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# Some second-order $\theta$ schemes combined with finite element method for nonlinear fractional Cable equation

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**Abstract** In this article, some second-order time discrete schemes covering parameter $\theta$ combined with Galerkin finite element (FE) method are proposed and analyzed for looking for the numerical solution of nonlinear Cable equation with time fractional derivative. At time $t_{k-\theta}$, some second-order $\theta$ schemes combined with weighted and shifted Grünwald difference (WSGD) approximation of fractional derivative are considered to approximate the time direction, and the Galerkin FE method is used to discretize the space direction. The stability of second-order $\theta$ schemes is derived and the second-order time convergence rate in $L^2$-norm is proved. Finally, some numerical calculations are implemented to indicate the feasibility and effectiveness for our schemes.

**Keywords** Second-order $\theta$ scheme · Nonlinear fractional Cable equation · Finite element algorithm · Stability · Error estimates

**Mathematics Subject Classification (2000)** 65M60 · 65N15 · 65N30

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## 1 Introduction

Fractional differential equations (FDEs) have been increasingly concerned by more and more researchers in scientific and engineering fields. Many important problems can be solved by considering the corresponding FDEs, which include fractional diffusion problems, fractional wave equations, fractional Cable equations and so forth. The solutions of FDEs are difficultly solved by some analytic methods, so some numerical methods based on the features of fractional derivatives and fractional equations have to be developed. In the literatures, these numerical methods cover FE methods [1,3,6–9,11–14,17,18,33], finite difference (FD) methods [4,5], spectral methods [19,22], finite volume (element) methods [2], discontinuous Galerkin methods [27,31] and so forth.

Here, we will develop some Galerkin FE algorithms to solve the nonlinear time fractional Cable equation

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{K}{}_0^R D_t^{\alpha} u(\mathbf{x},t) - {}_0^R D_t^{\beta} \Delta u(\mathbf{x},t) + g(u) = f(\mathbf{x},t), (\mathbf{x},t) \in \Omega \times J, \\ u(\mathbf{x},t) = 0, \mathbf{x} \in \partial\Omega, t \in \bar{J}, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \mathbf{x} \in \bar{\Omega}, \end{cases} \tag{1}$$

where $\Omega \subset R^d, d = 1,2$ and $J = (0,T]$ are spatial domain and temporal interval with $0 < T < \infty$, respectively. The initial value $u_0(\mathbf{x})$ and the source term $f(\mathbf{x},t)$ are given functions, $\mathcal{K}$ is a non-negative constant, the nonlinear reaction term $g(u)$ satisfies $|g(u)| \leq C|u|$ with $|g'(u)| \leq C$, where $C$ is a positive constant. And ${}_0^R D_t^{\gamma} w(\mathbf{x},t)$ is Riemann-Liouville (R-L) fractional-order derivative with order $\gamma \in (0,1)$ defined by

$${}_0^R D_t^{\gamma} w(\mathbf{x},t) = \frac{1}{\Gamma(1-\gamma)} \frac{\partial}{\partial t} \int_0^t \frac{w(\mathbf{x},s)}{(t-s)^{\gamma}} ds. \tag{2}$$

Fractional Cable equation, which is a class of vital mathematical model reflecting the anomalous electro-diffusion in nerve cells, has been theoretically and numerically discussed by some authors [19,20,25,22–24,26].

Usually, ones approximate time derivative at integer or fractional points by a lot of numerical schemes, such as backward Euler method (BEM), second-order Crank-Nicolson method (CNM), second-order backward difference method (BDM) and so forth. In [19], Lin et al. proposed the FD/spectral approximations, which are formulated by using spectral approximation in space, L1-formula for time fractional derivative and second-order BDM for temporal integer derivative, for solving the Cable equation with time fractional derivative. In [11,12], Liu et al. developed FE methods combined with second-order BDM for solving fourth-order nonlinear time fractional reaction-diffusion problems. Ding and Li [10] proposed a second-order midpoint approximation formula for R-L derivative, and formulated FD scheme based on CNM in time for solving time fractional Cable equation. Recently, in [30], Gao et al. proposed some FD schemes for linear time fractional sub-diffusion equations, and discussed the stability and convergence based on certain superconvergence at some point $t_{n-\frac{\alpha}{2}}$. Following the idea in [30], Wang et al. [26] derived the FE approximations combined with second-order discrete scheme at time $t_{n-\frac{\alpha}{2}}$ for solving nonlinear time fractional Cable equation. Based on Alikhanov's work [29], Sun et al. [28] proposed some temporal second order FD schemes for fractional wave equations, which rely on the fractional parameters.

Besides, Tian et al. [15] proposed the WSGD formula for approximating R-L space fractional derivatives. Compared with the $L1$-approximation, the WSGD formula can get the second-order convergence rate, which is not affected by the changed fractional parameters. Following this idea, Wang and Vong [16] applied the WSGD formula to approximating the Caputo fractional derivative in time, and formulated FD scheme to solve the modified time fractional sub-diffusion equation and time fractional diffusion-wave equation. Ji and Sun[32] used a FD scheme with WSGD formula to solve a linear fractional diffusion equation. In [21], Liu et al. studied a two-grid FE method combined WSGD approximation for a two-dimensional time fractional Cable equation, and made some comparisons in computational time. In [31], Liu et al. proposed a local discontinuous Galerkin (LDG) method with WSGD approximation for linear time fractional subdiffusion equation. In [27], Du et al. combined with WSGD formula with LDG method for solving fourth-order nonlinear time FDE.

In this paper, motivated by the works in [26,28–30], some second-order $\theta$ schemes combined with FE methods and WSGD approximation are proposed. For solving nonlinear time fractional Cable equation, we propose some new second-order $\theta$ schemes, in which we approximate the temporal integer derivative $\frac{\partial u}{\partial t}$ at any point $t_{n-\theta}$ ($\forall\theta \in [0, \frac{1}{2}]$) by some new second-order $\theta$ formulas, discretize the time fractional derivatives by second-order WSGD formula proposed by Tian et al. [15] and give some new second-order approximate formulas for nonlinear term. Compared to these discrete schemes [26,28,30], our methods can get the approximate result at any $t_{n-\theta}$ ($\forall\theta \in [0, \frac{1}{2}]$). Moreover, our methods can cover second-order CNM with $\theta = \frac{1}{2}$ and second-order BDM with $\theta = 0$.

Throughout this article, we will denote $C > 0$ as a constant, which is independent of the time step length $\Delta t$ and space mesh parameter $h$. The layout of the paper is as follows. In Section 2, we show some lemmas and do some analysis of stability for fully discrete scheme. In Section 3, we give some detailed error analysis in $L^2$-norm. In Section 4, we provide some numerical results to confirm the theoretical analysis. In Section 5, we give some conclusions.

## 2 Numerical approximation and stability

For obtaining fully discrete scheme, we insert the nodes $t_n = n\Delta t(n = 0, 1, 2, \cdots, N)$ in the time interval $[0, T]$, where $t_n$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ with mesh length $\Delta t = T/N$ for some positive integer $N$. We now define $\phi^n = \phi(t_n)$ for a smooth function $\phi$ on $[0, T]$.

To do the next study, we need to consider some lemmas on integer and fractional derivatives in time.

