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SOME RESULTS ON THE q-ANALOGUES OF THE INCOMPLETE FIBONACCI AND LUCAS POLYNOMIALS

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Abstract. In the present paper, we introduce new families of the q-Fibonacci and q-Lucas polynomials, which are represented here as the incomplete q-Fibonacci polynomials $F_n^k(x,s,q)$ and the incomplete q-Lucas polynomials $L_n^k(x,s,q)$, respectively. These polynomials provide the q-analogues of the incomplete Fibonacci and Lucas numbers. We give several properties and generating functions of each of these families q-polynomials. We also point out the fact that the results for the q-analogues which we consider in this article for $0<q<1$ can easily be translated into the corresponding results for the (p,q)-analogues (with $0<q<p\leqq 1$) by applying some obvious parametric variations, the additional parameter p being redundant.

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1. INTRODUCTION

The Fibonacci numbers are defined by the following recurrence relation:

$$F_{n+1}=F_n+F_{n-1} \qquad (n\in\{0,1,2,\cdots\})$$

with the initial conditions $F_0=0$ and $F_1=1$ and the Lucas numbers are defined by the same recurrence relation with the different initial conditions $L_0=2$ and $L_1=1$.

In existing literature, there are many extensions and generalizations of the Fibonacci numbers. For instance, Filipponi [13] defined the incomplete Fibonacci and Lucas numbers as follows:

$$F_n(k)=\sum_{j=0}^{k}\binom{n-1-j}{j} \qquad \left(0\leqq k\leqq \left\lfloor \tfrac{n-1}{2}\right\rfloor\right) \tag{1.1}$$

and

$$L_n(k)=\sum_{j=0}^{k}\frac{n}{n-j}\binom{n-j}{j} \qquad \left(0\leqq k\leqq \left\lfloor \tfrac{n}{2}\right\rfloor\right), \tag{1.2}$$

where $n = 1, 2, 3, \cdots$. We note that

$$F_n(\lfloor \tfrac{n-1}{2} \rfloor) = F_n$$

and

$$L_n(\lfloor \tfrac{n}{2} \rfloor) = L_n.$$

The generating functions of these numbers were studied by Pintér and Srivastava [17]. Many other authors have also studied this topic (see, for example, [8–12, 18, 19, 21, 22]).

For $0 < q < 1$, the q-integer is defined by

$$[n] := [n]_q = \frac{1-q^n}{1-q} \tag{1.3}$$

and the q-factorial is defined by

$$[n]! := \begin{cases} [n]\,.\,[n-1]\cdots[1] & (n = 1, 2, 3, \cdots) \\ 1 & (n = 0). \end{cases} \tag{1.4}$$

The q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]!\,[k]!} \qquad (0 \leqq k \leqq n) \tag{1.5}$$

with (see [2] and [20])

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix} = 0 \qquad (n < k).$$

The Heine's binomial formula is recalled here as follows (see [2, p. 2]):

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{j=1}^{\infty} \frac{[n]_q [n+1]_q \cdots [n+j-1]_q}{[j]_q!} x^j.$$

The q-difference operator D_q is defined as follows:

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)\,x}$$

if $x \neq 0$.

Cigler [6] introduced the q-Fibonacci polynomials which are defined below:

$$F_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^j\, x^{n-1-2j} \qquad (n \in \{0, 1, 2, \cdots\}). \tag{1.6}$$

Also, in [4], we have the following explicit formula for the q-Lucas polynomials:

$$L_n(x,s,q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j \, x^{n-2j} \qquad (n \in \{0,1,2,\cdots\}). \tag{1.7}$$

A recurrence relation for the q-Fibonacci polynomials is given by

$$F_n(x,s,q) = xF_{n-1}(x,s,q) + (q-1)\, sD_q F_{n-1}(x,s,q) + sF_{n-2}(x,s,q),$$

with the initial values $F_0(x,s,q) = 0$ and $F_1(x,s,q) = 1$. The q-Lucas polynomials satisfy the same recurrence relation as above:

$$L_n(x,s,q) = xL_{n-1}(x,s,q) + (q-1)\, sD_q L_{n-1}(x,s,q) + sL_{n-2}(x,s,q),$$

but with the initial values given by (see, for details, [4])

$$L_0(x,s,q) = 2 \qquad \text{and} \qquad L_1(x,s,q) = x.$$

The following formulas provide relationship between these polynomials:

$$L_n(x,s,q) = F_{n+1}(x,s,q) + sF_{n-1}(x,s,q)$$

and

$$L_n(x,qs,q) = F_{n+1}(x,s,q) + q^n sF_{n-1}(x,s,q).$$

For more details about the q-analogues of the Fibonacci polynomials, see [1, 3–5, 7, 14]. There are only a few studies for the q-Fibonacci and q-Lucas polynomials and for the extensions and generalizations of these polynomials.

