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Some results on zeros of palindromic and perturbed polynomials of even degree

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Abstract: *In this paper we give necessary and sufficient conditions for all zeros of palindromic polynomial of even degree* $R(z) = 1 + \lambda(z + z^2 + ... + z^{n-1}) + z^n$*, with* $\lambda \in \mathbb{R}$*, to be on the unit circle and we find* $\gamma \in \mathbb{R}$ *for which* $S(z) = R(z) + \gamma z^n$ *has all its zeros inside or on the unit circle .*

Keywords: *Palindromic Polynomials, Zeros of Polynomials, Perturbed Polynomials*

Let $P(z) = a_0 + a_1 z + ... + a_n z^n$ be a polynomial of degree n, $n \geq 1$, $a_i \in \mathbb{R}$, $i = 0, ..., n$. Then P is palindromic if $a_i = a_{n-i}$, for every $i = 0, 1, ..., n$. In this paper we give necessary and sufficient conditions for all zeros of palindromic polynomial of even degree n, $n \geq 1$, $R(z) = 1 + \lambda(z + z^2 + ... + z^{n-1}) + z^n$, with $\lambda \in \mathbb{R}$, to lie on the unit circle. Furthermore, we prove that the polynomial $S(z) = R(z) + \gamma z^n$, with $\gamma \geq \lambda - 2$ ($\gamma > 0, \lambda \geq 0$), has all its zeros in the closed unit disc. More details can be found in [1, 4].

1 Classical results

Theorem 1.1 (Eneström-Kakeya, real coefficients case)**.** *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial such that* $0 < a_0 \leq a_1 \leq \ldots \leq a_n$*. Then,* $P(z)$ *has all its zeros in the closed unit disc.*

Definition 1.2. Let the polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, $a_i \in \mathbb{R}$. Define the associated polynomial

$$P^*(z) = z^n P\left(\frac{1}{z}\right) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n = a_0 \prod_{j=1}^{n} (z - z_j^*),$$

whose zeros z_k^* are the inverses of the zeros z_k of $P(z)$, that is, $z_k^* = \frac{1}{\bar{z}_k}$.

Definition 1.3. If $P(z) = P^*(z)$, that is, $P(z) = z^n P\left(\frac{1}{z}\right)$, the polynomial $P(z)$ is said to be palindromic.

It is clear that if $P(z) = \sum_{i=0}^{n} a_i z^i$, $a_i \in \mathbb{R}$, $i = 0, ..., n$, is palindromic, then $a_i = a_{n-i}$, $i = 0, 1, ..., n$, as we mentioned above.

Definition 1.4. Given $P(z)$ with real coefficients, the sequence of polynomials $P_j(z)$ is defined by:

$$P_j(z) = \sum_{k=0}^{n-j} a_k^{(j)} z^k, \quad \text{where} \quad P_0(z) = P(z) \quad \text{and}$$

$$P_{j+1}(z) := a_0^{(j)} P_j(z) - a_{n-j}^{(j)} P_j^*(z),\ j = 0, 1, \ldots, n-1, \tag{1.1}$$

with $P_0^*(z) = P^*(z)$.

From (1.1), the coefficients of $P_{j+1}(z)$ satisfy the recurrence relation

$$a_k^{(j+1)} = a_0^{(j)} a_k^{(j)} - a_{n-j}^{(j)} a_{n-j-k}^{(j)},\ k = 0, 1, \ldots n-j \ \text{ and } \ j = 0, 1, \ldots n. \tag{1.2}$$

Definition 1.5. For each polynomial $P_j(z)$ we shall denote the constant term $a_0^{(j)}$ by δ_j and

$$\delta_{j+1} = a_0^{(j+1)} = |a_0^{(j)}|^2 - |a_{n-j}^{(j)}|^2,\ j = 0, 1, \ldots, n-1.$$

Lemma 1.6. *If P_j has p_j zeros in $|z| < 1$ and if $\delta_{j+1} \neq 0$, then P_{j+1} has*

$$p_{j+1} = \begin{cases} p_j, & \text{if} \quad \delta_{j+1} > 0 \\ n - j - p_j, & \text{if} \quad \delta_{j+1} < 0 \end{cases}$$

zeros in $|z| < 1$. Furthermore, P_{j+1} has the same zeros on $|z| = 1$ as P_j.

The proof of this lemma may be found in Marden [3], p. 195.

The next result is due to Schur [5, 6] and the proof follows from Lemma 1.6.