**Lemma 1** *With $v(t) \in C^3[0,T]$, at time $t_{n-\theta}$, the following second-order result for approximating first-order derivative for any $\theta \in [0, \frac{1}{2}]$ holds*

$$\frac{\partial v}{\partial t}(t_{n-\theta}) = \begin{cases} \partial_t[v^{n-\theta}] + O(\Delta t^2), \ n \geq 2; \\ \dfrac{v^1 - v^0}{\Delta t} + O(\Delta t), n = 1 \end{cases} \tag{3}$$

*where*

$$\partial_t[v^{n-\theta}] \triangleq \frac{(3-2\theta)v^n - (4-4\theta)v^{n-1} + (1-2\theta)v^{n-2}}{2\Delta t}.$$

**Lemma 2** *With $v(t) \in C^2[0,T]$, at time $t_{n-\theta}$, two important approximate formulas*

$$\begin{aligned} f(t_{n-\theta}) =& (1-\theta)f^n + \theta f^{n-1} + O(\Delta t^2) \\ \triangleq& f^{n-\theta} + O(\Delta t^2), \end{aligned} \tag{4}$$

*and*

$$\begin{aligned} g(v(t_{n-\theta})) =& (2-\theta)g(v^{n-1}) - (1-\theta)g(v^{n-2}) + O(\Delta t^2) \\ \triangleq& g[v^{n-\theta}] + O(\Delta t^2), \end{aligned} \tag{5}$$

*hold for any $\theta \in [0, \frac{1}{2}]$.*

**Proof**. At time $t_{n-\theta}$, we use the Taylor's formula for $f(t_n)$ and $f(t_{n-1})$ to easily get

$$f^{n-\theta} = (1-\theta)f^n + \theta f^{n-1} + O(\Delta t^2), \tag{6}$$

and

$$g(v(t_{n-\theta})) = (1-\theta)g(v^n) + \theta g(v^{n-1}) + O(\Delta t^2), \tag{7}$$

By using Taylor's formula for $g(v^{n-1})$ and $f(v^{n-2})$ at time $t_n$, we easily get

$$g(v^n) = 2g(v^{n-1}) - g(v^{n-2}) + O(\Delta t^2). \tag{8}$$

Substitute (8) into (7) to get (5).

**Lemma 3** *From [5], we easily find that the following first-order approximate scheme holds, for Riemann-Liouville fractional derivative with parameter $\gamma \in (0,1)$*

$${}_0^R D_t^\gamma v(t_n) = \Delta t^{-\gamma} \sum_{l=0}^{n} w_l^\gamma v^{k-l} + O(\Delta t), \tag{9}$$

*where $w_0^\gamma = 1, w_l^\gamma = (-1)^l \begin{pmatrix} \gamma \\ l \end{pmatrix} = \frac{\Gamma(l-\gamma)}{\Gamma(-\gamma)\Gamma(l+1)}$, $l \geq 1$. And it is easy to know that series $w_l^\gamma$ satisfy*

$$w_l^\gamma < 0, w_l^\gamma = (1 - \frac{\gamma+1}{l})w_{l-1}^\gamma, (l = 1, 2 \cdots), \sum_{l=1}^{\infty} w_l^\gamma = -1. \tag{10}$$

**Lemma 4** *Let $\bar{v}(t)$, Liouville fractional derivative ${}_{-\infty}D_t^{\gamma+2}\bar{v}(t)$ and the Fourier transform $\hat{\bar{v}}$ belong to $L^1(R)$. According to the Ref. [15], ones can arrive at*

$$\begin{aligned}{}_{-\infty}D_t^{\gamma}\bar{v}(t) =&\Delta t^{-\gamma}\sum_{i=0}^{\infty} w_i^{\gamma}\Big[\frac{\gamma-2q}{2(p-q)}\bar{v}(t-(i-p)\Delta t)\\ &+\frac{2p-\gamma}{2(p-q)}\bar{v}(t-(i-q)\Delta t)\Big]+O(\Delta t^2).\end{aligned} \tag{11}$$

*Further, by taking* $\bar{v}(t)=\begin{cases} v(t), & t\in[0,T],\\ 0, & t\in(-\infty,0).\end{cases}$ *and $(p,q)=(0,-1)$, the following approximate formula with second-order accuracy at time $t=t_n$ holds for $0<\gamma<1$*

$$\begin{aligned}{}_0^RD_t^{\gamma}v(t_n) =&\Delta t^{-\gamma}\sum_{i=0}^{n}\mathcal{A}_{\gamma}(i)v^{n-i}+O(\Delta t^2)\\ \triangleq& I_{\gamma}^n[v^n]+O(\Delta t^2),\end{aligned} \tag{12}$$

*with*

$$\mathcal{A}_{\gamma}(i)=\begin{cases}\frac{\gamma+2}{2}w_0^{\gamma}, & \text{when } i=0,\\ \frac{\gamma+2}{2}w_i^{\gamma}+\frac{-\gamma}{2}w_{i-1}^{\gamma}, & \text{when } i>0,\end{cases} \tag{13}$$

*where series $w_i^{\gamma}$ are defined in Lemma 3.*

**Lemma 5** *(See [16]) Let $\{\mathcal{A}_{\gamma}(i)\}$ be defined as in (13). Then for any positive integer $L$ and real vector $(v^0,v^1,\cdots,v^L)\in R^{L+1}$, it holds that*

$$\sum_{n=0}^{L}\sum_{i=0}^{n}\mathcal{A}_{\gamma}(i)\Big(v^{n-i},v^n\Big)\geq 0. \tag{14}$$

We now use the approximate formulas at time $t_{k-\theta}$ based on lemmas 1-4 to get semi-discrete formulation in the time direction
Case $n=1$:

$$\begin{aligned}&\Big(\frac{u^1-u^0}{\Delta t},v\Big)+\mathcal{K}\Big((1-\theta)I_{\alpha}^1[u^1]+\theta I_{\alpha}^0[u^0],v\Big)\\ &+\Big((1-\theta)I_{\beta}^1[\nabla u^1]+\theta I_{\beta}^0[\nabla u^0],\nabla v\Big)+(g(u^0),v)\\ =&(f^{1-\theta},v)+(\sum_{k=0}^{4}E_k^{1-\theta},v),\forall v\in H_0^1,\end{aligned} \tag{15}$$

Case $n\geq 2$:

$$\begin{aligned}&\Big(\partial_t[u^{n-\theta}],v\Big)+\mathcal{K}\Big((1-\theta)I_{\alpha}^n[u^n]+\theta I_{\alpha}^{n-1}[u^{n-1}],v\Big)\\ &+\Big((1-\theta)I_{\beta}^n[\nabla u^n]+\theta I_{\beta}^{n-1}[\nabla u^{n-1}],\nabla v\Big)+(g[u^{n-\theta}],v)\\ =&(f^{n-\theta},v)+(\sum_{k=0}^{4}E_k^{n-\theta},v),\ \forall v\in H_0^1,\end{aligned} \tag{16}$$

where

$$
\begin{aligned}
&E_0^{1-\theta} = \partial_t[v^1] - u_t(t_{1-\theta}) = O(\Delta t),\\
&E_1^{n-\theta} = \mathcal{K}{}_0^R D_t^\alpha u(t_{n-\theta}) - \mathcal{K} I_\alpha^{n-\theta}[u^{n-\theta}] = O(\Delta t^2),\\
&E_2^{n-\theta} = {}_0^R D_t^\beta \Delta u(t_{n-\theta}) - I_\beta^{n-\theta}[\Delta u^{n-\theta}] = O(\Delta t^2),\\
&E_3^{1-\theta} = g(u^0) - g(u^{1-\theta}) = O(\Delta t), E_4^{1-\theta} = f^1 - f^{1-\theta} = O(\Delta t),\\
&E_0^{n-\theta} = \partial_t[v^{n-\theta}] - \frac{\partial u}{\partial t}(t_{n-\theta}) = O(\Delta t^2),\\
&E_3^{n-\theta} = g[u^{n-\theta}] - g(u(t_{n-\theta})) = O(\Delta t^2),\ E_4^{n-\theta} = f^{n-\theta} - f(t_{n-\theta}) = O(\Delta t^2),\\
&I_\gamma^{n-\theta}[u^{n-\theta}] \triangleq (1-\theta) I_\gamma^n[u^n] + \theta I_\gamma^{n-1}[u^{n-1}].
\end{aligned}
\tag{17}
$$

Based on the time semi-discrete scheme, we get the following fully discrete scheme by choosing finite element space $V_h \subset H_0^1$.
Case $n = 1$:

$$
\begin{aligned}
&\Big(\frac{u_h^1 - u_h^0}{\Delta t}, v_h\Big) + \mathcal{K}\Big((1-\theta) I_\alpha^1[u_h^1] + \theta I_\alpha^0[u_h^0], v_h\Big)\\
&+ \Big((1-\theta) I_\beta^1[\nabla u_h^1] + \theta I_\beta^0[\nabla u_h^0], \nabla v_h\Big) + (g(u_h^0), v_h) = (f^{1-\theta}, v_h), \forall v_h \in V_h.
\end{aligned}
\tag{18}
$$