We choose to remark in passing that several authors (see, for example, [23] and [24]) studied the so-called (p,q)-Fibonacci and (p,q)-Lucas polynomials by introducing a seemingly redundant parameter p, since the so-called (p,q)-number $[n]_{p,q}$ is given (for $0 < q < p \leqq 1$) by

$$\begin{aligned} [n]_{p,q} &:= \begin{cases} \dfrac{p^n - q^n}{p-q} & (n \in \{1,2,3,\cdots\}) \\ 0 & (n = 0) \end{cases} \\ &= p^{n-1} \left(\frac{q}{p}\right)_n \\ &=: p^{n-1}\, [n]_{\frac{q}{p}} \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} [n]_q &:= \frac{1-q^n}{1-q} \\ &= p^{1-n} \left(\frac{p^n - (pq)^n}{p - (pq)} \right) \end{aligned}$$

$$= p^{1-n}\,[n]_{p,pq}, \tag{1.9}$$

which do exhibits the fact that the results for the q-analogues which we consider in this article for $0 < q < 1$ can easily be translated into the corresponding results for the (p,q)-analogues (with $0 < q < p \leqq 1$) by applying some obvious parametric variations, the additional parameter p being redundant.

The aim of this paper is to introduce and study q-analogues of (1.1) and (1.2). We thus investigate the incomplete q-Fibonacci and q-Lucas polynomials and derive some of their properties including their generating functions.

2. Main results

We define the incomplete q-Fibonacci and q-Lucas polynomials by using the explicit formulas as follows.

Definition 1. The incomplete q-Fibonacci polynomials are defined by

$$\mathcal{F}_n^k := F_n^k(x,s,q) = \sum_{j=0}^{k} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^j\, x^{n-1-2j} \tag{2.1}$$

for $0 \leqq k \leqq \left\lfloor \frac{n-1}{2} \right\rfloor$.

If we take $q \to 1-$ in (2.1), we get incomplete bivariate Fibonacci numbers in [22]. On the other hand, if we take $q \to 1-$ and $s = x = 1$ in (2.1), we get incomplete Fibonacci numbers studied in [13].

TABLE 1. Incomplete q-Fibonacci polynomials for $n = 1, \cdots, 8$ and $k = 0, 1, 2, 3, 4$.

	$F_n^k(x,s,q) = \mathcal{F}_n^k$			
n/k	0	1	2	3
1	1			
2	x			
3	x^2	x^2+qs		
4	x^3	$x^3+[2]\,qsx$		
5	x^4	$x^4+[3]\,qsx^2$	$x^4+[3]\,qsx^2+q^3s^2$	
6	x^5	$x^5+[4]\,qsx^3$	$x^5+[4]\,qsx^3+[3]\,q^3s^2x$	
7	x^6	$x^6+[5]\,qsx^4$	$x^6+[5]\,qsx^4+\frac{[3][4]}{[2]}\,q^3s^2x^4$	$x^6+[5]\,qsx^4+\frac{[3][4]}{[2]}\,q^3s^2x^4+q^6s^3$
8	x^7	$x^7+[6]\,qsx^5$	$x^7+[6]\,qsx^5+\frac{[4][5]}{[2]}\,q^3s^2x^3$	$x^7+[6]\,qsx^5+\frac{[4][5]}{[2]}\,q^3s^2x^3+[4]\,q^6s^3x$

We now define incomplete q-Lucas polynomials.

Definition 2. The incomplete q-Lucas polynomials are defined by

$$\mathcal{L}_n^k := L_n^k(x,s,q) = \sum_{j=0}^{k} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j\, x^{n-2j} \tag{2.2}$$

for $0 \leqq k \leqq \left\lfloor \frac{n}{2} \right\rfloor$.

