Some selected results of professor Baltasar Rodríguez-Salinas.

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Abstract

Lecture on some aspects of the research output of Baltasar
Rodríguez-Salinas, given at the Meeting on Mathematical Analysis
in his homage on the occasion of his seventieth birthday.

Contents

Professor Baltasar Rodríguez-Salinas Palero, whose seventieth birth-day we celebrate here in Avila, published his first papers in 1942 when he was 17 years old and still in secondary school in Alcalá de Henares.

Since then he has written more than one-hundred and fifty mathematical works. It would be impossible to give an account in a one-year course of all the beautiful results with which he enriched mathematics, and it is even more so in a one-hour lecture. I will therefore present only samples which are simple to state and easy to understand, omitting of

necessity many of his most profound theorems. During his whole career
Professor Rodríguez-Salinas has shown an uncommon ability to assimi-
late quickly new theories. His creativity ranges from algebra [55], [118]
and projective geometry [43], [88] to mathematical economics [142], from
Fourier analysis [15], [97] to oceanography [12]. In connection with the
last work let me mention that he has also a degree of "ingeniero ge-
ografo". He also wrote a number of essays on the history of mathemat-
ics. From the rich variety of fields in which don Baltasar has done
research, I selected three: 1) asymptotic expansions, 2) topological
vector spaces, 3) measure and integration.

Professor Rodríguez-Salinas has written thirty joint papers, most of
them since 1973. He is a generous collaborator who willingly shares his
ideas with others. With one exception his nine co-authors come from
the more than twenty doctoral students he has had at Zaragoza and
Madrid.

In what follows, the numbers in square brackets refer to the works
of Rodríguez-Salinas and the numbers in curly brackets to the works of
other authors listed at the end.

1 Asymptotic expansions, moment problem, quasi-analyticity

Baltasar Rodríguez-Salinas wrote his doctoral dissertation [19] on dif-
ferential equations under the direction of Tomás Rodríguez Bachiller in
1954. However, already at that time he was interested in questions re-
lated to asymptotic expansions whose study was initiated in Spain by
Ricardo San Juan Llosa.

Before presenting the results of don Baltasar it is necessary to intro-
duce some notation and to recall some classical facts. We shall denote
by A_α the angular domain $|\theta| < \alpha\frac{\pi}{2}$ of the plane of the complex variable
$z = x + iy = re^{i\theta}$, or of the Riemann surface of the function $\log z$ if
$\alpha > 2$. The power series $\sum_{k \in I\!N} a_k z^k$ is an asymptotic expansion at the
point $z = 0$ of the function f analytic in A_α if for every $n \in I\!N$ one has

$$|f(z) - \sum_{k=0}^{n} a_k z^k| = o(|z|^n) \tag{1}$$

as $z \to 0$ in A_α. It is very important to observe that this condition has nothing to do with the convergence of the series, in spite of the fact that the initiators of the theory, T. J. Stieltjes and Henri Poincaré, called them "semi-convergent series".

The function f determines the coefficients a_k but conversely distinct functions can have the same asymptotic expansion. For instance $\frac{z}{z+1}$ and $\frac{z}{z+1} + e^{-1/z}$ defined in A_2 have $\sum_{n \in I\!\!N} (-1)^n z^{n+1}$ as asymptotic expansion at 0. To investigate the uniqueness of the function with a given asymptotic expansion it is necessary to introduce explicit bounds for the expressions on the left hand side of (1). This leads to Watson's Problem: Let (m_n) be a sequence of strictly positive numbers. Let f_1 and f_2 be two analytic functions in A_α and assume that there exists a sequence (a_n) of complex numbers such that for certain constants $A > 0$ and $k > 0$, depending on f_j, one has

$$|f_j(z) - \sum_{i=0}^{n-1} a_i z^i| \le Ak^n m_n |z|^n$$

for $n \in I\!\!N$, $z \in A_\alpha$ and $j = 1, 2$. Under what condition on (m_n) can one assert that $f_1(z) = f_2(z)$?

Clearly the problem is equivalent to the following: Let (m_n) be a sequence of strictly positive numbers and g an analytic function in A_α which satisfies the inequalities $|g(z)| \le Ak^n m_n |z|^n$ for $n \in I\!\!N$ and $z \in A_\alpha$. What conditions on (m_n) imply that $g = 0$?

In order to state the solution of the problem, let us introduce the function

$$T(r) = \sup_{n \in I\!\!N} \frac{r^n}{m_n}, \quad r \ge 0.$$

Furthermore, let π be the highest convex polygonal line below the points $(n, \log m_n)$ of the euclidean plane. Let $\log m_n^c$ by the y-coordinate of the point on π with x-coordinate n. The sequence (m_n^c) is called the *logarithmically convex regularization* of (m_n) and is given also by

$$m_n^c = \sup_{r > 0} \frac{r^n}{T(r)}, \quad n \in I\!\!N.$$

Then we have the following theorem, which for $\alpha = 1$ is due to T. Carleman and A. Ostrowski and for a general α is the special case of a result of S. Mandelbrojt:

The inequalities $|g(z)| \leq Ak^n m_n |z|^n$, $n \in \mathbb{N}$, $z \in A_\alpha$ imply that $g(z) \equiv 0$ if and only if the following equivalent conditions are satisfied:

$$(C_\alpha) \int_0^\infty \frac{\log T(r)}{1 + r^{1+\frac{1}{\alpha}}} dr = \infty, \quad \sum \left(\frac{m_n^c}{m_{n+1}^c} \right)^{1/\alpha} = \infty.$$

This solution of Watson's Problem is applied to prove the Denjoy-Carleman theorem concerning quasi-analytic classes. Let again be (m_n) a sequence of strictly positive numbers and denote by $C\{m_n\}$ the set of all infinitely differentiable functions f in a given interval which with some constants $A > 0$ and $k > 0$, depending on f, satisfy

$$|f^{(n)}(t)| \leq Ak^n m_n$$

for $n \in \mathbb{N}$ and every t. The class $C\{m_n\}$ is *quasi-analytic* if from $f_j \in C\{m_n\}$, $j = 1, 2$ and $f_1^{(n)}(c) = f_2^{(n)}(c)$ for all $n \in \mathbb{N}$ at a point c of the interval it follows that f_1 and f_2 are equal in the whole interval. The theorem in question states that $C\{m_n\}$ is quasi-analytic if and only if condition (C_1) is satisfied.

It is an immediate consequence of the solution of Watson's Problem that (C_1) is a sufficient condition. Indeed, if there exists a non-identically zero function $f \in C\{m_n\}$ with $f^{(n)}(c) = 0$ for all $n \in \mathbb{N}$, then one can construct an infinitely differentiable, not identically zero function g on $[0, 1]$ such that $|g^{(n)}(t)| \leq m_n$ and $g^{(n)}(0) = g^{(n)}(1) = 0$ for $n \in \mathbb{N}$. Consider

$$F(\zeta) = \int_0^1 g(t) e^{-t\zeta} dt.$$

Integrating by parts n times we obtain

$$F(\zeta) = \frac{1}{\zeta^n} \int_0^1 g^{(n)}(t) e^{-t\zeta} dt$$

and thus $|F(\zeta)| \leq m_n |\zeta|^{-n}$. Since F is analytic for $\mathrm{Re}\,\zeta > 0$, the change of variables $\zeta \longmapsto 1/z$ shows that if (C_1) is satisfied then $F = 0$ and therefore $g(t) \equiv 0$.

Watson's Problem arises also in the theory of moments. Stieltjes asked under what conditions on the sequence (m_n) of strictly positive numbers does there exist an increasing function μ on $[0, \infty)$ such that for every $n \in \mathbb{N}$ the moment $\int_0^\infty t^n d\mu(t)$ is equal to m_n. Let us observe that

a necessary condition is $\underline{m_n^2 \leq m_{n-1}m_{n+1} \text{ for } n \geq 1}$, i.e., that $m_n^c = m_n$ for all $n \in I\!N$.

Already Stieltjes himself gave an example of two increasing, non-equivalent functions which have all their moments equal. We are thus led to the problem of uniqueness which can be stated as follows: What conditions on (m_n) imply that, if μ is a function of bounded variation on $[0, \infty)$ which satisfies

$$\int_0^\infty t^n d\mu(t) = 0 \quad \text{and} \quad \int_0^\infty t^n d|\mu|(t) \leq m_n, \quad n \in I\!N,$$

then μ is equivalent to a constant?