Case $n \geq 2$:

$$
\begin{aligned}
&\Big(\partial_t[u_h^{n-\theta}], v_h\Big) + \mathcal{K}\Big((1-\theta) I_\alpha^n[u_h^n] + \theta I_\alpha^{n-1}[u_h^{n-1}], v_h\Big) + \Big((1-\theta) I_\beta^n[\nabla u_h^n]\\
&+ \theta I_\beta^{n-1}[\nabla u_h^{n-1}], \nabla v_h\Big) + (g[u_h^{n-\theta}], v_h) = (f^{n-\theta}, v_h), \forall v_h \in V_h.
\end{aligned}
\tag{19}
$$

**Remark 1** *In [26,30], the finite element scheme and finite difference system are proposed at time $t_{n-\frac{\alpha}{2}}$, where $\alpha \in (0,1)$ is the parameter of time fractional derivative, that is to say that the time discrete schemes in [26,30] must be formulated at time $t_{n-\frac{\alpha}{2}}$ and were related to time fractional parameter $\alpha$. In our paper, the scheme (19) is proposed at time $t_{n-\theta}$, where $\theta \in [0, 1/2]$ is an arbitrary constant independent of time fractional parameter $\alpha$.*

**Remark 2** *(I). When taking $\theta = 0$, the scheme (19) is reduced to second-order backward difference/ finite element scheme*

$$
\begin{aligned}
&\Big(\frac{3u_h^n - 4u_h^{n-1} + u_h^{n-2}}{2\Delta t}, v_h\Big) + \mathcal{K}\Big(I_\alpha^n[u_h^n], v_h\Big) + \Big(I_\beta^n[\nabla u_h^n], v_{hx}\Big)\\
&\qquad + (2g(u_h^{n-1}) - g(u_h^{n-2}), v_h) = (f^n, v_h), \forall v_h \in V_h.
\end{aligned}
\tag{20}
$$

*(II). When taking $\theta = 1/2$, the scheme (19) is reduced to Crank-Nicolson finite difference/ finite element scheme*

$$\begin{aligned}\Big(\frac{u_h^n - u_h^{n-1}}{\Delta t}, v_h\Big) &+ \frac{\mathcal{K}}{2}\Big(I_\alpha^n[u_h^n] + I_\alpha^{n-1}[u_h^{n-1}], v_h\Big)\\ &+ \frac{1}{2}\Big(I_\beta^n[\nabla u_h^n] + I_\beta^{n-1}[\nabla u_h^{n-1}], \nabla v_h\Big)\\ &+ \Big(\frac{3}{2}g(u_h^{n-1}) - \frac{1}{2}g(u_h^{n-2}), v_h\Big) = \Big(\frac{f^n + f^{n-1}}{2}, v_h\Big), \forall v_h \in V_h.\end{aligned} \tag{21}$$

For analyzing the stability and error estimates, we need to consider the following lemma.

**Lemma 6** *For series $\{\chi^n\}$ $(n \geq 2)$, the following inequality holds*

$$\Big(\partial_t[\chi^{n-\theta}], \chi^{n-\theta}\Big) \geq \frac{1}{4\Delta t}(\mathbb{H}[\chi^n] - \mathbb{H}[\chi^{n-1}]), \tag{22}$$

$$\mathbb{H}[\chi^n] = (3-2\theta)\|\chi^n\|^2 - (1-2\theta)\|\chi^{n-1}\|^2 + (2-\theta)(1-2\theta)\|\chi^n - \chi^{n-1}\|^2, \tag{23}$$

*and*

$$\mathbb{H}[\chi^n] \geq \frac{1}{1-\theta}\|\chi^n\|^2, \tag{24}$$

*where $0 \leq \theta \leq 1/2$.*

**Proof**. We make use of a similar proof to the one in Ref. [28] to get the conclusion of lemma 6.

In what follows, we consider the following stable inequality.

**Theorem 1** *For $u_h^n \in V_h$, the stability for fully discrete system (18)-(19) holds*

$$\|u_h^n\|^2 \leq C(\|u_h^0\|^2 + \max_{0\leq i\leq n}\|f^i\|^2). \tag{25}$$

**Proof**. For the case $n \geq 2$, we choose $v_h = u_h^{n-\theta} = (1-\theta)u_h^n + \theta u_h^{n-1}$ in (19) and use lemma 6 to arrive at

$$\begin{aligned}&\frac{1}{4\Delta t}(\mathbb{H}[u_h^n] - \mathbb{H}[u_h^{n-1}]) + \mathcal{K}\Big((1-\theta)I_\alpha^n[u_h^n] + \theta I_\alpha^{n-1}[u_h^{n-1}], u_h^{n-\theta}\Big)\\ &+ \Big((1-\theta)I_\beta^n[\nabla u_h^n] + \theta I_\beta^{n-1}[\nabla u_h^{n-1}], \nabla u_h^{n-\theta}\Big) + (g[u_h^{n-\theta}], u_h^{n-\theta})\\ \leq&(f^{n-\theta}, u_h^{n-\theta}).\end{aligned} \tag{26}$$

Sum (26) from $n=2$ to $L$ and use inequality (24) to get

$$\begin{aligned} &\mathbb{H}(u_h^L)+4\mathcal{K}\Delta t\sum_{n=2}^{L}\Big((1-\theta)I_\alpha^n[u_h^n]+\theta I_\alpha^{n-1}[u_h^{n-1}],u_h^{n-\theta}\Big)\\ &+4\Delta t\sum_{n=2}^{L}\Big((1-\theta)I_\beta^n[\nabla u_h^n]+\theta I_\beta^{n-1}[\nabla u_h^{n-1}],\nabla u_h^{n-\theta}\Big)\\ \le&\mathbb{H}(u_h^1)+4\Delta t\sum_{n=2}^{L}(f^{n-\theta},u_h^{n-\theta})-4\Delta t\sum_{n=2}^{L}(g[u_h^{n-\theta}],u_h^{n-\theta}). \end{aligned} \tag{27}$$

For the next proof, we now consider the second term on the left hand side of (27). Now we use some notations to get

$$\begin{aligned} &4\mathcal{K}\Delta t\sum_{n=2}^{L}\Big((1-\theta)I_\alpha^n[u_h^n]+\theta I_\alpha^{n-1}[u_h^{n-1}],u_h^{n-\theta}\Big)\\ =&4\mathcal{K}\Delta t^{1-\alpha}\sum_{n=2}^{L}\Big((1-\theta)\sum_{i=0}^{n}\mathcal{A}_\alpha(i)u_h^{n-i}+\theta\sum_{i=0}^{n-1}\mathcal{A}_\alpha(i)u_h^{n-1-i},u_h^{n-\theta}\Big)\\ =&4\mathcal{K}\Delta t^{1-\alpha}\sum_{n=2}^{L}\Big(\sum_{i=0}^{n}\mathcal{A}_\alpha(i)[(1-\theta)u_h^{n-i}+\theta u_h^{n-1-i}],u_h^{n-\theta}\Big)\\ =&4\mathcal{K}\Delta t^{1-\alpha}\sum_{n=2}^{L}\sum_{i=0}^{n}\mathcal{A}_\alpha(i)\Big(u_h^{n-\theta-i},u_h^{n-\theta}\Big). \end{aligned} \tag{28}$$

By the similar derivation to (28), we have

$$\begin{aligned} &4\Delta t\sum_{n=2}^{L}\Big((1-\theta)I_\beta^n[\nabla u_h^n]+\theta I_\beta^{n-1}[\nabla u_h^{n-1}],\nabla u_h^{n-\theta}\Big)\\ =&4\Delta t^{1-\beta}\sum_{n=2}^{L}\sum_{i=0}^{n}\mathcal{A}_\beta(i)\Big(\nabla u_h^{n-\theta-i},\nabla u_h^{n-\theta}\Big). \end{aligned} \tag{29}$$

We now handle the three terms on the right hand side of (27). We use Cauchy-Schwarz inequality and Young inequality to arrive at

$$\begin{aligned} &\mathbb{H}[u_h^1]+4\Delta t\sum_{n=2}^{L}(f^{n-\theta},u_h^{n-\theta})-4\Delta t\sum_{n=2}^{L}(g[u_h^{n-\theta}],u_h^{n-\theta})\\ \le&\mathbb{H}[u_h^1]+2\Delta t\sum_{n=2}^{L}(\|f^{n-\theta}\|^2+\|u_h^{n-\theta}\|^2)+2\Delta t\sum_{n=2}^{L}(\|g[u_h^{n-\theta}]\|^2+\|u_h^{n-\theta}\|^2)\\ \le&\mathbb{H}[u_h^1]+C\Delta t\sum_{n=1}^{L}\|f^n\|^2+C\Delta t\sum_{n=0}^{L}\|u_h^n\|^2. \end{aligned} \tag{30}$$

