be a connected graph with diameter D. Then*

$$\lambda_1(G) + D \leq n - 1 + 2\cos\frac{\pi}{n+1}$$

and the equality holds if and only if $G \cong P_n$.

Proof. Let $M = M(n, D)$ be shown in Fig. 1. If $G \neq M$, then by Lemma 3.4, $\lambda_1(G) < \lambda_1(M)$. Thus, $\lambda_1(G) + D < \lambda_1(M) + D$. Therefore in the following, it is sufficient to show that $\lambda_1(M) + D \leq n - 1 + 2\cos\frac{\pi}{n+1}$, with equality if and only if $G \cong P_n$.

Let $M' = M[V(M) \backslash \{v_0, \ldots, v_{\lceil \frac{D}{2} \rceil -2}, v_{\lceil \frac{D}{2} \rceil +2}, \ldots, v_D\}]$. Then it is easy to see that $M' = K_{n-D} - v_{\lceil \frac{D}{2} \rceil -1} v_{\lceil \frac{D}{2} \rceil +1}$. Therefore, M' contains a Hamiltonian path with end vertices $v_{\lceil \frac{D}{2} \rceil -1}$, $v_{\lceil \frac{D}{2} \rceil +1}$, say $v_{\lceil \frac{D}{2} \rceil -1} P_1 v_{\lceil \frac{D}{2} \rceil +1}$. Then $P = v_0 v_1 \cdots v_{\lceil \frac{D}{2} \rceil -1} P_1 v_{\lceil \frac{D}{2} \rceil +1} \cdots v_D$ is a Hamiltonian path of $M(n, D)$. Let $M_1 = P$ and $M_2 = M \backslash E(M_1)$. It is clear that M_2 consists of a complete graph of order $n - D + 1$ deleting an Hamiltonian cycle and $D - 1$ isolated vertices. Then $\lambda_1(M_1) = 2\cos\frac{\pi}{n+1}$ and $\lambda_1(M_2) = n - D - 1$. Then by Lemma 3.2,

$$\lambda_1(M) \leq \lambda_1(M_1) + \lambda_1(M_2) = n - D - 1 + 2\cos\frac{\pi}{n+1}. \tag{3.1}$$

Therefore, $\lambda_1(M) + D \leq n - 1 + 2\cos\frac{\pi}{n+1}$.

If $G \cong P_n$, then $D = n - 1$ and $\lambda_1(G) = 2\cos\frac{\pi}{n+1}$. Therefore the equality holds.

For the converse, we may assume that $\lambda_1(G) + D \leq n - 1 + 2\cos\frac{\pi}{n+1}$. Then $G \cong M$ by Lemma 3.2 and the inequality (3.1) is equality. If $M_2 \neq \emptyset$, then $2 \leq D \leq n - 2$. Since $M_1 \cong P_n$ and M_2 consists of a complete graph of order $n - D + 1$ deleting a

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Hamiltonian cycle and $D-1$ isolated vertices. Therefore, by Corollary 3.3,

$$\lambda_1(M) < \lambda_1(M_1) + \lambda_1(M_2) = n - D - 1 + 2\cos\frac{\pi}{n+1},$$

that is, $\lambda_1(M) + D < n - 1 + 2\cos\frac{\pi}{n+1}$, a contradiction. Then $M_2 = \emptyset$ and $M = M_1 \cong P_n$. Thus, we complete the proof. $\square$

Corollary 3.6. *Let G be a connected graph with diameter D. Then*

$$\lambda_1(G) \le n - D - 1 + 2\cos\frac{\pi}{n+1}$$

and the equality holds if and only if $G \cong P_n$.

Lemma 3.7 ([11]). *Let $G_{n,\mu}$ be the set of graphs on n vertices with matching number μ. For any $G \in G_{n,\mu}$, we have*

(i) If $n = 2\mu$ or $n = 2\mu + 1$, then $\rho(G) \le \rho(K_n)$ with equality if and only if $G \cong K_n$.

(ii) If $2\mu + 2 \le n < 3\mu + 2$, then $\rho(G) \le 2\mu$ with equality if and only if $G \cong K_{2\mu+1} \cup \overline{K_{n-2\mu-1}}$.

(iii) If $n = 3\mu + 2$, then $\rho(G) \le 2\mu$ with equality if and only if $G \cong K_\mu \vee \overline{K_{n-\mu}}$ or $G \cong K_{2\mu+1} \cup \overline{K_{n-2\mu-1}}$.

(iiii) If $n > 3\mu + 2$, then $\rho(G) \le \frac{1}{2}(\mu - 1 + \sqrt{(\mu-1)^2 + 4\mu(n-\mu)})$ with equality if and only if $G \cong K_\mu \vee \overline{K_{n-\mu}}$.