TABLE 2. Incomplete q-Lucas polynomials for $n = 1, \cdots, 7$ and $k = 0, 1, 2, 3$.

n/k	0	1	2	3
			$U_n^k(x,s,q) = \mathcal{L}_n^k$	
1	x			
2	x^2	$x^2 + [2]\,s$		
3	x^3	$x^3 + [3]\,sx$		
4	x^4	$x^4 + [4]\,sx^2$	$x^4 + [4]\,sx^2 + \frac{[4]}{[2]}\,qs^2$	
5	x^5	$x^5 + [5]\,sx^3$	$x^5 + [5]\,sx^3 + [5]\,qs^2x$	
6	x^6	$x^6 + [6]\,sx^4$	$x^6 + [6]\,sx^4 + \frac{[3][6]}{[2]}\,qs^2x^2$	$x^6 + [6]\,sx^4 + \frac{[3][6]}{[2]}\,qs^2x^2 + [6]\,q^3s$
7	x^7	$x^7 + [7]\,sx^5$	$x^7 + [7]\,sx^5 + \frac{[4][7]}{[2]}\,qs^2x^3$	$x^7 + [7]\,sx^5 + \frac{[4][7]}{[2]}\,qs^2x^3 + [7]\,q^3s^3x$

In particular, if we take $q \to 1-$ in (2.2), then we get incomplete bivariate Lucas numbers studied in [22]. Moreover, if we take $q \to 1-$ and $s = x = 1$ in (2.2), then we get incomplete Lucas numbers studied in [13].

2.1. *Recurrence relations*

Theorem 1. *The recurrence relation of the incomplete q-Fibonacci polynomials is given by*

$$\mathcal{F}_{n+2}^{k+1} = x\mathcal{F}_{n+1}^{k+1} + (q-1)\,sD_q\mathcal{F}_{n+1}^{k} + s\mathcal{F}_n^k \tag{2.3}$$

for $0 \leqq k \leqq \frac{n-2}{2}$.

Proof. We find from (2.1) that

$$\begin{aligned} x\mathcal{F}_{n+1}^{k+1} &+ (q-1)\,sD_q\mathcal{F}_{n+1}^{k} + s\mathcal{F}_n^k \\ &= \sum_{j=0}^{k+1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n+1-2j} \\ &\quad + (q-1)\sum_{j=0}^{k} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} [n-2j]\, s^{j+1} x^{n-1-2j} \\ &\quad + \sum_{j=0}^{k} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^{j+1} x^{n-1-2j} \\ &= \sum_{j=0}^{k+1} \left\{ q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + (q-1)\,q^{\binom{j}{2}} \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix} [n-2j+2] \right. \\ &\quad \left. + q^{\binom{j}{2}} \begin{bmatrix} n-j \\ j-1 \end{bmatrix} \right\} s^j x^{n+1-2j}. \end{aligned}$$

It is known that (see [5])

$$q^{\binom{j+1}{2}} \begin{bmatrix} n-j+1 \\ j \end{bmatrix} =$$

$$= q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + (q-1)\, q^{\binom{j}{2}} \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix} [n-2j+2] + q^{\binom{j}{2}} \begin{bmatrix} n-j \\ j-1 \end{bmatrix}.$$

Therefore, the recurrence relation (2.3) is seen to hold true. □

In Theorem 1, by taking $q \to 1-$ and $x = s = 1$, we get the following result:

$$F_{n+2}(k+1) = F_{n+1}(k+1) + F_n(k) \qquad \left(0 \leqq k \leqq \tfrac{n-2}{2}\right),$$

where $F_n(k)$ are the incomplete Fibonacci numbers studied in [13].

The following theorem results, in part, from the recurrence relation (2.3).

Theorem 2. *The following non-homogeneous recurrence relation of the incomplete q-Fibonacci polynomials holds true*:

$$\mathcal{F}_{n+2}^{k} = \left(x + (q-1)\, s D_q\right) \mathcal{F}_{n+1}^{k} + s \mathcal{F}_{n}^{k} - q^{\binom{k}{2}} q^{n} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} s^{k+1} x^{n-2k-1}. \tag{2.4}$$

Proof. The proof of the non-homogeneous recurrence relation (2.4) follows from Definition 1 and the equation (2.3). □

Theorem 3. *It is asserted that*

$$\mathcal{L}_{n}^{k} = \mathcal{F}_{n+1}^{k} + s \mathcal{F}_{n-1}^{k-1} \qquad \left(0 \leqq k \leqq \frac{n}{2}\right). \tag{2.5}$$