Substitute (28)-(30) into (27) to get

$$\begin{aligned}&\mathbb{H}[u_h^L]+4\mathcal{K}\Delta t^{1-\alpha}\sum_{n=2}^{L}\sum_{i=0}^{n}\mathcal{A}_\alpha(i)\Big(u_h^{n-\theta-i},u_h^{n-\theta}\Big)\\&+4\Delta t^{1-\beta}\sum_{n=2}^{L}\sum_{i=0}^{n}\mathcal{A}_\beta(i)\Big(\nabla u_h^{n-\theta-i},\nabla u_h^{n-\theta}\Big)\\ \le&\mathbb{H}[u_h^1]+C\Delta t\sum_{n=1}^{L}\|f^n\|^2+C\Delta t\sum_{n=0}^{L}\|u_h^n\|^2.\end{aligned}\tag{31}$$

In what follows, we need to consider the case $n=1$. We now set $v_h=(1-\theta)u_h^1+\theta u_h^0$ in (18) and note that

$$\Big(\frac{u_h^1-u_h^0}{\Delta t},(1-\theta)u_h^1+\theta u_h^0\Big)=\frac{\|u_h^1\|^2-\|u_h^0\|^2}{2\Delta t}+\frac{1-2\theta}{2\Delta t}\|u_h^1-u_h^0\|^2,\tag{32}$$

to arrive at

$$\begin{aligned}&\frac{\|u_h^1\|^2-\|u_h^0\|^2}{2\Delta t}+\frac{1-2\theta}{2\Delta t}\|u_h^1-u_h^0\|^2+\mathcal{K}\Delta t^{-\alpha}\sum_{i=0}^{1}\mathcal{A}_\alpha(i)\Big(u_h^{1-\theta-i},u_h^{1-\theta}\Big)\\&+\Delta t^{-\beta}\sum_{i=0}^{1}\mathcal{A}_\beta(i)\Big(\nabla u_h^{1-\theta-i},\nabla u_h^{1-\theta}\Big)+(g(u_h^0),u_h^{1-\theta})=(f^{1-\theta},u_h^{1-\theta}).\end{aligned}\tag{33}$$

Noting that $1-2\theta\ge 0$, multiplying (34) by $2\Delta t$ using Cauchy-Schwarz inequality with Young inequality, we arrive at

$$\begin{aligned}&\|u_h^1\|^2+2\mathcal{K}\Delta t^{1-\alpha}\sum_{i=0}^{1}\mathcal{A}_\alpha(i)\Big(u_h^{1-\theta-i},u_h^{1-\theta}\Big)\\&+2\Delta t^{1-\beta}\sum_{i=0}^{1}\mathcal{A}_\beta(i)\Big(\nabla u_h^{1-\theta-i},\nabla u_h^{1-\theta}\Big)\\=&\|u_h^0\|^2+2\Delta t(g(u_h^0),u_h^{1-\theta})+2\Delta t(f^{1-\theta},u_h^{1-\theta})\\ \le&C\|u_h^0\|^2+2\Delta t\|u_h^1\|^2+C\Delta t(\|f^0\|^2+\|f^1\|^2).\end{aligned}\tag{34}$$

Simplifying (34), we arrive at

$$\begin{aligned}\|u_h^1\|^2\le&-2\mathcal{K}\Delta t^{1-\alpha}\sum_{i=0}^{1}\mathcal{A}_\alpha(i)\Big(u_h^{1-\theta-i},u_h^{1-\theta}\Big)\\&-2\Delta t^{1-\beta}\sum_{i=0}^{1}\mathcal{A}_\beta(i)\Big(\nabla u_h^{1-\theta-i},\nabla u_h^{1-\theta}\Big)\\&+C\|u_h^0\|^2+C\Delta t(\|f^0\|^2+\|f^1\|^2).\end{aligned}\tag{35}$$

So, we have

$$\begin{aligned}
&\mathbb{H}[u_h^1] = (3-2\theta)\|u_h^1\|^2 - (1-2\theta)\|u_h^0\|^2 + (\theta-2)(2\theta-1)\|u_h^1-u_h^0\|^2 \\
\le& -2\mathcal{K}\Delta t^{1-\alpha}\sum_{i=0}^{1}\mathcal{A}_\alpha(i)\Big(u_h^{1-\theta-i},u_h^{1-\theta}\Big) - 2\Delta t^{1-\beta}\sum_{i=0}^{1}\mathcal{A}_\beta(i)\Big(\nabla u_h^{1-\theta-i},\nabla u_h^{1-\theta}\Big) \\
&+C\|u_h^0\|^2 + C\Delta t(\|f^0\|^2+\|f^1\|^2),
\end{aligned} \tag{36}$$

Substitute (36) into (31) and use (24) to get

$$\begin{aligned}
&\|u_h^L\|^2 + 4\mathcal{K}\Delta t^{1-\alpha}\sum_{n=1}^{L}\sum_{i=0}^{n}\mathcal{A}_\alpha(i)\Big(u_h^{n-\theta-i},u_h^{n-\theta}\Big) \\
&+4\Delta t^{1-\beta}\sum_{n=1}^{L}\sum_{i=0}^{n}\mathcal{A}_\beta(i)\Big(\nabla u_h^{n-\theta-i},\nabla u_h^{n-\theta}\Big) \\
\le& C\|u_h^0\|^2 + C\Delta t\sum_{n=0}^{L}\|f^n\|^2 + C\Delta t\sum_{n=0}^{L}\|u_h^n\|^2.
\end{aligned} \tag{37}$$

Making use of lemma 5 and the discrete Gronwall inequality for sufficiently small $\Delta t$, we get

$$\|u_h^L\|^2 \le C\|u_h^0\|^2 + C\Delta t\sum_{n=0}^{L}\|f^n\|^2, \tag{38}$$

which shows the conclusion of theorem 1.

## 3 A priori error analysis

For considering a priori error estimates for finite element method, we need to give the projection operator and the estimate inequality.

**Lemma 7** *Define a Ritz projection operator* $\mathfrak{S}_h : H_0^1(\Omega) \to V_h$ *satisfying*

$$(\nabla(z-\mathfrak{S}_h z),\nabla z_h) = 0,\ \forall z_h \in V_h, \tag{39}$$

*with the estimate inequality*

$$\|z-\mathfrak{S}_h z\| + h\|z-\mathfrak{S}_h z\|_1 \le Ch^{m+1}\|z\|_{m+1},\ \forall z \in H_0^1(\Omega)\cap H^{m+1}(\Omega), \tag{40}$$

*where the norms are defined by* $\|z\|_l = \sqrt{\sum_{0\le|r|\le l}\int_\Omega |D^r z|^2}$.

For the convenience of error analysis in the following derivations, we now write

$$u(t_n) - u_h^n = (u(t_n)-\mathfrak{S}_h u^n) + (\mathfrak{S}_h u^n - u_h^n) = \eta^n + \xi^n.$$

The error equation is as follows
Case $n = 1$:

$$\begin{aligned}
&\Big(\frac{\xi^1-\xi^0}{\Delta t}, v_h\Big) + \mathcal{K}\Big((1-\theta)I_\alpha^1[\xi^1] + \theta I_\alpha^0[\xi^0], v_h\Big)\\
&+ \Big((1-\theta)I_\beta^1[\nabla\xi^1] + \theta I_\beta^0[\nabla\xi^0], \nabla v_h\Big)\\
=&\Big(\frac{\eta^1-\eta^0}{\Delta t}, v_h\Big) - (g(u^0) - g(u_h^0), v_h) - \mathcal{K}\Big((1-\theta)I_\alpha^1[\eta^1] + \theta I_\alpha^0[\eta^0], v_h\Big)\\
&+ (\sum_{k=0}^{4} E_k^{1-\theta}, v_h), \forall v_h \in V_h.
\end{aligned} \tag{41}$$

Case $n \geq 2$:

$$\begin{aligned}
&\Big(\partial_t[\xi^{n-\theta}], v_h\Big) + \mathcal{K}\Big((1-\theta)I_\alpha^n[\xi^n] + \theta I_\alpha^{n-1}[\xi^{n-1}], v_h\Big)\\
&+ \Big((1-\theta)I_\beta^n[\nabla\xi^n] + \theta I_\beta^{n-1}[\nabla\xi^{n-1}], \nabla v_h\Big)\\
=&-\Big(\partial_t[\eta^{n-\theta}], v_h\Big) - (g[u^{n-\theta}] - g[u_h^{n-\theta}], v_h) - \mathcal{K}\Big((1-\theta)I_\alpha^n[\eta^n]\\
&+ \theta I_\alpha^{n-1}[\eta^{n-1}], v_h\Big) + (\sum_{k=0}^{4} E_k^{n-\theta}, v_h), \forall v_h \in V_h.
\end{aligned} \tag{42}$$

In what follows, we will give the detailed proof of error estimates in $L^2$-norm.