Lemma 1.7. *If $0 < |a_0| < |a_n|$, then $P(z)$ has all its zeros in the closed unit disc if, and only if, $P_1^*(z)$ has all its zeros in the closed unit disc.*

Using the same notation presented in [2], let $\mathbf{a} = (a_1, a_2, ..., a_{n-1}) \in \mathbb{R}^{n-1}$ and $L : \mathbb{R}^{n-1} \to \mathbb{R}$ be a function defined by

$$L(\mathbf{a}) := \min_{y \in \mathbb{R}} \sum_{j=1}^{n-1} |a_j - y|.$$

With a permutation σ on $\{1, 2, ..., n-1\}$ for which $a_{\sigma(1)} \leq a_{\sigma(2)} \leq ... \leq a_{\sigma(n-1)}$ one has: if n is even, then $L(\mathbf{a}) = \sum_{j=1}^{n-1} |a_j - a_{\sigma(n/2)}|$; if n is odd, then $L(\mathbf{a}) = \sum_{j=1}^{n-1} |a_j - y|$ for every y in a closed interval $[a_{\sigma(\lfloor n/2 \rfloor)}, a_{\sigma(\lceil n/2 \rceil)}]$, where $\lfloor t \rfloor := \max(-\infty, t] \cap \mathbb{Z}$ and $\lceil t \rceil := \min[t, \infty) \cap \mathbb{Z}$. In addition, considering $\overline{m}(\mathbf{a})$ (resp. $\underline{m}(\mathbf{a})$) defined by $\overline{m}(\mathbf{a}) := a_{\sigma(\lceil n/2 \rceil)}$ (resp. $\underline{m}(\mathbf{a}) := a_{\sigma(\lfloor n/2 \rfloor)}$) then $\overline{m}(\mathbf{a}) = \underline{m}(\mathbf{a})$ when n is even.

Theorem 1.8. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a palindromic polynomial of degree n with $a_n > 0$, and let $\mathbf{a} = (a_1, a_2, ..., a_{n-1})$.*

1. *Suppose $\underline{m}(\mathbf{a}) + L(\mathbf{a}) \leq 2a_n$.*
   - *(a) If $P(1) \geq 0$, then all zeros of P lie on the unit circle. In this case, there are at least two zeros of the form $e^{i\theta}$ with $-\frac{2\pi}{n} \leq \theta \leq \frac{2\pi}{n}$.*
   - *(b) If $P(1) < 0$, then P has real zeros $\beta > 1$ and β^{-1} and the other zeros lie on the unit circle.*
2. *Suppose $\overline{m}(\mathbf{a}) \geq L(\mathbf{a}) + 2a_n$. Then one of the following holds:*
   - *(a) All the zeros of P lie on the unit circle. When n is odd, there are three or five zeros of the form $e^{i\theta}$ with $\frac{(n-1)\pi}{n} \leq \theta \leq \frac{(n+1)\pi}{n}$. When n is even, -1 is a zero with multiplicity 2 or 4.*
   - *(b) P has real zeros $\beta < -1$ and β^{-1} and the other zeros lie on the unit circle.*

The proof of this result may be found in [2].

2 Main Results

Theorem 2.1. *The zeros of the polynomial* $R(z) = 1 + \lambda(z + z^2 + ... + z^{n-1}) + z^n$, $\lambda \in \mathbb{R}$, *of even degree* $n > 1$, *lie on the unit circle if and only if* $-\frac{2}{n-1} \leq \lambda \leq 2$.

Proof. From Theorem 1.8, $\mathbf{a} = (\lambda, \lambda, ..., \lambda)$, $\overline{m}(\mathbf{a}) = \underline{m}(\mathbf{a}) = \lambda$ and $L(\mathbf{a}) = 0$.

If $\underline{m}(\mathbf{a}) + L(\mathbf{a}) \leq 2$, i.e., $\lambda \leq 2$, as $R(1) = 2 + (n-1)\lambda \geq 0$ when $\lambda \geq -\frac{2}{n-1}$, from item (1) (a) of Theorem 1.8 follows that all zeros of $R(z)$ lie on the unit circle when $-\frac{2}{n-1} \leq \lambda \leq 2$.

Furthermore, if $\underline{m}(\mathbf{a}) + L(\mathbf{a}) = \lambda \leq 2$ and $R(1) = 2 + (n-1)\lambda < 0$, i.e., $\lambda < -\frac{2}{n-1}$, $R(z)$ has one real root in $(1, \infty)$. In fact,

$$\lim_{z \to 1} R(z) = 2 + (n-1)\lambda < 0 \text{ and } \lim_{z \to +\infty} R(z) > 0,$$

that is, there is a signal change of $R(z)$ in $(1, \infty)$. This case is described in item (1) (b) of Theorem 1.8.