Proof. Using the equation (2.1), we have

$$\begin{aligned}
\mathcal{F}_{n+1}^{k} + s \mathcal{F}_{n-1}^{k-1} &= \sum_{j=0}^{k} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} + s \sum_{j=0}^{k-1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix} s^j x^{n-2-2j} \\
&= \sum_{j=0}^{k} q^{\binom{j}{2}} \left\{ q^{-\binom{j}{2}} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^j x^{n-2j} \\
&= \sum_{j=0}^{k} q^{\binom{j}{2}} \left\{ q^{j} \begin{bmatrix} n-j \\ j \end{bmatrix} + \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^j x^{n-2j} \\
&= \sum_{j=0}^{k} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
&= \mathcal{L}_{n}^{k}
\end{aligned}$$

for $0 \leqq k \leqq \frac{n}{2}$. □

Theorem 4. *The following recurrence relation holds true*:

$$L_{n}^{k}(x, qs, q) = \mathcal{F}_{n+1}^{k} + q^{n} s \mathcal{F}_{n-1}^{k-1} \qquad \left(0 \leqq k \leqq \frac{n}{2}\right). \tag{2.6}$$

Proof. We consider

$$\begin{aligned}
\mathcal{F}_{n+1}^{k}+q^{n}s\mathcal{F}_{n-1}^{k-1} &= \sum_{j=0}^{k} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^{j}x^{n-2j} \\
&\quad + q^{n}s\sum_{j=0}^{k-1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix} js^{j}x^{n-2-2j} \\
&= \sum_{j=0}^{k} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^{j}x^{n-2j} \\
&\quad + q^{n}\sum_{j=1}^{k} q^{\binom{j}{2}} \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} s^{j}x^{n-2j} \\
&= \sum_{j=1}^{k} q^{\binom{j}{2}} \left\{ q^{j}\begin{bmatrix} n-j \\ j \end{bmatrix} + q^{n}\begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^{j}x^{n-2j} \\
&= \sum_{j=1}^{k} q^{\binom{j}{2}} q^{j}\frac{[n]}{[n-j]}\begin{bmatrix} n-j \\ j \end{bmatrix} s^{j}x^{n-2j} \\
&= L_{n}(x,qs,q)
\end{aligned}$$

for $0 \leqq k \leqq \frac{n}{2}$. □

Theorem 5. *The recurrence relation of incomplete q-Lucas polynomials is given by*

$$\mathcal{L}_{n+2}^{k+1} = x\mathcal{L}_{n+1}^{k+1} + (q-1)sD_{q}\mathcal{L}_{n+1}^{k} + s\mathcal{L}_{n}^{k} \qquad \left(0 \leqq k \leqq \frac{n-1}{2}\right). \tag{2.7}$$

Proof. In view of (2.3) and (2.5), we find that

$$\begin{aligned}
\mathcal{L}_{n+2}^{k+1} &= \mathcal{F}_{n+3}^{k+1} + s\mathcal{F}_{n+1}^{k} \\
&= \left(x\mathcal{F}_{n+2}^{k+1} + (q-1)sD_{q}\mathcal{F}_{n+2}^{k} + s\mathcal{F}_{n+1}^{k}\right) \\
&\quad + s\left(x\mathcal{F}_{n}^{k} + (q-1)sD_{q}\mathcal{F}_{n}^{k-1} + s\mathcal{F}_{n}^{k-1}\right) \\
&= x\left(\mathcal{F}_{n+2}^{k+1} + s\mathcal{F}_{n}^{k}\right) + (q-1)sD_{q}\left(\mathcal{F}_{n+2}^{k} + \mathcal{F}_{n}^{k-1}\right) + s\left(\mathcal{F}_{n+1}^{k} + s\mathcal{F}_{n-1}^{k-1}\right) \\
&= x\mathcal{L}_{n+1}^{k+1} + (q-1)sD_{q}\mathcal{L}_{n+1}^{k}) + s\mathcal{L}_{n}^{k}
\end{aligned}$$

for $0 \leqq k \leqq \frac{n-1}{2}$. □

Theorem 6. *The following non-homogeneous recurrence relation of the incomplete q-Lucas polynomials holds true*:

$$\mathcal{L}_{n+2}^{k} = \left(x+(q-1)\,sD_q\right)\mathcal{L}_{n+1}^{k} + s\mathcal{L}_{n}^{k} - \frac{q^{\binom{k}{2}}q^{n}\left(q^{-k}[n+1]-1\right)}{[n-k]}\begin{bmatrix} n-k \\ k \end{bmatrix} s^{k+1}x^{n-2k}. \tag{2.8}$$