**Theorem 2** *Let $u(t_n)$ be the solution of system (15)-(16) and $u_h^n$ be the solution of system (18)-(19), respectively. For the sufficiently smooth solution $u(t) \in C^3[0,T]$ with $u_h^0 = \mathfrak{S}_h u_0$, there exists a constant $C$ independent of space-time mesh pair $(h, \Delta t)$ such that*

$$\|u(t_n) - u_h^n\| \leq C[\Delta t^2 + h^{m+1}]. \tag{43}$$

**Proof**. Take $v_h = \xi^{n-\theta} = (1-\theta)\xi^n + \theta\xi^{n-1}$ in (42), sum $n$ from 2 to $L$ and use (22)-(36) to arrive at

$$\begin{aligned}
&\frac{\mathbb{H}[\xi^L] - \mathbb{H}[\xi^1]}{4\Delta t} + \mathcal{K}\sum_{n=2}^{L}\Big((1-\theta)I_\alpha^n[\xi^n] + \theta I_\alpha^{n-1}[\xi^{n-1}], \xi^{n-\theta}\Big)\\
&+ \sum_{n=2}^{L}\Big((1-\theta)I_\beta^n[\nabla\xi^n] + \theta I_\beta^{n-1}[\nabla\xi^{n-1}], \nabla\xi^{n-\theta}\Big)\\
\leq& -\sum_{n=2}^{L}\Big(\partial_t[\eta^{n-\theta}], \xi^{n-\theta}\Big) - \sum_{n=2}^{L}(g[u^{n-\theta}] - g[u_h^{n-\theta}], \xi^{n-\theta})\\
&- \mathcal{K}\sum_{n=2}^{L}\Big((1-\theta)I_\alpha^n[\eta^n] + \theta I_\alpha^{n-1}[\eta^{n-1}], \xi^{n-\theta}\Big) + \sum_{n=2}^{L}(\sum_{k=0}^{4} E_k^{n-\theta}, \xi^{n-\theta}).
\end{aligned} \tag{44}$$

Now we estimate every terms on the right hand side of (44). We use Cauchy-Schwarz inequality as well as Young inequality to get

$$\begin{aligned}
-\sum_{n=2}^{L}\Big(\partial_t[\eta^{n-\theta}],\xi^{n-\theta}\Big) \leq& \sum_{n=2}^{L}\|\partial_t[\eta^{n-\theta}]\|\|\xi^{n-\theta}\| \\
\leq& \frac{(3-2\theta)}{\Delta t}\int_{t_0}^{t_L}\|\eta_t\|^2 ds + \frac{1}{2}\sum_{n=2}^{L}\|(1-\theta)\xi^n+\theta\xi^{n-1}\|^2 \\
\leq& \frac{(3-2\theta)}{\Delta t}\int_{t_0}^{t_L}\|\eta_t\|^2 ds + \sum_{n=1}^{L}\|\xi^n\|^2.
\end{aligned} \tag{45}$$

Use the triangle inequality, Cauchy-Schwarz inequality and Young inequality to get

$$\begin{aligned}
&-\sum_{n=2}^{L}(g[u^{n-\theta}]-g[u_h^{n-\theta}],\xi^{n-\theta}) \\
\leq& \sum_{n=2}^{L}\|(2-\theta)[g(u^{n-1})-g(u_h^{n-1})]-(1-\theta)[g(u^{n-2})-g(u_h^{n-2})]\|\|\xi^{n-\theta}\| \\
\leq& \sum_{n=2}^{L}(\|(2-\theta)g'(u^{\mu 1})(\eta^{n-1}+\xi^{n-1})\|+\|(1-\theta)g'(u^{\mu 2})(\eta^{n-2}+\xi^{n-2})\|)\|\xi^{n-\theta}\| \\
\leq& C\sum_{n=0}^{L}(\|\eta^n\|^2+\|\xi^n\|^2).
\end{aligned} \tag{46}$$

Using Cauchy-Schwarz inequality, Young inequality and lemma 7 with the similar technique in [23], we arrive at

$$\begin{aligned}
&-\mathcal{K}\sum_{n=2}^{L}\Big((1-\theta)I_\alpha^n[\eta^n]+\theta I_\alpha^{n-1}[\eta^{n-1}],\xi^{n-\theta}\Big) \\
=&\mathcal{K}\Delta t^{-\alpha}\sum_{n=2}^{L}\Big((1-\theta)\sum_{i=0}^{n}\mathcal{A}_\alpha(i)\eta^{n-i}+\theta\sum_{i=0}^{n-1}\mathcal{A}_\alpha(i)\eta^{n-1-i},\xi^{n-\theta}\Big) \\
\leq&\mathcal{K}\sum_{n=2}^{L}\Big((1-\theta)\|\Delta t^{-\alpha}\sum_{i=0}^{n}\mathcal{A}_\alpha(i)\eta^{n-i}\|\|\xi^{n-\theta}\|+\theta\|\Delta t^{-\alpha}\sum_{i=0}^{n-1}\mathcal{A}_\alpha(i)\eta^{n-1-i}\|\|\xi^{n-\theta}\|\Big) \\
=&\mathcal{K}\sum_{n=2}^{L}\Big((1-\theta)\|{}_0^R D_{t_n}^\alpha\eta+O(\Delta t^2)\|\|\xi^{n-\theta}\|+\theta\|{}_0^R D_{t_{n-1}}^\alpha\eta+O(\Delta t^2)\|\|\xi^{n-\theta}\|\Big) \\
\leq&C\mathcal{K}(h^{m+1}+\Delta t^2)\sum_{n=1}^{L}\|\xi^n\|\leq C\sum_{n=1}^{L}(h^{2m+2}+\Delta t^4)+\frac{1}{2}\sum_{n=1}^{L}\|\xi^n\|^2.
\end{aligned} \tag{47}$$

Make use of Cauchy-Schwarz inequality and Young inequality to arrive at

$$\sum_{n=2}^{L}(\sum_{k=0}^{4}E_k^{n-\theta},\xi^{n-\theta}) \leq C\sum_{n=2}^{L}(\Delta t^4+\|\xi^n\|^2). \tag{48}$$

Substitute (45)-(48) to (44) to get

$$\begin{aligned}&\frac{\mathbb{H}[\xi^L]-\mathbb{H}[\xi^1]}{4\Delta t}+\mathcal{K}\sum_{n=2}^{L}\Big((1-\theta)I_\alpha^n[\xi^n]+\theta I_\alpha^{n-1}[\xi^{n-1}],\xi^{n-\theta}\Big)\\&+\sum_{n=2}^{L}\Big((1-\theta)I_\beta^n[\nabla\xi^n]+\theta I_\beta^{n-1}[\nabla\xi^{n-1}],\nabla\xi^{n-\theta}\Big)\\ \leq&\frac{(3-2\theta)}{\Delta t}\int_{t_0}^{t_L}\|\eta_t\|^2ds+C\sum_{n=0}^{L}(\|\eta^n\|^2+\|\xi^n\|^2)+C\sum_{n=1}^{L}(h^{2m+2}+\Delta t^4).\end{aligned} \tag{49}$$