Theorem 3.8. *Let G be a graph on $n \ge 3$ vertices with spectral radius $\lambda_1(G)$ and matching number μ. Then*

$$\lambda_1(G) - \mu \le n - 1 - \lfloor n/2 \rfloor$$

and the equality holds if and only if $G \cong K_n$ or $G \cong K_{n-1} \cup K_1$ and n is even.

Proof. Let G be a graph with matching number μ. Then $\mu \le \lfloor n/2 \rfloor$, thus we distinguish the following three cases.

Case 1. $\mu = \lfloor n/2 \rfloor$.

Then,

$$\lambda_1(G) - \mu = \lambda_1(G) - \lfloor n/2 \rfloor \le n - 1 - \lfloor n/2 \rfloor. \tag{3.2}$$

Case 2. $\frac{n-2}{3} \le \mu \le n/2 - 1$.

Then by Lemma 3.7 (ii) and (iii), $\lambda_1(G) \le 2\mu$. Therefore, we obtain

$$\lambda_1(G) - \mu \le \mu \le n/2 - 1 \le n - 1 - \lfloor n/2 \rfloor. \tag{3.3}$$

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Case 3. $\mu \leq \frac{n}{3} - 1$.

Similar to the proof in [20], $\lambda_1(G) - \mu < n - 1 - \lfloor n/2 \rfloor$.

If $G \cong K_n$, then $\lambda_1(G) = n - 1$ and $\mu = \lfloor n/2 \rfloor$, thus the equality holds. If $G \cong K_{n-1} \cup K_1$ and n is even, then $\lambda_1(G) = n - 2$ and $\mu = n/2 - 1$, thus the equality holds.

For the converse, we may suppose that $\lambda_1 - \mu = n - 1 - \lfloor n/2 \rfloor$. Then all the inequalities in (3.2) and (3.3) are equalities. Since the inequality (3.2) is equality, $\lambda_1(G) = n - 1$ and $\mu = \lfloor n/2 \rfloor$, then $G \cong K_n$. Since the inequality (3.3) is equality. Then $G \cong K_{2\mu+1} \cup \overline{K_{n-2\mu-1}}$ by Lemma 3.7 (ii), and $\lambda_1(G) = 2\mu$, $\mu = n/2 - 1$ and n is even. Thus, $G \cong K_{n-1} \cup K_1$ and n is even. □

THEOREM 3.9. *Let G be a graph on $n \geq 6$ vertices with spectral radius $\lambda_1(G)$ and matching number μ. Then*

$$\frac{\lambda_1(G)}{\mu} \leq \sqrt{n-1}$$

and equality holds if and only if $G \cong K_{1,n-1}$.

Proof. If G is empty, then the result follows immediately. If G is not empty, then we have $1 \leq \mu \leq \lfloor n/2 \rfloor$. Thus, we consider the following two cases.

Case 1. $n/3 - 1 < \mu \leq \lfloor n/2 \rfloor$.

Then by Lemma 3.7 (i), (ii) and (iii), $\lambda_1(G) \leq 2\mu$, thus $\frac{\lambda_1(G)}{\mu} \leq 2 < \sqrt{n-1}$ since $n \geq 6$.

Case 2. $1 \leq \mu \leq n/3 - 1$.

If $\mu = 1$, then $G \cong K_3 \cup \overline{K_{n-3}}$ or $G \cong K_{1,i} \cup \overline{K_{n-i-1}}$ for $1 \leq i \leq n-1$. Thus, $\lambda_1(G) \leq \lambda_1(K_{1,n-1}) = \sqrt{n-1}$ since $n \geq 6$. Equality holds if and only if $G \cong K_{1,n-1}$.

If $2 \leq \mu \leq n/3 - 1$, similar to the proof in [20], $\frac{\lambda_1(G)}{\mu} < \sqrt{n-1}$.

Therefore, $\frac{\lambda_1(G)}{\mu} \leq \sqrt{n-1}$ with equality if and only if $G \cong K_{1,n-1}$. □

REMARK 3.10. The order $n \geq 6$ of the graph in Theorem 3.9 is needed. By a direct computation, when $n = 3$, $\frac{\lambda_1(G)}{\mu} \leq 2$ with equality if and only if $G \cong K_3$; when $n = 4$, $\frac{\lambda_1(G)}{\mu} \leq 2$ with equality if and only if $G \cong K_3 \cup K_1$; when $n = 5$, $\frac{\lambda_1(G)}{\mu} \leq 2$ with equality if and only if $G \cong K_5$ or $K_{1,4}$ or $K_3 \cup \overline{K_2}$.

4. The minimum least eigenvalue among all quasi-tree graphs. A graph G is called *minimizing* (respectively, *maximizing*) in a certain class of graphs if the least eigenvalue (respectively, spectral radius) of G attains the *minimum* (respectively,

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maximum) among all graphs in the class.