Proof. It is easy to derive non-homogeneous recurrence relation (2.8) by using (2.2) and (2.7). □

2.2. *Summation formulas*

Theorem 7. *The following summation formula for the incomplete q-Fibonacci polynomials holds true*:

$$\sum_{j=0}^{h-1}\frac{1}{x^{j}}\left(s\mathcal{F}_{n+j}^{k}+(q-1)\,sD_q\mathcal{F}_{n+1+j}^{k}\right) = \frac{1}{x^{h-1}}\mathcal{F}_{n+1+h}^{k+1} - x\mathcal{F}_{n+1}^{k+1}, \tag{2.9}$$

where $x \neq 0$ *and* $n \geqq 2k+2$.

Proof. Our proof uses the principle of mathematical induction on h. For $h=1$, the equation (2.9) holds true. Suppose that the equation (2.9) holds true for some integer $h>1$. Then, by using the equation (2.1), we get

$$\begin{aligned}
&\sum_{j=0}^{h}\frac{1}{x^{j}}\left(s\mathcal{F}_{n+j}^{k}+(q-1)\,sD_q\mathcal{F}_{n+1+j}^{k}\right)\\
&\quad= \sum_{j=0}^{h-1}\frac{1}{x^{j}}\left(s\mathcal{F}_{n+j}^{k}+(q-1)\,sD_q\mathcal{F}_{n+1+j}^{k}\right)\\
&\qquad+\frac{1}{x^{h}}\left(s\mathcal{F}_{n+h}^{k}+(q-1)\,sD_q\mathcal{F}_{n+1+h}^{k}\right)\\
&\quad= \frac{1}{x^{h-1}}\mathcal{F}_{n+1+h}^{k+1} - x\mathcal{F}_{n+1}^{k+1} + \frac{1}{x^{h}}\left(s\mathcal{F}_{n+h}^{k}+(q-1)\,sD_q\mathcal{F}_{n+1+h}^{k}\right)\\
&\quad= \frac{1}{x^{h}}\left(x\mathcal{F}_{n+1+h}^{k+1}+s\mathcal{F}_{n+h}^{k}+(q-1)\,sD_q\mathcal{F}_{n+1+h}^{k}\right) - x\mathcal{F}_{n+1}^{k+1}\\
&\quad= \frac{1}{x^{h}}\mathcal{F}_{n+h+2}^{k+1} - x\mathcal{F}_{n+1}^{k+1},
\end{aligned}$$

which completes the proof of the assertion (2.9) by the principle of mathematical induction on h. □

Theorem 8. *A summation formula for the incomplete q-Lucas polynomials is given by*

$$\sum_{j=0}^{h-1} \frac{1}{x^j} \left(s\mathcal{L}_{n+j}^{k} + (q-1)\, sD_q \mathcal{L}_{n+1+j}^{k} \right) = \frac{1}{x^{h-1}} \mathcal{L}_{n+1+h}^{k+1} - x\mathcal{L}_{n+1}^{k+1}, \tag{2.10}$$

where $x \neq 0$ and $n \geqq 2k+1$.

Proof. For using the principle of mathematical induction on h, we suppose that the assertion (2.10) is true for some integer $h > 1$. We thus find from the equation (2.7) that

$$\begin{aligned}
&\sum_{j=0}^{h} \frac{1}{x^j} \left(s\mathcal{L}_{n+j}^{k} + (q-1)\, sD_q \mathcal{L}_{n+1+j}^{k} \right) \\
&\quad = \sum_{j=0}^{h-1} \frac{1}{x^j} \left(s\mathcal{L}_{n+j}^{k} + (q-1)\, sD_q \mathcal{L}_{n+1+j}^{k} \right) \\
&\qquad + \frac{1}{x^h} \left(s\mathcal{L}_{n+h}^{k} + (q-1)\, sD_q \mathcal{L}_{n+1+h}^{k} \right) \\
&\quad = \frac{1}{x^{h-1}} \mathcal{L}_{n+1+h}^{k+1} - x\mathcal{L}_{n}^{k+1} + \frac{1}{x^h} \left(s\mathcal{L}_{n+h}^{k} + (q-1)\, sD_q \mathcal{L}_{n+1+h}^{k} \right) \\
&\quad = \frac{1}{x^h} \left(x\mathcal{L}_{n+1+h}^{k+1} + s\mathcal{L}_{n+h}^{k} + (q-1)\, sD_q \mathcal{L}_{n+1+h}^{k} \right) - x\mathcal{L}_{n+1}^{k+1} \\
&\quad = \frac{1}{x^h} \mathcal{L}_{n+h+2}^{k+1} - x\mathcal{L}_{n+1}^{k+1},
\end{aligned}$$