Multiply (49) by $4\Delta t$ to get

$$\begin{aligned}&\mathbb{H}[\xi^L]+\mathcal{K}\sum_{n=2}^{L}\Big((1-\theta)I_\alpha^n[\xi^n]+\theta I_\alpha^{n-1}[\xi^{n-1}],\xi^{n-\theta}\Big)\\&+\sum_{n=2}^{L}\Big((1-\theta)I_\beta^n[\nabla\xi^n]+\theta I_\beta^{n-1}[\nabla\xi^{n-1}],\nabla\xi^{n-\theta}\Big)\\ \leq&\mathbb{H}[\xi^1]+(12-8\theta)\int_{t_0}^{t_L}\|\eta_t\|^2ds+C\Delta t\sum_{n=0}^{L}(\|\eta^n\|^2+\|\xi^n\|^2)\\&+C\Delta t\sum_{n=1}^{L}(h^{2m+2}+\Delta t^4).\end{aligned} \tag{50}$$

Taking $v_h=\xi^{1-\theta}=(1-\theta)\xi^1+\theta\xi^0$ in (41) and noting that the formula (32), we have

$$\begin{aligned}&\frac{\|\xi^1\|^2-\|\xi^0\|^2}{2\Delta t}+\mathcal{K}\Big((1-\theta)I_\alpha^1[\xi^1]+\theta I_\alpha^0[\xi^0],\xi^{1-\theta}\Big)\\&+\Big((1-\theta)I_\beta^1[\nabla\xi^1]+\theta I_\beta^0[\nabla\xi^0],\nabla\xi^{1-\theta}\Big)\\ \leq&\Big(\frac{\eta^1-\eta^0}{\Delta t},\xi^{1-\theta}\Big)-(g(u^0)-g(u_h^0),\xi^{1-\theta})\\&-\mathcal{K}\Big((1-\theta)I_\alpha^1[\eta^1]+\theta I_\alpha^0[\eta^0],\xi^{1-\theta}\Big)+(\sum_{k=0}^{4}E_k^{1-\theta},\xi^{1-\theta}).\end{aligned} \tag{51}$$

Multiply (51) by $2\Delta t$ and use Cauchy-Schwarz inequality and Young inequality to get

$$
\begin{aligned}
&\|\xi^1\|^2 + 2\Delta t\mathcal{K}\Big((1-\theta)I_\alpha^1[\xi^1] + \theta I_\alpha^0[\xi^0], \xi^{1-\theta}\Big) \\
&+ 2\Delta t\Big((1-\theta)I_\beta^1[\nabla\xi^1] + \theta I_\beta^0[\nabla\xi^0], \nabla\xi^{1-\theta}\Big) \\
\leq&\|\xi^0\|^2 + 2\Delta t\Big(\frac{\eta^1-\eta^0}{\Delta t}, \xi^{1-\theta}\Big) - 2\Delta t(g(u^0) - g(u_h^0), \xi^{1-\theta}) \\
&- 2\Delta t\mathcal{K}\Big((1-\theta)I_\alpha^1[\eta^1] + \theta I_\alpha^0[\eta^0], \xi^{1-\theta}\Big) + 2\Delta t(\sum_{k=0}^{4} E_k^{1-\theta}, \xi^{1-\theta}) \\
\leq& Ch^{2m+2} + (\frac{1}{2} + C\Delta t)\|\xi^1\|^2 + C(\Delta t^4 + \Delta t^6).
\end{aligned}
\tag{52}
$$

Combine (50) with (52) and use (40) to get

$$
\begin{aligned}
&\mathbb{H}[\xi^L] + 4\mathcal{K}\Delta t^{1-\alpha}\sum_{n=1}^{L}\sum_{i=0}^{n}\mathcal{A}_\alpha(i)\Big(\xi^{n-\theta-i}, \xi^{n-\theta}\Big) \\
&+ 4\Delta t^{1-\beta}\sum_{n=1}^{L}\sum_{i=0}^{n}\mathcal{A}_\beta(i)\Big(\nabla\xi^{n-\theta-i}, \nabla\xi^{n-\theta}\Big) \\
\leq& C(h^{2m+2} + \Delta t^4 + \Delta t\sum_{n=0}^{L}\|\xi^n\|^2).
\end{aligned}
\tag{53}
$$

Note that inequality (24) and use Cronwall lemma and lemma 5 to get

$$
\|\xi^L\|^2 \leq C(h^{2m+2} + \Delta t^4). \tag{54}
$$

Combine (54) with (40), and use triangle inequality to get the conclusion of theorem.

**Remark 3** *In [21], from the conclusion ones can see that there is the term $\Delta t^{-\alpha}h^{m+1}$, which results in the conditional convergence results in error theory. In this paper, we have dropped the influences of the term $\Delta t^{-\alpha}$ and obtained the unconditional error results.*

## 4 Some numerical results

In this section, we choose some numerical examples to verify the spatial convergence rate and temporal convergence rate, respectively. In subsection 4.1 and subsection 4.2, to test the convergence rate with order 2, we choose two-dimensional example and one-dimensional examples, respectively.

## 4.1 Two-dimensional example

Firstly, we need to verify the spatial convergence order by a numerical example.
**Example 4.1.1.** In (1), we take $\mathcal{K} = 1$, $g(u) = u^3 - u$, the spatial domain $[0, 1] \times [0, 1]$, the temporal interval $[0, 1]$ and the exact solution

$$u(\mathbf{x}, t) = t^2 \sin(2\pi x)\sin(2\pi y), \mathbf{x} = (x, y), \tag{55}$$

then get the source term

$$\begin{aligned} f(\mathbf{x}, t) =& \Big[2t - t^2 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 16\pi^2 \frac{t^{2-\beta}}{\Gamma(3-\beta)}\Big] \sin(2\pi x)\sin(2\pi y) \\ &+ t^6 \sin^3(2\pi x)\sin^3(2\pi y). \end{aligned}$$

Now we choose the continuous bilinear function space $V_h$ with $Q(x, y) = a_0 + a_1 x + a_2 y + a_3 xy$. In Table 1, for the given parameters $\theta = 0, 0.1, 0.3, 0.5$, we choose the fixed time step $\tau = 1/100$ and changed space mesh parameters $h = 1/20, 1/30, 1/40$, then arrive at the errors and convergence rates with parameter pair $(\alpha, \beta) = (0.01, 0.99), (0.5, 0.5)$ and $(0.99, 0.01)$, respectively. From the rate of convergence computed in Table 1, we can find that the approximate order is close to 2, which is in agreement with the theoretical results in space.

**Table 1** The spatial $L^2$-errors with $\Delta t = 1/100$

| | $(\alpha,\beta)$ | $(0.01, 0.99)$ | | $(0.5, 0.5)$ | | $(0.99, 0.01)$ | |
|---|---|---|---|---|---|---|---|
| | $h$ | $\|u - u_h\|$ | Rate | $\|u - u_h\|$ | Rate | $\|u - u_h\|$ | Rate |
| $\theta = 0$ | 1/20 | 4.1596e-003 | - | 4.2983e-003 | - | 4.4810e-003 | - |
| | 1/30 | 1.8706e-003 | 1.9710 | 1.9404e-003 | 1.9615 | 2.0184e-003 | 1.9669 |
| | 1/40 | 1.0573e-003 | 1.9832 | 1.1026e-003 | 1.9648 | 1.1433e-003 | 1.9757 |
| $\theta = 0.1$ | 1/20 | 4.1619e-003 | - | 4.2981e-003 | - | 4.4803e-003 | - |
| | 1/30 | 1.8729e-003 | 1.9693 | 1.9402e-003 | 1.9617 | 2.0177e-003 | 1.9674 |
| | 1/40 | 1.0596e-003 | 1.9800 | 1.1024e-003 | 1.9651 | 1.1426e-003 | 1.9767 |
| $\theta = 0.3$ | 1/20 | 4.1665e-003 | - | 4.2977e-003 | - | 4.4789e-003 | - |
| | 1/30 | 1.8774e-003 | 1.9661 | 1.9398e-003 | 1.9620 | 2.0162e-003 | 1.9685 |
| | 1/40 | 1.0641e-003 | 1.9736 | 1.1019e-003 | 1.9657 | 1.1411e-003 | 1.9787 |
| $\theta = 0.5$ | 1/20 | 4.1727e-003 | - | 4.2973e-003 | - | 4.4774e-003 | - |
| | 1/30 | 1.8836e-003 | 1.9617 | 1.9393e-003 | 1.9623 | 2.0147e-003 | 1.9102 |
| | 1/40 | 1.0702e-003 | 1.9650 | 1.1015e-003 | 1.9663 | 1.1396e-003 | 1.9808 |