In order to answer this question, one introduces the Stieltjes transform of μ:

$$F(z) = \int_0^\infty \frac{d\mu(t)}{z - t}.$$

From the identity

$$\frac{1}{z-t} = \frac{1}{z} + \frac{t}{z^2} + \cdots + \frac{t^{n-1}}{z^n} + \frac{t^n}{z^n(z-t)}$$

it follows that

$$F(z) = \frac{1}{z^n} \int_0^\infty \frac{t^n d\mu(t)}{z - t}$$

and so

$$|F(z)| \leq \frac{m_n}{|z|^n}$$

in the region where $|z - t| > 1$ for $0 \leq t < \infty$.

We are essentially in the case $\alpha = 2$ of Watson's Problem, hence the theorem stated above implies the result due to Carleman (but contained implicitly in an earlier work of Marcel Riesz) according to which the Stieltjes moment problem is "determined" if (assuming licitly that $m_n = m_n^c$)

$$\int_1^\infty \frac{\overline{\log T(r)}}{r^{\frac{3}{2}}} dr = \infty, \quad \text{or} \quad \sum \left(\frac{m_n}{m_{n+1}}\right)^{1/2} = \infty.$$

After this long introduction I arrive at the theorems of Rodríguez-Salinas and the first ones I want to mention are related to the Stieltjes

problem [17]. He proves that if f is an integrable function on $(0, \infty)$ such that

$$\int_0^\infty t^n f(t)\, dt = 0 \tag{2}$$

and

$$\int_0^\infty t^n |f(t)|\, dt \leq m_n \tag{3}$$

for $n \in \mathbb{N}$ and the sequence (m_n) satisfies condition (C_2), then $f(t) = 0$ for almost every t. Conversely, if

$$\int_1^\infty \frac{\log T(r)}{r^{\frac{3}{2}}}\, dr < \infty$$

then there exists a non-identically vanishing continuous function f in $(0, \infty)$ which satisfies conditions (2) and (3). He proves furthermore that if instead of (2) the moments

$$a_n = \int_0^\infty t^n f(t)\, dt, \quad n \in \mathbb{N},$$

satisfy

$$\limsup_{n \to \infty} \sqrt[n]{|a_n|} = \rho < \infty,$$

then (3) and (C_2) imply that $f(t) = 0$ for $t > \rho$.

Let us denote by $K\{m_n; A_\alpha\}$ the class of analytic functions f in A_α for which there exist a sequence (a_n) of complex numbers and constants $A > 0$, $k > 0$ such that

$$\left| f(z) - \sum_{i=0}^{n-1} a_i z^i \right| \leq A k^n m_n |z|^n \tag{4}$$

for $z \in A_\alpha$ and $n \in \mathbb{N}$. Following San Juan we say that $K\{m_n; A_\alpha\}$ is a *semi-analytic* class if every $f \in K\{m_n; A_\alpha\}$ is uniquely determined by the sequence (a_n). The theorem quoted above asserts that $K\{m_n; A_\alpha\}$ is semi-analytic if and only if (C_α) is satisfied. As a generalization of Liouville's theorem Rodríguez-Salinas proves [44] that $K\{m_n; A_\alpha\}$ contains only constants if and only if one has

$$\liminf \frac{\sqrt[n]{m_n}}{n^{\alpha-2}} = 0 \qquad \text{if} \quad 0 < \alpha \leq 2,$$

$$(C_{\alpha-2}) \qquad\qquad\qquad \text{if} \quad \alpha > 2.$$

San Juan and Rodríguez-Salinas considered the analogues of quasi-analytic classes and of moment problems when the functions are analytic in an angular sector A_α and are not defined only on the real half-axis.

Let us denote by $C\{m_n; A_\alpha\}$ the class of functions f which are analytic in A_α, have angular derivatives of all orders at the point 0 and satisfy the inequalities $|f^n(z)| \leq Ak^n m_n$ for $z \in A_\alpha$, $n \in I\!N$ and some $A > 0$, $k > 0$, depending on f. We say that $C\{m_n; A_\alpha\}$ is *quasi-analytic* if from $f \in C\{m_n; A_\alpha\}$ and $f^{(n)}(0) = 0$ for $n \in I\!N$ it follows that f is identically zero. The theorem of Rodríguez-Salinas [44] states that $C\{m_n; A_\alpha\}$ is quasi-analytic if and only if (m_n) satisfies $(C_{\alpha+1})$. For $C\{m_n; A_\alpha\}$ to contain only constants it is necessary and sufficient to have

$$\liminf_{n \to \infty} \frac{\sqrt[n]{m_n}}{n^{\alpha-1}} = 0 \qquad \text{if} \quad 0 < \alpha \leq 1,$$

$$(C_{\alpha-1}) \qquad\qquad\qquad \text{if} \quad \alpha > 1.$$

Let f be an analytic function in A_α and suppose that its absolute moments on the real axis

$$b_n = \int_0^\infty x^n |f(x)| dx$$

satisfy an estimate $b_n \leq m_n$. San Juan, Ferrán Sunyer i Balaguer and R. P. Boas initiated the search for conditions on (m_n) which insure that $f(z) \equiv 0$. Rodríguez-Salinas proves [21], among others, the following theorems, where the ordinary moments are denoted by

$$a_n = \int_0^\infty x^n f(x) dx.$$

1. Let f be an analytic function in A_α, $\alpha < 1$, such that

$$\limsup_{z \to \infty} \frac{\log |f(z)|}{|z|} \leq h < \infty.$$

If

$$\limsup_{n \to \infty} \frac{\sqrt[a_n]{|a_n|}}{n} = 0 \quad \text{and} \quad \sum_{n \in I\!N} \frac{n}{\sqrt[n]{b_n}} = \infty,$$

then $f(z) \equiv 0$.

2. Let f be an entire analytic function of exponential type $\leq \rho$, i.e.

$$\limsup_{z \to \infty} \frac{\log |f(z)|}{|z|} \leq \rho.$$

If

$$\limsup_{n \to \infty} \frac{\sqrt[n]{|a_n|}}{n} = \sigma < \frac{1}{e\rho} \quad \text{and} \quad \sum_{n \in I\!N} \frac{n}{\sqrt[n]{b_n}} = \infty,$$

then $f(z) \equiv 0$.

3. Let f be an analytic function in A_α such that

$$\limsup_{z \to \infty} \frac{\log |f(z)|}{H(|z|)} \leq h < \infty,$$

where H is a positive increasing function of $r > 0$ satisfying

$$\int_1^\infty \frac{H(r)}{r^{1+\frac{1}{\alpha}}} dr < \infty.$$

If (m_n) is a sequence of positive numbers such that

$$\liminf_{n \to \infty} \frac{\sqrt[\alpha n]{m_n}}{n} = 0$$

and $b_n \leq m_n$, then $f(z) \equiv 0$.

4. Conversely, if

$$\liminf_{n \to \infty} \frac{\sqrt[\alpha n]{m_n}}{n} > 0$$

then there exists a non identically zero bounded analytic function in A_α for which $b_n \leq m_n$.

5. Let f be an analytic function in A_α such that

$$\limsup_{z \to \infty} \frac{\log |f(z)|}{|z|^{1/\alpha}} \leq h < \infty.$$

If (m_n) is a sequence of positive numbers such that

$$\limsup_{n \to \infty} \frac{(\log n) \sqrt[\alpha n]{m_n}}{\alpha n} < \frac{\pi}{2eh}$$

and $b_n \leq m_n$, then $f(z) \equiv 0$.

6. Let f be an analytic function of exponential type $\leq \rho$ in A_α. Assume that (a_n) satisfies the condition

$$\lim_{n \to \infty} \frac{\sqrt[\alpha n]{|a_n|}}{n} = 0 \qquad \text{if } \alpha < 1,$$

$$\limsup_{n \to \infty} \frac{(\log n) \sqrt[n]{|a_n|}}{n} < \frac{\pi}{2e\rho} \qquad \text{if } \alpha = 1,$$

$$\limsup_{n \to \infty} \sqrt[n]{|a_n|} < \frac{\cos[(2-\alpha)\frac{\pi}{2}]}{e\rho} \qquad \text{if } 1 < \alpha \leq 2,$$

$$\limsup_{n \to \infty} \sqrt[n]{|a_n|} < \frac{1}{e\rho} \qquad \text{if } \alpha \geq 2.$$

Let (m_n) be a sequence of strictly positive numbers which satisfies condition (C_2). If $b_n \leq m_n$ for $n \in I\!N$, then $f(z) \equiv 0$ [17].