which completes the proof of the assertion (2.10) by the principle of mathematical induction on h. □

We now derive some summation formulas for the incomplete q-Lucas polynomials.

Lemma 1. *It is asserted that*

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j} = \frac{1}{2} \left(n\mathcal{L}_n - x \frac{d}{dx} \mathcal{L}_n \right). \tag{2.11}$$

Proof. We observe that

$$\frac{d}{dx} \mathcal{L}_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j \,(n-2j)\, x^{n-2j-1}$$

$$
\begin{aligned}
&= n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j-1} \\
&\quad -2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j-1} \\
&= n x^{-1} \mathcal{L}_n - 2x^{-1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j},
\end{aligned}
$$

which proves the assertion (2.11) of Lemma 1. □

The following theorem asserts the summation formula of the incomplete q-Lucas polynomials.

Theorem 9. *Let $\mathcal{L}_n^k$ be the nth incomplete q-Lucas polynomial. Then*

$$
\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k = \left(\left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{L}_n + \frac{x}{2} \frac{d}{dx} \mathcal{L}_n. \tag{2.12}
$$

Proof. By using Definition 2 of the incomplete q-Lucas polynomials, we get

$$
\begin{aligned}
\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k &= \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n \right\} + \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + q^{\binom{1}{2}} \frac{[n]}{[n-1]} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} s x^{n-2} \right\} \\
&\quad + \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + q^{\binom{1}{2}} \frac{[n]}{[n-1]} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} s x^{n-2} + q^{\binom{2}{2}} \frac{[n]}{[n-2]} \begin{bmatrix} n-2 \\ 2 \end{bmatrix} s^2 x^{n-4} \right\} \\
&\quad + \cdots + \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + \cdots + q^{\binom{\lfloor \frac{n}{2} \rfloor}{2}} \frac{[n]}{[n-\lfloor \frac{n}{2} \rfloor]} \begin{bmatrix} n-\lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \right\} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 - j \right) q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} j \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j},
\end{aligned}
$$

which, in view of Lemma 1, yields

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k = \left(\lfloor \tfrac{n}{2} \rfloor - \frac{n}{2} + 1 \right) \mathcal{L}_n + \frac{x}{2} \frac{d}{dx} \mathcal{L}_n .$$

□

3. Generating functions

In this section, we obtain the generating functions of the incomplete q-Fibonacci and the q-Lucas polynomials.

Lemma 2 (see [17]). *Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following non-homogeneous recurrence relation:*

$$s_n = x s_{n-1} + y s_{n-2} + r_n \quad (n > p), \tag{3.1}$$

where $\{r_n\}$ is a given complex sequence. Then the generating function $S^k(x,y;t)$ of the sequence $\{s_n\}$ is given by

$$S^k(x,y;t) = \left(s_0 - r_0 + (s_1 - x s_0 - r_0) t + G(t)\right)\left(1 - xt - yt^2\right)^{-1}, \tag{3.2}$$

where $G(t)$ denotes the generating function of the given sequence $\{r_n\}$.

Theorem 10. *The generating function of the incomplete q-Fibonacci polynomials is given by*

$$\begin{aligned} U^k(x,s,q;t) = t^{2k+1} \Bigg(& \mathcal{F}_{2k+1} + \left(\mathcal{F}_{2k+2} - \left(x + (q-1) s D_q\right) \mathcal{F}_{2k+1}\right) t \\ & + q^{\binom{k+2}{2}} s^{k+1} t^2 \frac{1}{(1-xtq)_q^{k+1}} \Bigg) \left(1 - \left(x + (q-1) s D_q\right) t - st^2\right)^{-1}, \end{aligned} \tag{3.3}$$

where $\mathcal{F}_k = F_k(x,s,q)$ are the q-Fibonacci polynomials.