## 4.2 One-dimensional examples

Further, for the sake of testing the rate of convergence in time direction and checking the influence of the parameters for the errors, we provided some one-dimensional examples. Here, for implementing the numerical computation, we take FE space covering piece linear basis functions. In (1), by choosing the parameter $\mathcal{K} = 1$, the nonlinear term $g(u) = u^2$ and the exact solution $u(x,t) = t^{\lambda}\sin(\pi x), \forall (x,t) \in [0,L]\times[0,T]$, we obtain the known function

$$f(x,t) = \Big[\lambda t^{\lambda-1} + \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}t^{\lambda-\alpha} + \pi^2\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}t^{\lambda-\beta}\Big]\sin(\pi x) + t^{2\lambda}\sin^2(\pi x), \forall (x,t) \in [0,L]\times[0,T].$$

**Example 4.2.1.** By taking $L = 4$, $T = 1$, $\lambda = 2.5$, the exact solution is $u(x,t) = t^{2.5}\sin(\pi x)$, $(x,t) \in [0,4]\times[0,1]$. In Table 2, we compute the errors in $L^2$-norm and convergence rate in time with different parameters $\alpha$, $\beta$, $\theta$ and changed time mesh parameters $\Delta t = 1/20, 1/40, 1/80$ and the fixed parameter $h = 1/100$. From these computed results in Table 2, it is easy to see that with the fixed parameter $\theta$, we get the second-order time convergence rate with changed fractional parameters $(\alpha,\beta) = (0.01, 0.01)$, $(0.5, 0.5)$, and $(0.99, 0.99)$, which show that the WSGD approximation has second-order accuracy in time, which is not impacted by the parameters $(\alpha,\beta)$. At the same time, for given fixed parameters $(\alpha,\beta)$, we also arrive at the second-order approximation in time, which show the $\theta$ schemes also have stable time second-order convergence rate which keeps the same results to our theory.

For checking the influence of parameters $\alpha$ and $\beta$, we give the contour plots of $u - u_h$ with the fixed $\theta = 0.3$, space-time mesh length $(h,\Delta t) = (\frac{1}{100}, \frac{1}{40})$ in Figs. 1-4. By the comparison between Fig. 1 and Fig 2, ones can see that when the smaller parameter $\alpha = 0.01$ with changed parameter $\beta = 0.01, 0.09$ are taken, the contour plots of $u - u_h$ has the larger changes. Similarly, from the comparison between Fig. 1 and Fig. 3, ones can also see the similar results. From Fig. 2 and Fig. 4, ones can find that for the chosen bigger parameter $\beta = 0.99$, the contour plots of $u - u_h$ are greatly affected. However, for the fixed bigger $\alpha = 0.99$, the values of $u - u_h$ has the smaller changes for changed parameters $\beta = 0.01, 0.99$ in Fig. 3 and Fig. 4.

**Table 2** The temporal $L^2$-errors with $h = 1/100$

| | $(\alpha,\beta)$ | $(0.01, 0.01)$ | | $(0.5, 0.5)$ | | $(0.99, 0.99)$ | |
|---|---|---|---|---|---|---|---|
| | $\Delta t$ | $\|u-u_h\|$ | Rate | $\|u-u_h\|$ | Rate | $\|u-u_h\|$ | Rate |
| $\theta = 0$ | 1/20 | 7.2901E-03 | - | 5.4710E-03 | - | 5.3071E-03 | |
| | 1/40 | 2.0121E-03 | 1.8572 | 1.5048E-03 | 1.8622 | 1.4021E-03 | 1.9203 |
| | 1/80 | 5.3798E-04 | 1.9031 | 4.0754E-04 | 1.8845 | 3.7426E-04 | 1.9055 |
| $\theta = 0.1$ | 1/20 | 6.6843E-03 | - | 5.0349E-03 | - | 5.0931E-03 | |
| | 1/40 | 1.8298E-03 | 1.8691 | 1.3737E-03 | 1.8739 | 1.3369E-03 | 1.9297 |
| | 1/80 | 4.8806E-04 | 1.9065 | 3.7196E-04 | 1.8849 | 3.5637E-04 | 1.9074 |
| $\theta = 0.3$ | 1/20 | 5.4016E-03 | - | 4.1258E-03 | - | 4.6767E-03 | |
| | 1/40 | 1.4544E-03 | 1.8930 | 1.1081E-03 | 1.8965 | 1.2138E-03 | 1.9460 |
| | 1/80 | 3.8684E-04 | 1.9106 | 3.0109E-04 | 1.8799 | 3.2310E-04 | 1.9094 |
| $\theta = 0.5$ | 1/20 | 4.0169E-03 | - | 3.1819E-03 | - | 4.2931E-03 | |
| | 1/40 | 1.0644E-03 | 1.9160 | 8.4282E-04 | 1.9166 | 1.1061E-03 | 1.9565 |
| | 1/80 | 2.8381E-04 | 1.9071 | 2.3208E-04 | 1.8606 | 2.9454E-04 | 1.9090 |

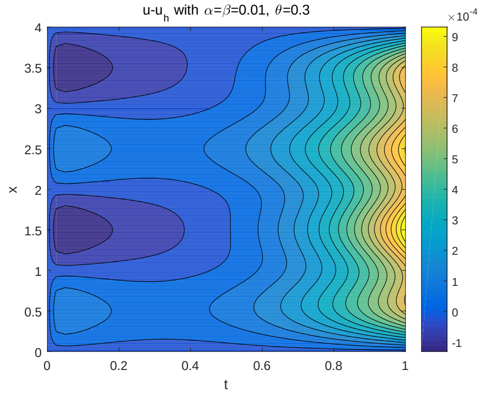


**Fig. 1** The contour plots of $u-u_h$ with $h=\frac{1}{100}$, $\Delta t=\frac{1}{40}$

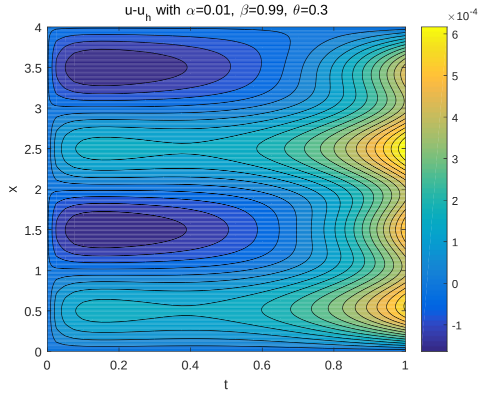


**Fig. 2** The contour plots of $u-u_h$ with $h=\frac{1}{100}$, $\Delta t=\frac{1}{40}$

**Example 4.2.2.** By choosing $L=2$, $T=1$, $\lambda=2$, the fixed spatial mesh $h=1/200$ and the changing time step $\Delta t=1/20, 1/40, 1/80$, we get the errors in $L^2$-norm and time convergence rate, which is close to 2. From the computed results in Table 3, ones can see that with the changed fractional parameter pairs $(\alpha,\beta)=(0.01,0.01),(0.5,0.5),(0.99,0.99)$ and different parameters $\theta=0,0.2,0.4,0.5$, the convergence rate in the current form of exact solution is consistent with the theoretical results.

In Figs. 5-8, we show the contour plots of $u-u_h$ by taking fixed parameters $(\alpha,\beta)=(0.5,0.5)$, space-time mesh length $(h,\Delta t)=(\frac{1}{100},\frac{1}{40})$ and changed parameters $\theta=0,0.2,0.4,0.5$ to check the impact of parameters. Ones can see that with the same magnitude $10^{-4}$, the maximum values of $|u-u_h|$ are reduced slightly with the decrease of parameter $\theta$.