7. Let f be an analytic function in A_α of order less than $1/(\alpha+2)$, i.e.

$$|f(z)| \leq A e^{|z|^\sigma}$$

for $z \in A_\alpha$ and some $A > 0$, $\sigma < 1/(\alpha+2)$. Assume that

$$\limsup_{n \to \infty} \sqrt[n]{|a_n|} < \infty.$$

Let (m_n) be a sequence of positive numbers for which

$$\liminf_{n \to \infty} \frac{\sqrt[n]{m_n}}{n^{\alpha+2}} = 0.$$

If $b_n \leq m_n$, for $n \in I\!N$, then $f(z) \equiv 0$ [17].

8. If (m_n) is a sequence of strictly positive numbers such that

$$\liminf_{n \to \infty} \frac{\sqrt[n]{m_n}}{n^{\alpha+2}} > 0,$$

then there exists a non identically zero bounded analytic function in A_α such that $a_n = 0$ and $b_n \leq m_n$ for $n \in I\!N$ [17].

Let $C\{m_n^{(1)}\}$ and $C\{m_n^{(2)}\}$ be two quasi-analytic classes. Carleman asked whether from the hypotheses $f_1 \in C\{m_n^{(1)}\}$, $f_2 \in C\{m_n^{(2)}\}$ and

$f_1^{(n)}(c) = f_2^{(n)}(c)$ for $n \in I\!N$ does it follow that the functions, f_1 and f_2 are identically equal. It was a triumph of Spanish mathematics when in 1936 San Juan gave a negative answer to this question. A little later Mandelbrojt proved, using the method of San Juan, that every infinitely differentiable function is the sum of two infinitely differentiable functions which belong to quasi-analytic classes. If f is not identically zero such that $f^{(n)}(c) = 0$ for $n \in I\!N$ and $f = f_1 - f_2$ where f_1 and f_2 belong to necessarily distinct quasi-analytic classes then $f_1^{(n)}(c) = f_2^{(n)}(c)$ for $n \in I\!N$ but $f_1 \neq f_2$. In his doctoral dissertation {2} Th. Bang improved Mandelbrojt's theorem by proving the following result: Let $C\{m_n^{(1)}\}$ and $C\{m_n^{(2)}\}$ be two classes of infinitely differentiable functions defined on the whole real line. In this case one may assume that each sequence $(m_n^{(j)})$ is logarithmically convex. Let π_j be the convex polygonal line with vertices $(n, \log m_n^{(j)})$. Every function of $C\{m_n^{(1)} + m_n^{(2)}\}$ is the sum of a function in $C\{m_n^{(1)}\}$ and of a function in $C\{m_n^{(2)}\}$ if and only if the highest minorant of π_1 and π_2 is convex.

San Juan and Rodríguez-Salinas found analogues of Bang's theorem for semi-analytic classes. In particular, don Baltasar studied the case of angular domains [44]. For two sequences $(\nu_n^{(1)})$ and $(\nu_n^{(2)})$ of strictly positive numbers he introduced the following hypotheses:

(H_1) The sequences $(\nu_n^{(1)})$ and $(\nu_n^{(2)})$ are logarithmically convex.

(H_2) The highest minorant of the two polygonal lines, which have the points $(n, \log \nu_n^{(1)})$ and $(n, \log \nu_n^{(2)})$ as respective vertices, is convex.

Then we have:

1. Case $\alpha = 2$. Let $m_n = m_n^{(1)} + m_n^{(2)}$. If (m_n) does not satisfy (C_2) and the sequences $(m_n^{(1)})$ and $(m_n^{(2)})$ satisfy (H_1) and (H_2) then there exist two distinct analytic functions f_1 and f_2 belonging to the classes $K\{m_n^{(1)}; A_2\}$ and $K\{m_n^{(2)}; A_2\}$, respectively, which have the same asymptotic expansion at 0.

2. Case $0 < \alpha < 2$. Let $0 \leq \delta \leq 2 - \alpha$ and $m_n = m_n^{(1)} + m_n^{(2)}$. Assume that (m_n) does not satisfy (C_α) and that the sequences $(n^{\delta n} m_n^{(1)})$ and $(n^{\delta n} m_n^{(2)})$ satisfy the hypotheses (H_1) and (H_2). Then one can find functions $f_1 \neq f_2$ belonging to $K\{m_n^{(1)}; A_2\}$ and $K\{m_n^{(2)}; A_2\}$, respectively, which have the same asymptotic expansion at 0.

Rodríguez-Salinas has a theorem also in the case $\alpha > 2$ and results about regions which are not angular sectors.

Bang asked whether there can exist two distinct functions belonging to two quasi-analytic classes which coincide in a whole interval and are analytic there. Rodríguez-Salinas gave a positive answer to this question [30]. He proves first the following theorem:

Let $(m_n^{(1)})$ and $(m_n^{(1)})$ be two sequences of strictly positive numbers, $\alpha > 0$, and $m_n = (m_n^{(1)} + m_n^{(1)})/(\alpha n)!$ where $(\alpha n)! = \Gamma(\alpha n + 1)$. Assume that (m_n) does not satisfy (C_1) and that the sequences $(m_n^{(1)}/(\alpha n)!)$ and $(m_n^{(2)}/(\alpha n)!)$ satisfy (H_1) and (H_2). Then there exists a pair of functions $f_1 \neq f_2$ defined for $-\infty < t < \infty$, analytic for $t \neq 0$, equal for $t < 0$ and belonging to the classes $C\{m_n^{(1)}\}$ and $C\{m_n^{(2)}\}$, respectively.

Then he shows that the bounds $m_n^{(j)}(j = 1, 2)$ can be chosen in such a way that the classes $C\{m_n^{(j)}\}$ are quasi-analytic.

It was also Carleman who posed the problem of equivalence of classes of infinitely differentiable functions, that is, to give conditions on (m_n) and $(\tilde{m}_n)$ which ensure that $C\{m_n\} = C\{\tilde{m}_n\}$. The solution was given in 1940 by Henri Cartan and Mandelbrojt who introduced for this purpose new kinds of regularizations of sequences, distinct from the logarithmically convex regularization and depending on the type of interval on which the functions are defined {19}. For instance, in the case of a finite open interval they introduce the function

$$S(r) = \max_{n \leq r} \frac{r^n}{m_n}, \qquad r \geq 0,$$

and define the exponential regularization of (m_n) through the intermediary of logarithms by

$$m_n^{\circ} = \sup_{r \geq n} \frac{r^n}{S(r)}, \qquad n \in \mathbb{N}.$$

Rodríguez-Salinas considered [27] the same problem for the classes $K\{m_n; A_\alpha\}$ and later for classes $K\{m_n; \Omega\}$ of analytic functions defined in certain general regions Ω and satisfying inequalities of the type (4). He announced his result in 1961 in a four page long note [36] but published the details only in 1987 in six articles [107]-[112] whose total length is 89 pages. To obtain his theorem Rodríguez-Salinas had to introduce an even more refined kind of regularization and study questions related to conformal representation and to the bounds of derivatives of polynomials. Here is the statement of the theorem:

Let Ω be a bounded connected set in the complex plane which consists of more than one point and such that the origin 0 belongs to the boundary $\partial\Omega$ of Ω. Let B be the unbounded component of $C\overline{\Omega}$ and $w = w(z)$ the conformal representation of B onto the exterior $\{w : |w| > 1\}$ of the unit disk such that $w(\infty) = \infty$ and $\lim_{z\to\infty} w'(z) = c_0 > 0$. Define the functions

$$\sigma(r) = \sup_{|z|=r} \log|w(z)|$$

and

$$\varphi(t) = \inf_{r>0} \frac{e^{t\sigma(r)}}{r}.$$

For $\rho > 0$ let Ω_ρ be the component of $\Omega \cap \{z : |z| < \rho\}$ which contains 0 on its boundary and let w_ρ, σ_ρ, φ_ρ be the analogous functions corresponding to Ω_ρ.

Suppose that there exist two constants $b > a \geq 1$ such that $a \leq \sigma(2r)/\sigma(r) \leq b$ for $0 < r \leq \sup\{|z| : z \in \Omega\}$. The point 0 is said to be *semi-regular* if there exist two positive numbers λ and ρ_0 such that $\sigma_\rho(r) \leq \lambda\sigma(r)/\sigma(\rho)$ for $r \leq \rho \leq \rho_0$. The point 0 is *regular* if there exists furthermore a sequence $(P_k(z))$ of polynomials such that $|P_k(z)| \leq 1$ for $z \in \Omega$ and

$$\frac{|P_k^{(n)}(0)|}{n!} \geq q^n \varphi(\frac{k}{n})^n$$

for $k/h \geq n \geq h$, where $0 < q \leq 1$ and $h = h(\Omega)$ is sufficiently large.