If $\lambda > 2$ ($\overline{m}(\mathbf{a}) > L(\mathbf{a}) + 2$), $R(z)$ has one real root in $(-\infty, -1)$. In fact,

$$\lim_{z \to -\infty} R(z) > 0 \text{ and } \lim_{z \to -1} R(z) = 2 - \lambda < 0,$$

that is, there is a signal change of $R(z)$ in $(-\infty, -1)$. Observe that this case is described in item (2) (b) of Theorem 1.8.

So, for n even, we prove that the zeros of $R(z)$ lie on the unit circle if, and only if, $-\frac{2}{n-1} \leq \lambda \leq 2$. □

Remark 2.2. If n is even and $\lambda = 2$, we have $R(-1) = 0$ and $z = -1$ is a zero of multiplicity 2 of $R(z)$, as described in item (2) (a) of Theorem 1.8.

Theorem 2.3. *The perturbed polynomial*

$$S(z) = R(z) + \gamma z^n = 1 + \lambda(z + z^2 + ... + z^{n-1}) + (1+\gamma)z^n, \ (\lambda \geq 0, \gamma > 0, n \ \textit{even})$$

has all zeros in the closed unit disc if $\gamma \geq \lambda - 2$ *and has at least one zero outside the closed unit disc if* $\gamma < \lambda - 2$.

Proof. For $\lambda = 0$, we have $S(z) = 1 + (1+\gamma)z^n$ and the proof is immediate.

From here, we consider $\lambda > 0$.

We write the polynomials $S(z)$ and $S_1(z)$ in the form

$$S(z) = s_n z^n + s_{n-1} z^{n-1} + \ldots + s_0,$$

where $s_n = 1 + \gamma$, $s_i = \lambda$, $i = 1, \ldots, n-1$, and $s_0 = 1$, and

$$S_1(z) = s_{n-1}^{(1)} z^{n-1} + s_{n-2}^{(1)} z^{n-2} + \ldots + s_0^{(1)},$$

where the coefficients $s_k^{(1)}$, $k = 0, 1, \ldots, n-1$, are defined by equation 1.2 using $j = 0$. So,

$$s_k^{(1)} = s_0 s_k - s_n s_{n-k}.$$

Substituting the values of s_k, $k = 0, ..., n$, we have

$$s_{n-1}^{(1)} = s_{n-2}^{(1)} = ... = s_1^{(1)} = -\gamma\lambda < 0 \text{ and } s_0^{(1)} = -\gamma(\gamma + 2) < 0.$$

Note that, as $\gamma > 0$, $0 < 1 < 1 + \gamma$, i.e., $0 < s_0 < s_n$, Lemma 1.7 can be applied to conclude that the zeros of $S(z)$ lie in the closed unit disc if and only if the zeros of $S_1^*(z)$ do.

Observe that

$$-S_1^*(z) = |s_{n-1}^{(1)}| + |s_{n-2}^{(1)}| z + ... + |s_1^{(1)}| z^{n-2} + |s_0^{(1)}| z^{n-1}.$$

If $|s_0^{(1)}| \geq |s_1^{(1)}| > 0$, the coefficients of $-S_1^*(z)$ are ordered and by the Eneström-Kakeya Theorem, the zeros of $-S_1^*(z)$ lie in $|z| \leq 1$. As the zeros of $S_1^*(z)$ and $-S_1^*(z)$ are the same, the zeros of $S_1^*(z)$ lie in $|z| \leq 1$ too.

But

$$|s_0^{(1)}| - |s_1^{(1)}| = \gamma(\gamma + 2 - \lambda) \geq 0.$$

Then, $|s_0^{(1)}| \geq |s_1^{(1)}|$ is equivalent to $\gamma \geq \lambda - 2$.

So, for $\gamma \geq \lambda - 2$, $S(z)$ has all its zeros in $|z| \leq 1$.

Now we prove that, if $\gamma < \lambda - 2$, $S(z)$ has at least one zero outside the unit disc.

As

$$|s_0^{(1)}| - |s_{n-1}^{(1)}| = \gamma(\gamma + 2 - \lambda),$$

$|s_0^{(1)}| < |s_{n-1}^{(1)}|$ is equivalent to $\gamma < \lambda - 2$.

By the Vieta's formula, we have

$$\zeta_1\zeta_2 ... \zeta_{n-1} = (-1)^{n-1}\frac{s_{n-1}^{(1)}}{s_0^{(1)}},$$

where ζ_i, $i = 1, ..., n-1$, are the zeros of $S_1^*(z)$.