Based on the above discussions on the calculated results in Tables 1-3 for Example 4.1.1, Examples 4.2.1-4.2.2 and the contour plots in Figs. 1-8 for

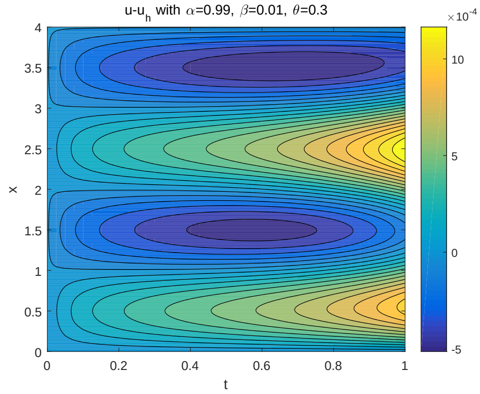


**Fig. 3** The contour plots of $u-u_h$ with $h=\frac{1}{100}$, $\Delta t=\frac{1}{40}$

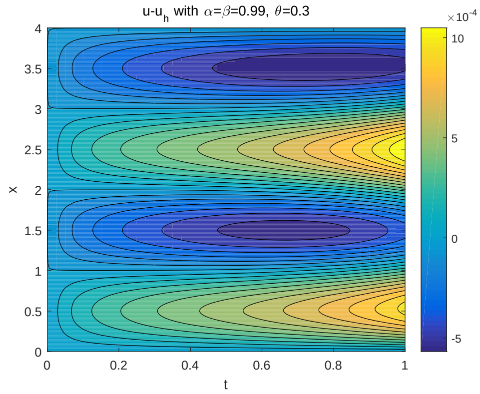


**Fig. 4** The contour plots of $u-u_h$ with $h=\frac{1}{100}$, $\Delta t=\frac{1}{40}$

Examples 4.2.1-4.2.2, ones can know that the second-order convergence rate can be obtained by our methods, also see that both second-order backward difference method with $\theta=0$ and Crank-Nicolson method with $\theta=0.5$ are the special cases of our second-order $\theta$ schemes. From Tables 2-3, we can see clearly that the errors in $L^2$-norm decrease gradually with the increase of parameter $\theta$ and also find the impact of parameters $(\alpha,\beta,\theta)$ for the values of $u-u_h$ in Figs. 1-8.

## 5 Conclusions and advancements

In this paper, we propose some second-order $\theta$ schemes combined with FE method, which can solve well the numerical solution for nonlinear time fractional Cable equation. We give detailed proof of stability of scheme and error estimate. On the purpose of testing theoretical results, we calculate the con-

**Table 3** The temporal $L^2$-errors with $h = 1/200$

| | $(\alpha,\beta)$ | $(0.01, 0.99)$ | | $(0.5, 0.5)$ | | $(0.99, 0.01)$ | |
|---|---|---|---|---|---|---|---|
| | $\Delta t$ | $\|u-u_h\|$ | Rate | $\|u-u_h\|$ | Rate | $\|u-u_h\|$ | Rate |
| $\theta = 0$ | 1/20 | 2.2845E-03 | - | 2.0956E-03 | - | 1.9169E-03 | - |
| | 1/40 | 6.1366E-04 | 1.8964 | 5.5997E-04 | 1.9039 | 5.0316E-04 | 1.9297 |
| | 1/80 | 1.5983E-04 | 1.9409 | 1.4675E-04 | 1.9320 | 1.3080E-04 | 1.9437 |
| $\theta = 0.2$ | 1/20 | 1.8774E-03 | - | 1.7472E-03 | - | 1.6910E-03 | - |
| | 1/40 | 4.9753E-04 | 1.9159 | 4.6085E-04 | 1.9227 | 4.3781E-04 | 1.9495 |
| | 1/80 | 1.2892E-04 | 1.9483 | 1.2051E-04 | 1.9351 | 1.1329E-04 | 1.9502 |
| $\theta = 0.4$ | 1/20 | 1.4458E-03 | - | 1.3918E-03 | - | 1.4736E-03 | - |
| | 1/40 | 3.7808E-04 | 1.9352 | 3.6259E-04 | 1.9405 | 3.7680E-04 | 1.9675 |
| | 1/80 | 9.7611E-05 | 1.9536 | 9.4940E-05 | 1.9333 | 9.7190E-05 | 1.9549 |
| $\theta = 0.5$ | 1/20 | 1.2205E-03 | - | 1.2156E-03 | - | 1.3662E-03 | - |
| | 1/40 | 3.1710E-04 | 1.9445 | 3.1497E-04 | 1.9484 | 3.4894E-04 | 1.9692 |
| | 1/80 | 8.1816E-05 | 1.9545 | 8.2723E-05 | 1.9289 | 8.9970E-05 | 1.9554 |

vergence accuracy in both time and space by some numerical examples, which are two-dimensional case and one-dimensional cases, respectively.

In the near future, we will apply the discussed schemes to solving other nonlinear evolution FDEs, such as nonlinear space FDEs, nonlinear space-time FDEs. Furthermore, ones can see clearly that we can solve a large numbers of nonlinear integer order evolution equations by our schemes, also develop some new second-order schemes combined with other numerical methods including FD methods, spectral methods, discontinuous methods and so on.

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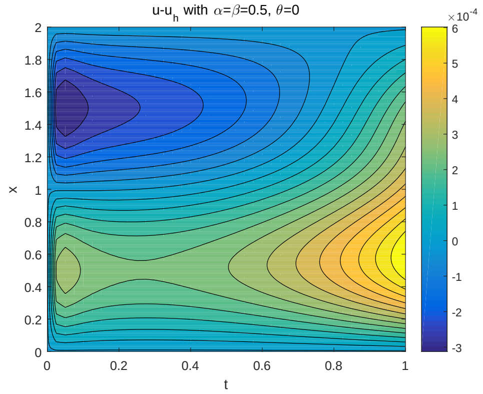


**Fig. 5** The contour plots of $u-u_h$ with $h=\frac{1}{100}$, $\Delta t=\frac{1}{40}$

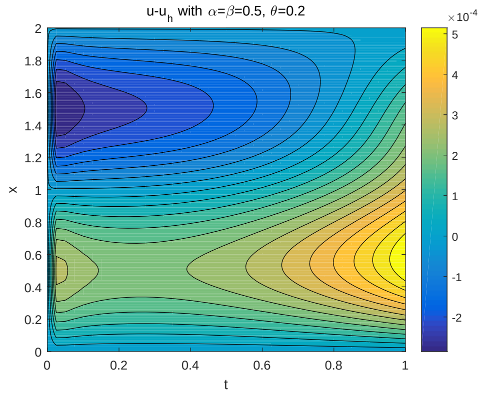


**Fig. 6** The contour plots of $u-u_h$ with $h=\frac{1}{100}$, $\Delta t=\frac{1}{40}$


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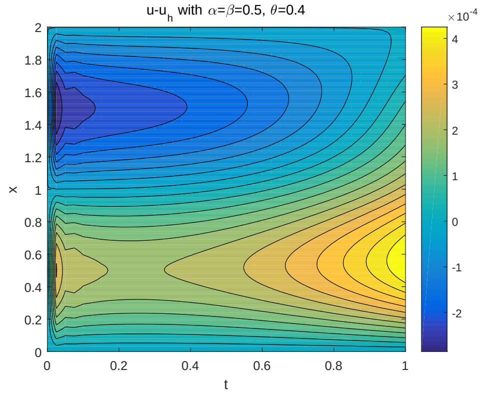


**Fig. 7** The contour plots of $u-u_h$ with $h=\frac{1}{100}$, $\Delta t=\frac{1}{40}$

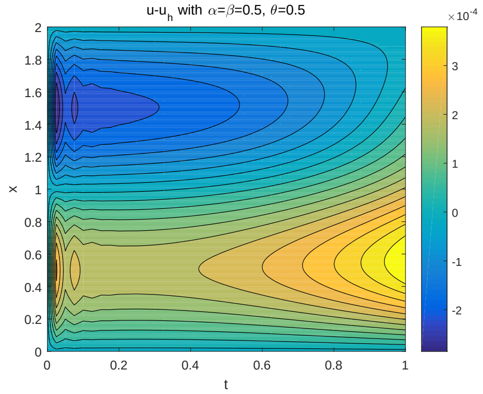


**Fig. 8** The contour plots of $u-u_h$ with $h=\frac{1}{100}$, $\Delta t=\frac{1}{40}$


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