The regularization $(\log m_n^*)$ of $(\log m_n)$ by means of $\log\varphi(t)$ is defined by

$$S(t) = \max_{n \leq t} \frac{\varphi(\frac{t}{n})^n}{m_n}$$

and

$$m_n^* = \sup_{t \geq n} \frac{\varphi(\frac{t}{n})^n}{S(t)}.$$

If 0 is a regular point, then $K\{m_n; \Omega\} = K\{\tilde{m}_n; \Omega\}$ if and only if there exist two positive constants a and b such that

$$a^n \leq \frac{\tilde{m}_n^*}{m_n^*} \leq b^n \quad \text{for} \quad n \in I\!N.$$

Rodríguez-Salinas has a similar result for unbounded domains.

To conclude this section I would like to mention without details three other contributions of don Baltasar:

1. Convexity inequalities for the bounds of the derivatives [23] and for the coefficients and bounds of asymptotic expansions [24] of a function analytic in A_α.

2. A generalization of a Phragmén-Lindelöf type theorem of L. Ahlfors and M. Heins [16].

3. An analogue for several variables of the Denjoy-Carleman theorem on quasi-analytic classes [22].

2 Topological vector spaces

Following the terminology introduced by A. Grothendieck {12}, a Banach space E is said to have the *Dieudonné property* if for every Banach space F every continuous linear map $T : E \to F$, which maps sequences that are Cauchy for the weakened topology $\sigma(E, E')$ into $\sigma(F, F')$-convergent sequences, also maps bounded subsets of E into $\sigma(F, F')$-compact subsets of F (i.e. T is weakly compact).

Rodríguez-Salinas obtains [101] a large class $\underset{\sim}{C}$ of Banach spaces such that if $E \in \underset{\sim}{C}$, then the space $C(X; E)$ of continuous functions defined on the compact space X with values in E, equipped with the topology of uniform convergence, has the Dieudonné property. To define $\underset{\sim}{C}$ he says that E has property (A) if for some norm equivalent to the original norm on E there exists a Suslin subspace B_0 of $E^{I\!N}$, consisting of Cauchy sequences for $\sigma(E, E')$, such that for each $e' \in E'$ there exists $(e_n) \in B_0$ with $\|e_n\| \leq 1 + \frac{1}{n}$ for $n \in I\!N$ and $\lim_{n\to\infty}\langle e', e_n\rangle = \|e'\|$.

The space E belongs to $\underset{\sim}{C}$ if every separable subspace of E is contained in another separable subspace which has property (A).

If the topological dual E' of E is separable, then E belongs to $\underset{\sim}{C}$. This implies a theorem of Fernando Bombal and Pilar Cembranos {5} according to which if E' is separable, then $C(X; E)$ has the Dieudonné property.

The class $\underset{\sim}{C}$ has strong stability properties:

1. If $E_k \in \underset{\sim}{C}$ for $1 \leq k \leq n$, then $\prod_{k=1}^{n} E_k \in \underset{\sim}{C}$.

2. If F is the image of $E \in \underset{\sim}{C}$ under a surjective continuous linear

map, then $F \in \underset{\sim}{C}$.

N. Bourbaki introduced barrelled locally convex spaces to character-ize those spaces for which the principle of uniform boundedness holds. It is a simple but important result that every Baire locally convex space is barrelled. Later new classes between Baire and barrelled spaces were introduced with the purpose to find new closed graph theorems, stability theorems, and to distinguish between (LF)-spaces. Most of them are discussed in the Research Note of Michael Kunzinger {16}.

A locally convex space E is said to be *suprabarrelled* by Manuel Valdivia {24} and a (db)-space by W. J. Robertson, I. Tweddle and F. E. Yeomans {21} if it satisfies the following condition: If (E_k) is an increasing sequence of subspaces of E such that $\cup_{k \in I\!\!N} E_k = E$, then there exists $p \in I\!\!N$ such that E_p (hence also E_k for $k \geq p$) is $\underline{\underline{d}}$ense in E and $\underline{\underline{b}}$ arrelled.

Rodríguez-Salinas [80] refined this classification. He says that a bar-relled space is barrelled of class 0. Then he defines *barrelled spaces of class* $\alpha \geq 1$ by transfinite induction. If α has a predecessor, then E is barrelled of class α if, given an increasing sequence (E_k) of subspaces of E such that $\cup_{k \in I\!\!N} E_k = E$, there exists $p \in I\!\!N$ such that E_p is dense in E and barrelled of class $\alpha - 1$. Thus the suprabarrelled spaces are precisely the barrelled spaces of class 1. If α is a limit ordinal then E is barrelled of class α if it is barrelled of class β for every $\beta < \alpha$.

For a finite n there is a direct description of barreledness of class n [81], {17} which establishes a remarkable link with M. de Wilde's spaces with webs. An n-web is a collection $\{E_{k_1 \ldots k_p}\}$ of subspaces of E where $k_j \in I\!\!N$ for $1 \leq j \leq p$ and $1 \leq p \leq n$, such that (E_k) is an increasing cover of E and for each $(p-1)$-tuple $k_1, \ldots, k_{p-1}$ $(2 \leq p \leq n)$ the sequence $(E_{k_1, \ldots, k_{p-1}k})_k$ is an increasing cover of $E_{k_1, \ldots, k_{p-1}}$. The space E is barrelled of class n if given an n-web there exists some n-tuple $r_1, \ldots, r_n$ of integers such that $E_{r_1, \ldots, r_n}$ is dense in E and barrelled.

This description leads to superbarrelled spaces introduced by don Baltasar in a recent article [146]. A *web* is a family $\{E_{k_1, \ldots, k_p}\}$ of sub-spaces of E as above, except that now p may take any integral value ≥ 1. The space E is *superbarrelled* if given a web $\{E_{k_1, \ldots, k_p}\}$ there exists a sequence (r_p) of integers such that each subspace $E_{r_1, \ldots, r_p}$ is dense in E and barrelled. He proves that a superbarrelled space is barrelled of class α for every α, and that a barrelled space is superbarrelled if and only if

it is barrelled of class ω_1, where ω_1 is the first uncountable ordinal.

On the other hand, E is the *totally barrelled* {25} (or σ-barrelled in the terminology of don Baltasar [84]) if, given a sequence (E_k) of subspaces of E such that $\cup_{k \in I\!N} E_k = E$, there exists $p \in I\!N$ such that E_p is barrelled and its closure $\overline{E_p}$ has finite codimension in E. We have the following implications for $\alpha > \beta \geq 1$:

Baire $\Rightarrow$ totally barrelled $\Rightarrow$ superbarrelled $\Rightarrow$ barrelled of class α $\Rightarrow$ barrelled of class β $\Rightarrow$ suprabarrelled $\Rightarrow$ barrelled.

Let Σ be a σ-algebra of subsets of some set X. Denote by $\ell_0^\infty(X, \Sigma)$ the vector space generated by the characteristic functions χ_A of the sets A belonging to Σ, equipped with the maximum-norm $\|f\| = \max_{x \in X} |f(x)|$. If $X = I\!N$ and Σ is the σ-algebra 2^X of all subsets of $I\!N$, then one usually writes ℓ_0^∞ instead of $\ell_0^\infty(I\!N, 2^{I\!N})$. J. Dieudonné proved that ℓ_0^∞ is barrelled and Grothendieck that so is $\ell_0^\infty(X, \Sigma)$. Valdivia {23} proved that $\ell_0^\infty(X, \Sigma)$ is suprabarrelled and Rodríguez-Salinas [81] that it is barrelled of class 2. J. C. Ferrando and Manuel López Pellicer {9}, {17}, {11} proved that $\ell_0^\infty(X, \Sigma)$ is barrelled of class n for every $n \in I\!N$, and Rodríguez-Salinas points out in [146] that López Pellicer actually proved in {18} that $\ell_0^\infty(X, \Sigma)$ is superbarrelled. However, J. Arias de Reyna {1} proved that $\ell_0^\infty(X, \Sigma)$ is not totally barrelled.