Proof. From (2.1) and (2.4), we get

$$\mathcal{F}_n^k = 0 \quad (0 \leqq n \leqq 2k+1)$$

and

$$\mathcal{F}_{2k+1}^k = \mathcal{F}_{2k+1} \qquad \text{and} \qquad \mathcal{F}_{2k+2}^k = \mathcal{F}_{2k+2}.$$

Also, for $n \geqq 2k+3$, we have

$$\begin{aligned} \mathcal{F}_{n+2k+1}^k = & \left(x + (q-1) s D_q\right) \mathcal{F}_{n+2k}^k \\ & + s \mathcal{F}_{n+2k-1}^k - q^{\binom{k}{2}} q^{n+2k-1} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} s^{k+1} x^{n-2}. \end{aligned}$$

Let

$$s_n = \mathcal{F}_{n+2k+1}^{k}$$

and

$$r_0 = r_1 = 0 \qquad \text{and} \qquad r_n = q^{\binom{k}{2}} q^{n+2k-1} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} s^{k+1} x^{n-2}.$$

We then find that

$$G(t) = q^{\binom{k}{2}} q^{2k+1} s^{k+1} t^2 \frac{1}{(1-xtq)_q^{k+1}}$$

and

$$S^k(x,s,q;t) = \Bigg(\mathcal{F}_{2k+1}^{k} + \left(\mathcal{F}_{2k+2}^{k} - \left(x + (q-1) s D_q\right) \mathcal{F}_{2k+1}^{k} \right) t + q^{\binom{k}{2}} q^{2k+1} s^{k+1} t^2 \frac{1}{(1-xtq)_q^{k+1}} \Bigg) \cdot \left(1 - \left(x + (q-1) s D_q\right) t - s t^2\right)^{-1}.$$

Therefore, we have

$$U^k(x,s,q;t) = t^{2k+1} S^k(x,s,q;t).$$

□

When $q \to 1-$ in Theorem 10, then we obtain the generating function of the incomplete bivariate Fibonacci polynomials (see [22]).

Theorem 11. *The generating function of the incomplete q-Lucas polynomials is given by*

$$V^k(x,s,q;t) = t^{2k} \Bigg(\mathcal{L}_{2k} + \left(\mathcal{L}_{2k+1} - \left(x + (q-1) s D_q\right) \mathcal{L}_{2k} \right) t + q^{\binom{k+1}{2}} s^{k+1} t^2 \left(q^{k+1} (1-xt) + 1 \right) \frac{1}{(1-xtq)_q^{k+1}} \Bigg) \cdot \left(1 - \left(x + (q-1) s D_q\right) t - s t^2\right)^{-1}, \tag{3.4}$$

where $\mathcal{L}_k = L_k(x,s,q)$ are the q-Lucas polynomials.

Proof. Considering the relation between the incomplete q-Fibonacci and q-Lucas polynomials, we get

$$V^k(x,s,q;t) = \sum_{n=0}^{\infty} \mathcal{L}_n^k t^n$$

$$= \sum_{n=0}^{\infty} \mathcal{F}_{n+1}^{k} t^{n} + s \sum_{n=0}^{\infty} \mathcal{F}_{n+1}^{k-1} t^{n}$$
$$= t^{-1} U^{k}(x,s,q;t) + st U^{k-1}(x,s,q;t).$$

□

4. CONCLUSION

In the present study, we have introduce the q-analogues of the incomplete Fibonacci and Lucas polynomials which satisfy essentially analogous recursion formulas and recurrence relations with the familiar q-Fibonacci and q-Lucas polynomials. Applications and some generalizations of the q-Fibonacci polynomials are given earlier in [15, 16], which contain some nice results for the q-Fibonacci polynomials. Also, Erkuş-Duman and Tuglu [12] studied various families of multilinear and multilateral generating functions for the generalized bivariate Fibonacci and Lucas polynomials. These works motivate the derivations of similar results for the incomplete q-Fibonacci and q-Lucas polynomials which we have investigated in this paper. By means of the relationships (1.8) and (1.9), we have exhibited the fact that the results for the q-analogues which we consider in this article for $0 < q < 1$ and the corresponding results for the (p,q)-analogues (with $0 < q < p \leqq 1$) are essentially equivalent, requiring only some obvious parametric variations, the additional parameter p being redundant.

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