LEMMA 4.1. *The graph $K_{2,n-2}$ is the unique maximizing graph among all bipartite quasi-tree graphs.*

Proof. Let G be a quasi-tree maximizing graph among all bipartite quasi-tree graphs. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of G. Assume that $G' = G - v_1$ is a tree. Let $V(G) = S \cup T$, where S and T are the partitions of $V(G)$ such that S and T are independent sets. Let $x_u = \{\max x_i | v_i \in S\backslash\{v_1\}\}$. Without loss of generality, we may assume that $\sum_{v_i \in S\backslash\{v_1\}} x_i \leq \sum_{v_i \in T\backslash\{v_1\}} x_i$. We first prove the following claim.

Claim 1. $v_1 \in S$ and $v_1 v_i \in E(G)$ for all $v_i \in T$.

By contradiction, we suppose that $v_1 \in T$. Note that G is a maximizing graph among all bipartite quasi-trees, then $v_1 v_i \in E(G)$ for all $v_i \in S$. Let $G^\star = G - \{v_1 v_i | v_i \in S\} + \{v_1 v_i | v_i \in T\}$. Therefore, $G^\star$ is also a bipartite quasi-tree graph. But

$$\rho(G^\star) \geq X^t A(G^\star) X = X^t A(G) X - 2x_1 \sum_{v_i \in S} x_i + 2x_1 \sum_{v_i \in T} x_i \geq X^t A(G) X = \rho(G).$$

If $\rho(G^\star) = \rho(G)$, then X is also the Perron vector of $G^\star$. Since $\rho(G^\star)X = \rho(G)X$, thus $(A(G^\star)x)_i = (A(G)x)_i$ for $i = 1, \ldots, n$. But on the other hand, for $v_i \in S$, $(A(G)x)_i = x_1 + \sum_{v_j \in N(v_i)} x_j$, and $A(G^\star)x_i = \sum_{v_j \in N(v_i)} x_j$, a contradiction. Hence, $\rho(G^\star) > \rho(G)$. This contradicts the maximality of G.

Claim 2. No vertex of $S\backslash\{u\}$ in G' has a neighbor with degree one.

If not, suppose that $w \in S\backslash\{u\}$ has a neighbor, say w' with $d_{G'}(w') = 1$. Then let $G^\star = G - ww' + uw'$. Obviously, $G^\star$ is a bipartite quasi-tree graph. But

$$\rho(G^\star) \geq X^t A(G^\star) X = X^t A(G) X - 2x_w x_{w'} + 2x_u x_{w'} \geq X^t A(G) X = \rho(G).$$

Similar to the proof of Claim 1, we have $\rho(G^\star) > \rho(G)$, a contradiction.

Claim 3. No vertex of T in G' has a neighbor with degree one.

If not, suppose that $w \in T$ has a neighbor, say w' with $d_{G'}(w') = 1$. Then we let $G^\star = G - ww' + uw' + v_1 w'$. Obviously, $G^\star$ is a bipartite quasi-tree graph. But since $\sum_{v_i \in S\backslash\{v_1\}} x_i \leq \sum_{v_i \in T\backslash\{v_1\}} x_i$, we have $x_1 \geq x_w$. Then

$$\rho(G^\star) \geq X^t A(G^\star) X = X^t A(G) X - 2x_w x_{w'} + 2x_1 x_{w'} + 2x_u x_{w'} > X^t A(G) X = \rho(G),$$

a contradiction.

Claim 4. The degree of the vertex of T in G' is one.

If not, suppose that $w \in T$ and $d_{G'}(w) \geq 2$. Without loss of generality, suppose w_1, w_2 are two neighbors of w in G'. Let $P_1 = ww_1 \cdots$ and $P_2 = ww_2 \cdots$ be the

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longest path passing the vertex w, w_1 and w, w_2 in G', respectively. Then the end vertices of P_1 and P_2 must be leaves of G' (since G' is a tree). But by Claims 2 and 3, the neighbor of leaves must be u. Then there is a cycle in G', this contradicts that G' is a tree.

Then by Claims 1–4, $G \cong K_{2,n-2}$. Thus, we complete the proof. $\square$

THEOREM 4.2. *The graph $K_{2,n-2}$ is the unique minimizing graph among all quasi-tree graphs.*

Proof. Let G be a minimizing graph among all quasi-tree graphs, and assume that $G - v_1$ is a tree. Let X be a least vector of G. Denote $V_+ = \{v_i | x_i \geq 0\}, V_- = \{v_i | x_i < 0\}$. If $v_1 \in V_+$, then we delete all edges between v_1 and the vertex of V_+. Similarly, if $v_1 \in V_-$, then we delete all edges between v_1 and the vertex of V_-. We get a bipartite graph, denoted by G_0. Obviously, G_0 is a quasi-tree, and $\lambda_n(G) \geq \lambda_n(G_0)$. So it is sufficient to determine the minimizing graph among all bipartite quasi-tree graphs. If G is a bipartite graph, then $\lambda_n(G) = -\lambda_1(G)$. Then by Lemma 4.1, the graph $K_{2,n-2}$ is the unique minimizing graph among all bipartite quasi-tree graphs. $\square$

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REFERENCES

[1] M. Aouchiche. *Comparaison Automatisée d'Invariants en Théorie des Graphes.* Ph.D. Thesis, École Polytechnique de Montréal, February 2006.