A locally convex space F is said to be Γ_r by Valdivia and infra-s by N. Adasch if it has the following property: For every barrelled space E every linear map $T : E \longrightarrow F$, whose graph is closed, is continuous. These authors gave various intrinsic characterizations of Γ_r-spaces {15}. Rodríguez-Salinas [80] calls the Γ_r-spaces also $\Gamma_r^{(0)}$-spaces and presents the following inductive definition: If the ordinal number $\alpha \geq 1$ has a predecessor, then a $\Gamma_r^{(\alpha)}$-*space* is a union of an increasing sequence of $\Gamma_r^{(\alpha-1)}$-subspaces. If α is a limit ordinal, then F is a $\Gamma_r^{(\alpha)}$-space provided that it is a $\Gamma_r^{(\beta)}$-space for some $\beta < \alpha$. He then proves the following theorem: Let E be a barrelled space of class α and F a $\Gamma_r^{(\alpha)}$-space. If $T : E \to F$ is a linear map with closed graph, then T is continuous and there exists a subspace F_0 of F such that F_0 is a Γ_r-space and $T(E) \subset F_0$.

Analogously to the above definition, given a fixed locally convex space E one says that F is a $\Gamma_r(E)$-space if every linear map $T : E \to F$ with closed graph is continuous. Rodríguez-Salinas says [80] that the linear map $T : E \to F$ is subcontinuous if for every series $\sum_{k \in I\!N} x_k$ in E

which is subseries convergent (i.e. for every strictly increasing sequence $(k_j)_{j\in I\!N}$ of positive integers the series $\sum_{j\in I\!N} x_{k_j}$ converges) the series $\sum_{k\in I\!N} Tx_k$ converges unconditionally in F and one has $T(\sum_{k\in I\!N} x_k) = \sum_{k\in I\!N} Tx_k$. He obtains, among others, the following theorems:

1. Let E and F be two locally convex spaces. Assume that F is sequentially complete, is a $\Gamma_r(\ell_0^\infty)$-space and does not contain a subspace isomorphic to ℓ^∞. If $T : E \to F$ is a linear map with closed graph, then T is subcontinuous.

2. Let E and F be two locally convex spaces and let (F_k) be an increasing sequence of subspaces of F such that $\cup_{k\in I\!N} F_k = F$. Assume that for every $k \in I\!N$ there exists a topology $\mathcal{T}_k$ on F_k, finer than the topology induced by F and such that F_k equipped with $\mathcal{T}_k$ is a sequentially complete space, is a Γ_r-space and contains no subspace isomorphic to ℓ^∞. If $T : E \to F$ is a linear map with closed graph, then there is a $p \in I\!N$ such that $T(E) \subset F_p$ and the map $T : E \to F[\mathcal{T}_p]$ is subcontinuous.

The beautiful proofs employ the map $S : \ell_0^\infty \to E$ defined by $(a_k) \mapsto \Sigma a_k x_k$ for a given subseries convergent series Σx_k. If F is a $\Gamma_r(\ell_0^\infty)$-space, then $T \circ S$ is continuous and $m(A) = T(S(\chi_A))$ is a bounded measure with values in F to which the theorem of J. Diestel and Barbara Faires {8} can be applied. Inspired by these results, Ferrando and López Pellicer {10} proved another closed graph theorem whose conclusion is that a certain linear map is subcontinuous.

De Wilde's theory of spaces with C-webs is widely known nowadays {15}. One of the basic theorems of de Wilde states that if E is a Fréchet space or an ultrabornological space and F a space with a C-web, then every linear map $E \to F$ with closed graph is continuous. Rodríguez-Salinas [85] shows that if E is a metrizable totally barrelled space and the C-web on F is absolutely convex, then the same conclusion holds.

The Austrian mathematician E. Helly was the first to prove the theorem on the extension of linear forms which now bears the name of H. Hahn and St. Banach {7}. He used in his proof the following fact, the particular case for $n = 1$ of his famous theorem on convex sets: If $[a_\alpha, b_\alpha]$ is a family of finite closed intervals in $I\!R$, such that any two have a point in common, then $\cap_\alpha [a_\alpha, b_\alpha] \neq \emptyset$. Replacing the intervals by balls, L. Nachbin characterized the real Banach spaces F which have the following property: Given an arbitrary real Banach space E, a linear subspace

E_0 of E and a continuous linear map $f_0 : E_0 \to F$, there exists a linear map $f : E \to F$ such that $f(x) = f_0(x)$ for $x \in E_0$ and $\|f\| = \|f_0\|$. A. W. Ingleton proved an analogous theorem for ultrametric normed spaces. In this case the condition can be stated as follows: Every family of closed balls, totally ordered by inclusion, has a non-empty intersection. Indeed, in an ultrametric space if two balls have a point in common then one is contained in the other. In collaboration with L. Bou García [64] don Baltasar proved a general principle from which the three mentioned results follow easily:

Let E and F be two vector spaces over an arbitrary commutative field K. Let $\mathcal{A}$ be a non-empty collection of subsets of F which has the following properties:

a) $\mathcal{A}$ is invariant with respect to translations;

b) if $\mathcal{B}$ is a collection of sets belonging to $\mathcal{A}$, such that two sets in $\mathcal{B}$ have a point in common, then the intersection of all the sets belonging to $\mathcal{B}$ is not empty.

Let ψ be a map from E into $\mathcal{A}$ such that 1) $\psi(x+y) \subset \psi(x) + \psi(y)$ for $x, y \in E$, 2) $\psi(\alpha x) = \alpha \psi(x)$ for $\alpha \in K$, $x \in E$. Let E_0 be a subspace of E and $f_0 : E_0 \to F$ a linear map such that $f_0(x) \in \psi(x)$ for each $x \in E_0$. Then there exists a linear map $f : E \to F$ such that $f(x) = f_0(x)$ for $x \in E_0$ and $f(x) \in \psi(x)$ for every $x \in E$.

Rodríguez-Salinas was very interested in the problem of extension of linear maps. He wrote an excellent expository account [61] which summarizes his own research and that of others, and has a very extensive list of references. He lectured on it in 1966 in Namur [52] and wrote a monograph [60] on the connection between the extension of additive maps and of measures. I would like to relate one of his numerous results which is interesting also from the point of view of abstract algebra [41].

Let X be an additively written commutative group. A real-valued function f defined on a subgroup $D(f)$ of X is *additive* in X if $f(x+y) = f(x) + f(y)$ for $x, y \in D(f)$. A function $p : X \to \mathbf{R}$ is *subadditive* on X if $p(x + y) \leq p(x) + p(y)$ for $x, y \in X$. A set P of subadditive functions on X is a (Z_1) set if for every totally ordered subset P_0 of P the function $\inf\{p : p \in P_0\}$ also belongs to P.

If f_1 and f_2 are two additive functions in X with domains $D(f_1)$ and $D(f_2)$, we write $f_1 \subset f_2$ if $D(f_1) \subset D(f_2)$ and $f_1(x) = f_2(x)$ for every $x \in D(f_1)$. A set F of additive functions in X is a (Z_2) set if for every

subset F_0 of F, which is totally ordered with respect to $\subset$, the least upper bound of F_0 belongs to F. Given a set P of subadditive functions on X and an additive function f in X, we denote by $P(f)$ the set of all the functions $p \in P$ such that the function $f[p]$ defined by

$$f[p](x) = \inf_{y \in D(f)} [p(x + y) - f(y)]$$

belongs to P.

Let now P be a (Z_1) set of subadditive functions on X and F a (Z_2) set of additive functions in X. If $p_0 \in P$, $f_0 \in F$ and $p_0 \in P(f_0)$, then there exist functions $p \in P$ and $f \in F$ such that

1.   a) $p \le p_0$,

     b) if $q \in P(f_0)$ and $q \le p$, then $q = p$.

2.   a) $f_0 \subset f$,

     b) $f(y) = p(x + y) - p(y)$ for $x \in X$, $y \in D(f)$,

     c) if $g \in F$, $p \in P(g)$ and $f \subset g$, then $g = f$.

From this abstract extension theorem Rodríguez-Salinas deduces the following Hahn-Banach theorem for modules. Let A be a not necessarily commutative ring with unit element and $r : A \to \mathbf{R}$ a non-constant homomorphism of rings. Let X be a left A-module. An additive function f in X defined on a submodule Y of X is *A-modular* if $f(\alpha x) = r(\alpha)f(x)$ for $\alpha \in A$, $x \in Y$. We chose an (r)-generating semi-ring A^+, i.e. a subset A^+ of A such that $A^+ + A^+ \subset A^+$, $A^+.A^+ \subset A^+$, $r(A^+) \subset (0, \infty)$ and $A \doteq A^+ - A^+$. A subadditive function p on X is A^+-*semimodular* if $p(\alpha x) \le r(\alpha)p(x)$ for $\alpha \in A^+$, $x \in X$.