So, if $\gamma < \lambda - 2$, follows that

$$|\zeta_1\zeta_2 ... \zeta_{n-1}| = \left|\frac{s_{n-1}^{(1)}}{s_0^{(1)}}\right| > 1.$$

Then, at least one zero of $S_1^*(z)$ lie outside the unit disc and, consequently, $S(z)$ has at least one zero outside the unit disc. □

Remark 2.4. For $\gamma = 0$ we have $S(z) = R(z)$ and the zeros of $S(z)$ lie on the unit circle under the conditions of Theorem 2.1.

3 Numerical Examples

Example 3.1. Let us consider the polynomial $R(z) = 1 + \frac{5}{3}\left(z + z^2 + z^3\right) + z^4$. Figure 1 displays the zeros of $R(z)$ (represented by •) and $S(z)$ for $\gamma = 0.5$ (represented by *). Note that the conditions of Theorem 2.1 are satisfied and the zeros of $R(z)$ lie on the unit circle. From Theorem 2.3 the zeros of the perturbed polynomial $S(z)$, for all $\gamma \geq 0$, lie inside or on the unit circle.

Example 3.2. Let us consider the polynomial $R(z) = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + 2z^5 + z^6$. Figure 2 displays the zeros of $R(z)$ (represented by •) and $S(z)$ for $\gamma = 0.8$ (represented by *). The conditions of Theorem 2.1 are satisfied and the zeros of $R(z)$ lie on the unit circle (from Remark 2.2, $z = -1$ is a zero of multiplicity 2 of $R(z)$). From Theorem 2.3 the zeros of the perturbed polynomial $S(z)$, for all $\gamma \geq 0$, lie inside or on the unit circle.

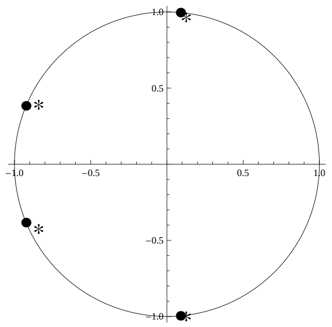

Figure 1: Zeros of $S(z) = 1 + \frac{5}{3}\left(z + z^2 + z^3\right)(1 + \gamma)z^4$ for $\gamma = 0$ (dots) and $\gamma = 0.5$ (stars).

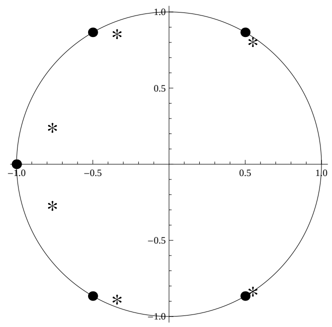

Figure 2: Zeros of $S(z) = 1+2z+2z^2+2z^3 + 2z^4 + 2z^5 + (1 + \gamma)z^6$ for $\gamma = 0$ (dots) and $\gamma = 0.8$ (stars).

Example 3.3. Let us consider the polynomial $R(z) = 1 + 4(z + z^2 + z^3) + z^4$. Figure 3 displays the zeros of $R(z)$ (represented by $\bullet$) and $S(z)$ for $\gamma = 2$ (represented by *) and $\gamma = 4$ (represented by +). As $\lambda = 4 > 2$, from Theorem 2.1 $R(z)$ has one real zero in $(-\infty, -1)$. From Theorem 2.3, the zeros of $S(z)$ lie inside or on the unit circle when $\gamma \geq 2$ and for $0 < \gamma < 2$, $S(z)$ has at least one zero outside the unit circle.

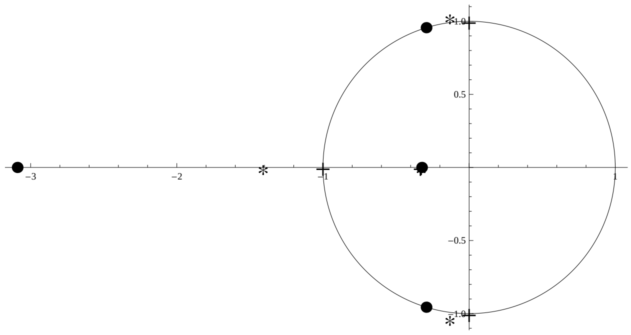

Figure 3: Zeros of $S(z) = 1 + 4(z + z^2 + z^3) + (1 + \gamma)z^4$ for $\gamma = 0$ (dots), $\gamma = 1$ (stars) and $\gamma = 2$ (plus).

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