[2] M. Aouchiche, G. Caporossi, and P. Hansen. Variable neighborhood search for extremal graphs, 20: Automated comparison of graph invariants. *MATCH Commun. Math. Comput. Chem.*, 58:365–384, 2007.

[3] M. Aouchiche and P. Hansen. A survey of automated conjectures in spectral graph theory, *Linear Algebra Appl.*, 432:2293–2322, 2010.

[4] M. Aouchiche, P. Hansen, and D. Stevanović. Variable neighborhood search for extremal graphs, 17: Further conjectures and results about the index. *Discussiones Mathematicae Graph Theory* 29:15–37, 2009.

[5] F.K. Bell, D. Cvetković, P. Rowlinson, and S.K. Simić. Graph for which the least eigenvalues is minimal, I. *Linear Algebra Appl.*, 429:234–241, 2008.

[6] F.K. Bell, D. Cvetković, P. Rowlinson, and S.K. Simić. Graph for which the least eigenvalues is minimal, II. *Linear Algebra Appl.*, 429:2168–2179, 2008.

[7] D. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs*, third edition. Barth, Heidelberg, 1995.

[8] E.R. van Dam. Graphs with given diameter maximizing the spectral radius. *Linear Algebra Appl.*, 426:454–457, 2007.

[9] K.C. Das. Conjectures on index and algebraic connectivity of graphs. *Linear Algebra Appl.*, 433:1666–1673, 2010.

[10] Y.-Z. Fan, Y. Wang, and Y.-B. Gao. Minimizing the least eigenvalues of unicyclic graphs with application to spectral spread. *Linear Algebra Appl.*, 429:577–588, 2008.

[11] L. Feng, G. Yu, and X.-D. Zhang. Spectral radius of graphs with given matching number, *Linear Algebra Appl.*, 422:133–138, 2007.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 670-682, September 2014

[12] P. Hansen and D. Stevanović. On bags and bugs. *Discrete Appl. Math.*, 156:986–997, 2008.

[13] C. Li, H. Wang, and P.V. Mieghem. Bounds for the spectral radius of a graph when nodes are removed. *Linear Algebra Appl.*, 437:319–323, 2012.

[14] H. Liu and M. Lu. On the spectral radius of quasi-tree graphs. *Linear Algebra Appl.*, 428:2708–2714, 2008.

[15] R. Liu, M. Zhai, and J. Shu. The least eigenvalues of unicyclic graphs with n vertices and k pendant vertices. *Linear Algebra Appl.*, 431:657–665, 2009.

[16] P. Van Mieghem et al. Decreasing the spectral radius of a graph by link removals. *Phys. Rev. E*, 84:016101, 2011.

[17] V. Nikiforov. The spectral radius of graphs without paths and cycles of specified length. *Linear Algebra Appl.*, 432:2243–2256, 2010.

[18] O. Perron. Zur theorie der matrices. *Math. Ann.*, 64:248–263, 1907.

[19] M. Petrović, B. Borovićanin, and T. Aleksić. Bicyclic graphs for which the least eigenvalue is minimum. *Linear Algebra Appl.*, 430:1328–1335, 2009.

[20] D. Stevanović. Resolution of AutoGraphiX conjectures relating the index and matching number of graphs. *Linear Algebra Appl.*, 433:1674–1677, 2010.

[21] Y.-Y. Tan and Y.-Z. Fan. The vertex (edge) independence number, vertex (edge) cover number and the least eigenvalue of a graph. *Linear Algebra Appl.*, 433:790–795, 2010.

[22] Y. Wang and Y.-Z. Fan. The least eigenvalue of a graph with cut vertices. *Linear Algebra Appl.*, 433:19–27, 2010.

[23] R. Xing and B. Zhou. On least eigenvalues and least eigenvectors of real symmetric matrices and graphs. *Linear Algebra Appl.*, 438:2378–2384, 2013.

[24] M.-L. Ye, Y.-Z. Fan, and D. Liang. The least eigenvalue of graphs with given connectivity. *Linear Algebra Appl.*, 430:1375–1379, 2009.

[25] B.-X. Zhu. The least eigenvalue of a graph with a given domination number. *Linear Algebra Appl.*, 437:2713–2718, 2012.