Theorem *Let P be the set of all A^+-semimodular subadditive functions on X and F the set of all A-modular additive functions in X. If $p_0 \in P$ and $f_0 \in F$ satisfy $f_0(x) \le p_0(x)$ for $x \in D(f_0)$, then there exist $p \in P$ and $f \in F$ such that:*

1. $p \le p_0$ and $f_0 \subset f$;

2. $f(y) = p(x + y) - p(y) = p(y)$ for $x \in X$, $y \in D(f)$;

3. if $q \in P$ satisfies $f_0(x) \le q(x)$ for every $x \in D(f_0)$ and $q \le p$, then $q = p$;

4. if $g \in F$ satisfies $g(x) \leq p(x)$ for every $x \in D(g)$, then $g \subset f$.

We are thus faced with the problem to know when is $D(f)$ the whole group X. To answer this question Rodríguez-Salinas introduces a generalization of solvable groups. He says that a group G is *perfect* if given a finite family $\{\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n\}$ of elements of G there exists another finite family $\{\xi_1, \ldots, \xi_r; \eta_1, \ldots, \eta_s\}$ and a bijective map φ from $\{(\rho, \mu) : 1 \leq \rho \leq r, 1 \leq \mu \leq m\}$ onto $\{(\sigma, \nu); 1 \leq \sigma \leq s, 1 \leq \nu \leq n\}$ such that for $\varphi(\rho, \nu) = (\sigma, \nu)$ one has $\xi_\rho \alpha_\mu = \eta_\sigma \beta_\nu$. Obviously every abelian group is perfect. A group G is *accessible* if it has a, possibly transfinite, normal series (G_λ) such that all the quotient groups are perfect. With this definition don Baltasar proves that $D(f) = X$ if there exists a multiplicative subgroup G of A^+ which satisfies the following conditions:

1. G is an accessible group;

2. Every element $a \in A^+$ can be written in the form $a = \sum_{i=1}^n c_i \alpha_i$, where the c_i belong to the center of A^+ and $\alpha_i \in G$.

I have neither the time nor the competence to report on the papers don Baltasar has written in collaboration with Francisco L. Hernández [113], [117], [128], [131], [139] on the geometry of Orlicz spaces. Fortunately they have published an expository lecture [120] which summarizes some of their results.

The subdivision in three parts of this lecture is artificial since many concepts discussed in this section have a close relation to measure theory. For instance, the algebraic dual of $\ell_0^\infty(X, \Sigma)$ is identified with the space of finitely additive measures m defined on Σ by means of $m(A) = \langle f, \chi_A \rangle$ for $f \in \ell_0^\infty(X, \Sigma)^*$ and $A \in \Sigma$. Rodríguez-Salinas [60] investigates together with the extension of linear maps also the extension of a measure defined on a Boole algebra.

The Dieudonné-Grothendieck theorem $\{8\}$ states that if $m : \Sigma \to F$ is a measure defined on a σ-algebra Σ of subsets of a set X with values in a Banach space F and if for x' belonging to a total subset of F' the scalar-valued measures $x' \circ m$, defined by $A \mapsto \langle m(A), x' \rangle$, are bounded and finitely additive, then m is a bounded vector measure. Rodríguez-Salinas [82] proves that the theorem is true if and only if F is a $\Gamma_r(\ell_0^\infty(x, \Sigma))$-space.

3   Theory of measure and integration

This is the largest and possibly the most important part of the oeuvre of
Baltasar Rodríguez-Salinas, and I will be able to cover only a very small
portion of it. Furthermore, these works consist mainly of great theories
and it is difficult to single out isolated theorems. I will concentrate on
describing briefly two of these theories: 1) Radon measures of type $(\mathcal{H})$,
and 2) Integration of functions with values in a locally convex space.

1) Until about 1964 there were essentially two theories of measure:
1. Abstract measures as discussed e.g. in the book of Paul Halmos
{13}, and 2. Radon measures on locally compact spaces developed by
Henri Cartan and André Weil, and codified in the book by N. Bourbaki
{6}. It became, however, apparent that, mainly because of their appli-
cations to probability theory, it is necessary to define Radon measures
also on spaces that are not locally compact, e.g. on infinite-dimensional
topological vector spaces. In 1964 Laurent Schwartz first in a series of
lectures in Lisbon and then in a course at the Paris University developed
a theory of Radon measures on arbitrary topological spaces {22}. In his
definition a Radon measure on a topological space E is an abstract mea-
sure μ which, among others, has the property that $\mu(K) < \infty$ for every
compact subset K of E. Thus the counting measure μ on $\mathbf{R}$, for which
$\mu(A)$ is the number of elements in A if A is finite and $\mu(A) = \infty$ if A is
infinite, is not a Radon measure in the sense of Schwartz.

Rodríguez-Salinas introduced in collaboration with Pedro Jiménez
Guerra the concept of Radon measures of type $(\mathcal{H})$ [62], [68], [69], [76].
Let E be a topological space and μ a positive measure on E, i.e. a set
function defined on a σ-algebra Σ of subsets of E such that $0 \leq \mu(A) \leq$
∞ for $A \in \Sigma$ and $\mu(\cup_n A_n) = \Sigma_n \mu(A_n)$ if (A_n) is a sequence of pairwise
disjoint sets in Σ. Assume that Σ contains the σ-algebra $\mathcal{B}$ of Borel
subsets of E. A subset K of E is said to be μ-*compact* if given a cover
$(G_\lambda)_{\lambda \in \Lambda}$ of K by open sets and $\epsilon > 0$, there exists a finite subset Φ of
the index set Λ such that $\mu(K \cap (\cup_{\lambda \in \Phi} G_\lambda)^c) \leq \epsilon$.

If there is a collection $\mathcal{H}$ of closed subsets H of E which verifies the
following conditions:

1. every $H \in \mathcal{H}$ is μ-compact,

2. $\mu(H) < \infty$ for every $H \in \mathcal{H}$,

3. for every $B \in \mathcal{B}$ one has

$$\mu(B) = \sup\{\mu(H) : B \subset H \in \mathcal{H}\},$$

then μ is said to be *Radon measure of type* $(\mathcal{H})$.

A possible choice for $\mathcal{H}$ is the collection $\mathcal{F}$ of all finite subsets of E. Clearly the counting measure is a Radon measure of type $(\mathcal{F})$ on any topological space. Other possibilities are: the collection $\mathcal{K}$ of all compact sets (this is the choice of Schwartz), the collection $\mathcal{K}_m$ of all metrizable compact sets, and the collection $\mathcal{F}_\varphi$ of all closed subsets with finite measure. A Radon measure of type $(\mathcal{H})$ is not assumed necessarily to be locally finite.

Let E be a not necessarily Hausdorff topological space and μ^* an outer measure on E, that is a set function defined on the collection $\mathcal{P}(E)$ of all subsets of E which satisfies the following conditions:

1. $0 \leq \mu^*(X) \leq \infty$ for $X \subset E$,

2. $\mu^*(\emptyset) = 0$,

3. $\mu^*(X) \leq \mu^*(Y)$ if $X \subset Y$, $X, Y \in \mathcal{P}(E)$,

and μ^* is countably subadditive, i.e.

4. $\mu^*(\cup_n X_n) \leq \Sigma_n \mu^*(X_n)$ if (X_n) is a sequence of subsets of E.

Rodríguez-Salinas and Jiménez Guerra say that μ^* is a *topological outer measure* if furthermore:

a) every Borel subset of E is μ^*-measurable in the sense of Carathédory, i.e.

$$\mu^*(X) = \mu^*(X \cap B) + \mu^*(X \cap B^c) \quad \text{for} \quad X \in \mathcal{P}(E), B \in \mathcal{B},$$

b) μ^* is locally finite, i.e. every point $x \in E$ has a neighborhood V such that $\mu^*(V) < \infty$,

c) if (G_λ) is an increasingly directed family of open subsets of E then

$$\mu^*(\cup_\lambda G_\lambda) = \sup_\lambda \mu^*(G_\lambda),$$

d) for every set $X \subset E$ one has

$$\mu^*(X) = \inf\{\mu^*(G) : X \subset G \in \mathcal{G}\}.$$

where $\mathcal{G}$ denotes the collection of all open subsets of E.

They call the restriction μ of μ^* to $\mathcal{B}$ (or to the collection of all μ^*-measurable sets) a *topological measure*. They also introduce the *essential outer measure* $\mu^\bullet$ associated with μ^* by setting

$$\mu^\bullet(X) = \sup\{\mu^*(X \cap G) : G \in \mathcal{G}_0\},$$

where $\mathcal{G}_0$ is the collection of those $G \in \mathcal{G}$ for which $\mu^*(G) < \infty$.

Let now μ be a locally finite Radon measure of type $(\mathcal{H})$, where $\mathcal{H}$ is an increasingly directed collection of closed subsets of E. For any $X \subset E$ the authors set

$$\mu_\bullet(X) = \sup\{\mu(H) : X \supset H \in \mathcal{H}\},$$

and for $H \in \mathcal{H}$ and $X \subset H$ they define

$$\mu_H^*(X) = \mu(H) - \mu_\bullet(H \cap X^c) = \inf\{\mu(B) : X \subset B \in \mathcal{B}\}.$$

Then μ_H^* is a topological outer measure on H and

$$\mu^\bullet(X) = \sup\{\mu_H^*(X \cap H) : H \in \mathcal{H}\}$$

is the essential outer measure associated with the topological outer measure

$$\mu^*(X) = \inf\{\mu(G) : X \subset G \in \mathcal{G}\}.$$

Using these concepts and techniques Rodríguez-Salinas and Jiménez Guerra generalize the results in the book of Laurent Schwartz. Thus where Schwartz imposes the condition that the compact subsets of E are metrizable, they consider Radon measures of type $(\mathcal{K}_m)$. They are able to define the measure induced on any subset of E not just a measurable one. They prove the existence of a "concassage" for a locally σ-finite Radon measure of type $(\mathcal{H})$, furthermore a Lebesgue-Nikodým theorem, the existence of tensor products, and most importantly theorems on the existence of the projective limit of a family of measures.

Let μ be a finite Radon measure of type $(\mathcal{H})$ and E a Hausdorff topological space. Consider a sequence $(C_n)_{n \in \mathbb{N}}$ of countable sets, a sequence $(p_n)_{n \in \mathbb{N}}$ of surjective maps $p_n : C_{n+1} \to C_n$, and a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of maps $\varphi_n : C_n \to \mathcal{P}(E)$. Rodríguez-Salinas says that E is a μ-*Suslin space* [71] if the following conditions are satisfied:

1. $\varphi_{n+1}(c) \subset \varphi_n(p_n(c))$ for $c \in C_{n+1}$, $n \in \mathbb{N}$,

2. the sets

$$E - \cup\{\varphi_0(c) : c \in C_0\}$$

and

$$\varphi_n(c) - \cup\{\varphi_{n+1}(d) : d \in C_{n+1}, p_n(d) = c\}, \quad c \in C_n, \quad n \in \mathbb{N}$$

have μ-measure zero,

3. let (c_n) be a coherent sequence, i.e. $c_n \in C_n$ and $p_n(c_{n+1}) = c_n$; then the sets $\varphi_n(c_n)$ converge to a point $x \in E$ contained in each set $\varphi_n(c_n)$.

He says that E is a *μ-Lusin space* if furthermore:

(*) each φ_n is injective and the sets $\varphi_n(c)$, $c \in C_n$, are pairwise disjoint.

Each set $\varphi_n(c)$ is then μ-measurable. The space E is *semi-μ-Suslin* if (C_n, p_n, φ_n) satisfies conditions 1. and 2. above and furthermore:

0) each set $\varphi_n(c)$, $c \in C_n$, $n \in \mathbb{N}$, is μ-measurable,

3') if (c_n) is a coherent sequence and $x \in \cap_n \varphi_n(c_n)$, then $\varphi_n(c_n)$ converges to x.

The space E is *semi-μ-Lusin* if furthermore (*) is satisfied. We have the following implications:

$$\begin{array}{ccccc}
\text{Suslin} & \Longrightarrow & \mu\text{-Suslin} & \Longrightarrow & \text{semi-}\mu\text{-Suslin} \\
\Uparrow & & \Uparrow & & \Uparrow \\
\text{Lusin} & \Longrightarrow & \mu\text{-Lusin} & \Longrightarrow & \text{semi-}\mu\text{-Lusin.}
\end{array}$$

Theorem [71]. *Let E be a Hausdorff space and μ a positive Borel measure on E such that $\mu(E) = 1$. The following are equivalent:*

1. *E is μ-Suslin.*

2. *μ is a Radon measure of type $(\mathcal{K}_m)$.*

3. *There exists a sequence (C_n, p_n, φ_n) such that each C_n consists of compact, metrizable, pairwise disjoint subsets of E and φ_n is the identity.*

4. E is μ-Lusin.

Let $\mathcal{L}^\infty$ be the algebra of essentially bounded measurable functions, L^∞ the usual quotient space modulo the functions almost everywhere equal to zero, and π the canonical surjection $\mathcal{L}^\infty \rightarrow L^\infty$. An algebra homomorphism $\rho : \mathcal{L}^\infty \rightarrow \mathcal{L}^\infty$ is a *lifting* {14} if it satisfies the following conditions:

(I) $\rho(f)$ is almost everywhere equal to f;

(II) if f is almost everywhere equal to g, then $\rho(f) = \rho(g)$.

The name "lifting" is due to the fact that ρ permits to define a section $\sigma : L^\infty \rightarrow \mathcal{L}^\infty$ of π by setting $\sigma(\pi(f)) = \rho(f)$. A lifting ρ is *almost strong* if there exists a set $A \subset E$ such that $\mu^*(A^c) = 0$ and for all bounded continuous functions f the restrictions of f and of $\rho(f)$ to A are equal. Don Baltasar proves [71] that if E is a semi-μ-Lusin space then every lifting is almost strong.

A Radon measure μ of type $(\mathcal{H})$ is *strict* if for every Borel set B one has

$$\mu(B) = \inf\{\mu(G) : B \subset G \in \mathcal{G}\}.$$

Let E be a uniform space and h a function which associates with each surrounding (= entourage) U a strictly positive number $h(U)$ such that $U \subset V$ implies $h(U) \leq h(V)$ and $\lim_U h(U) = 0$, where the limit is with respect to the decreasingly directed set of surroundings. For $x \in E$ let $U[x]$ be the neighborhood $\{y \in E : (x,y) \in U\}$ of x and write $U[A] = \cup_{x \in A} U[x]$. Generalizing a result of P. Mattila {20}, don Baltasar proves [91] the following result concerning the differentiation of measures on the uniform space E: If μ is a strict Radon measure of type $(\mathcal{H})$ on E, the set $A \subset E$ has σ-finite μ-measure and $f \in L^p(\mu)$, then

$$\lim_U \frac{1}{h(U)} \int_A \int_{U[x]} |f(y) - f(x)|^p d\mu(y) d\mu(x) = 0.$$

Rodríguez-Salinas and Jiménez Guerra [90] prove Fubini theorems of which I quote the following: Let X and Y be two topological spaces, μ a Radon measure of type $(\mathcal{H}_1)$ on X, and ν a Radon measure of type $(\mathcal{H}_2)$ on Y. Assume that the tensor product $\mu \otimes \nu$ exists; this is for instance the case when each $(\mathcal{H}_k)(k = 1, 2)$ is increasingly directed and

every $H \in \mathcal{H}_k$ is quasiregular for the induced topology. Let f be a $\mu \otimes \nu$-integrable function on $X \times Y$. Then $f_x : y \mapsto f(x, y)$ is ν-integrable for μ-almost every $x \in X$ and

$$\int_{X \times Y} f(x, y) d(\mu \otimes \nu) = \int_X \left(\int_Y f(x, y) d\nu(y) \right) d\mu(x).$$

2) Let E be a vector space equipped with a locally convex Hausdorff topology $\mathcal{T}$. Let me recall that a subset M of E is precompact if every ultrafilter on M is a Cauchy filter. Denote by $\mathcal{M}$ the collection of all precompact subsets of E. On the topological dual E' of E the topology $\mathcal{T}^0$ polar to $\mathcal{T}$ is the topology of uniform convergence on the sets $M \in \mathcal{M}$. The space E is said to be *polar semi-reflexive* if the topological dual of E' equipped with $\mathcal{T}^0$ coincides as a vector space with E. For this it is necessary and sufficient that every precompact subset of E be relatively compact $\{15\}$.

Rodríguez-Salinas constructs a theory of integration for functions $f : (\Omega, \Sigma, \mu) \to E$, where Σ is a σ-algebra of subsets of a set Ω, and E is a polar semi-reflexive space. He supposes first [75] that the measure μ defined on Σ is finite, and later [93] considers arbitrary measures. Denote by S_0 the space of Σ-measurable functions f which assume only finitely many values in E, i.e. of the form $f = \sum_{i=1}^n y_i \chi_{A_i}$, where $A_i \in \Sigma$ and $0 \neq y_i \in E$. Let S be the space which consists of the uniform limits of nets of functions belonging to S_0. It is easy to see that if $f \in S$, then $f(\Omega)$ is precompact in E. Let Σ_0 denote the collection of those $A \in \Sigma$ for which $\mu(A) < \infty$. Don Baltasar says that $f : \Omega \to E$ is μ-*measurable* if for every $A \in \Sigma_0$ there exists a net (f_i) of functions belonging to S which converges almost uniformly to f on A, i.e. for every $\epsilon > 0$ there exists a subset B of A such that $\mu(B) \leq \epsilon$ and (f_i) converges uniformly to f on $A \cap B^c$. The function f is *totally μ-measurable* if the same net (f_i) can be used for every $A \in \Sigma_0$. Denote by $M = M(\Sigma, \mu; E)$ the vector space of μ-measurable functions, and by $M_t = M_t(\Sigma, \mu; E)$ that of totally μ-measurable functions. Clearly $S_0 \subset S \subset M_t \subset M$.

The measure space (Ω, Σ, μ) is *strictly localizable* if there is a partitioning $\Omega = \cup_{\lambda \in \Lambda} \Omega_\lambda$ into pairwise disjoint sets $\Omega_\lambda \in \Sigma_0$ such that for every set $A \in \Sigma_0$ there exists a countable subset Λ_0 of the index set Λ for which $\mu(A \cap (\cup_{\lambda \in \Lambda_0} \Omega_\lambda)^c) = 0$. Rodríguez-Salinas proves that if (Ω, Σ, μ) is strictly localizable, then $M_t = M$. He is introduces two further concepts of measurability:

The function $f : \Omega \to E$ is $\overline{\mu}$-*measurable* if for every $A \in \Sigma_0$ there exists a net of μ-*measurable* functions which converges to f uniformly on A. The function f is *totally* $\overline{\mu}$-*measurable* if there exists a net of totally μ-*measurable* functions which converges to f, uniformly on every set $A \in \Sigma_0$.

If f is $\overline{\mu}$-*measurable* then f is essentially ω-precompact, i.e. for every $A \in \Sigma_0$ and every neighborhood V of 0 in E there exists a subset Z of A with $\mu(Z) = 0$ and a countable subset N of E such that $f(A \cap Z^c) \subset N + V$.

For $f = \sum_{i=1}^{n} y_i \chi_{A_i} \in S_0$ the integral on $A \in \Sigma$ is defined by

$$\int_A f d\mu = \sum_{i=1}^{n} y_i \mu(A \cap A_i)$$

provided that $\sum_{i=1}^{n} \mu(A \cap A_i) < \infty$, and f is said to be integrable if it is integrable on Ω, i.e. if $\sum_{i=1}^{n} \mu(A_i) < \infty$. For $f \in S$ and $A \in \Sigma_0$ one defines

$$\int_A f d\mu = \lim_i \int_A f_i d\mu,$$

where (f_i) is a net of functions in S_0 which converges uniformly to f. The limit exists and is independent of the choice of the net. One has

$$\int_A f d\mu \in \mu(A) \cdot f(\Omega)^{00};$$

since $f(\Omega)$ is precompact and E is polar semi-reflexive the bipolar $f(\Omega)^{00}$ belongs to E and so does the integral.

A function $f \in S$ is *integrable* if the following conditions are satisfied:

1) There exists a subset $\Omega_0 \in \Sigma$ of Ω such that f vanishes outside Ω_0 and Ω_0 is strictly localizable, i.e. it has a partitioning $(\Omega_\lambda)_{\lambda \in \Lambda}$ into pairwise disjoint sets $\Omega_\lambda \in \Sigma_0$ with the above described properties.

2) For every $A \in \Sigma$ the family $(\int_{A \cap \Omega_\lambda} f d\mu)_{\lambda \in \Lambda}$ is summable in E.

The *integral* of f on $A \in \Sigma$ is then defined by

$$\int_A f d\mu = \sum_{\lambda \in \Lambda} \int_{A \cap \Omega_\lambda} f d\mu.$$

It is independent of the choice of Ω_0 and of the Ω_λ.

At this point Rodríguez-Salinas supposes that the measure μ is essential, i.e. that for every $A \in \Sigma$ one has

$$\mu(A) = \sup\{\mu(A \cap K) : K \in \Sigma_0\}.$$

The function $f : \Omega \to E$ is said to be μ-*integrable* (resp. *totally μ-integrable*) if:

1) There exists a strictly localizable subset $\Omega_0 \in \Sigma$ of Ω such that $f(t) = 0$ for $t \in \Omega_0^c$.

2) f is μ-measurable (resp. totally μ-measurable).

3) f is *Pettis-integrable*, i.e. for every $x' \in E'$ the numerical function $t \mapsto \langle f(t), x' \rangle$ is μ-integrable and for all $A \in \Sigma$ there exists $m_f(A) \in E$ such that

$$\int_A \langle f(t), x' \rangle d\mu(t) = \langle m_f(A), x' \rangle.$$

Then the integral of f on A is $m_f(A)$.

Rodríguez-Salinas introduces furthermore the concept of a $\overline{\mu}$-*integrable* function $f : \Omega \to E$ by means of the following conditions:

0) f is $\overline{\mu}$-measurable.

1) There exists a strictly localizable subset $\Omega_0 \in \Sigma$ of Ω such that $f(t) = 0$ for $t \in \Omega_0^c$.

2) For every $A \in \Sigma_0$ the function f is the uniform limit on A of a net (f_i) of μ-integrable functions such that $\lim_i \int_A f_i d\mu$ exists in E. Then the limit does not depend on the choice of the approximating net and one sets

$$\int_A f d\mu = \lim_i \int_A f_i d\mu.$$

3) If $(\Omega_\lambda)_{\lambda \in \Lambda}$ is the partition of Ω_0 which figures in the definition of the strict localizability of Ω_0, then for all $A \in \Sigma$ the family $(\int_{A \cap \Omega_\lambda} f d\mu)_{\lambda \in \Lambda}$ is summable in E.

In this case the $\overline{\mu}$-*integral* of f on $A \in \Sigma$ is, of course, defined by

$$\int_A f d\mu = \sum_{\lambda \in \Lambda} \int_{A \cap \Omega_\lambda} f d\mu.$$

Every $\overline{\mu}$-integrable function is Pettis integrable but there exists $\overline{\mu}$-integrable functions that are not μ-measurable.

Let $\mathcal{P}$ be the collection of all continuous seminorms on E. Two functions f and g from Ω into E are μ-*equivalent* (in symbol: $f \cong g$) if

$$\mu(\{t \in \Omega : p(f(t) - g(t)) \neq 0\}) = 0$$

for all $p \in \mathcal{P}$. The *essential range* of $f : \Omega \to E$ on $A \in \Sigma$ is defined by

$$\mathrm{er}_A(f) = \{x \in E : \mu(\{t \in A : p(f(t) - x) \leq 1\}) \neq 0, \quad \forall p \in \mathcal{P}\}.$$

Among the several lifting theorems Rodríguez-Salinas proves in Part IV of [93] let me quote the following:

Let (Ω, Σ, μ) be a strictly localizable measure space, $\rho : S \to S$ a lifting and f a μ-measurable function. Then there exists a μ-measurable function g such that $f \cong g$, $\rho(g) = g$ and for every $A \in \Sigma_0$ with $\mu(A) > 0$ one has $g(t) \in \mathrm{er}_A(f) = \mathrm{er}_A(g)$ for almost every $t \in A$. Furthermore $g(A) \subset \mathrm{er}_A(g) \cup \{0\}$ if $\rho(A) \supset A$.

Between 1980 and 1984 Rodríguez-Salinas wrote a number of articles on the Radon-Nikodým theorem for vector-valued measures, all published in the Revista of the Royal Academy of Madrid [77], [78], [94], [96], [98], [99]. I can only give a small sample of his results. Let (Ω, Σ, μ) be a complete measure space with $\mu(\Omega) < \infty$. Let E be a locally convex Hausdorff space and $m : \Sigma \to E$ a vector measure. Denote by Σ^+ the collection of those $A \in \Sigma$ for which $\mu(A) > 0$. Set

$$A_\Omega(m) = \left\{ \frac{m(A)}{\mu(A)} : A \in \Sigma^+ \right\}.$$

The measure m is *controlled* by μ if m is μ-continuous and $A_\Omega(m)$ is bounded. The space E has the Radon-Nikodým property if for every (Ω, Σ, μ) and every $m : \Sigma \to E$ controlled by μ there exists a $\overline{\mu}$-integrable function $f : \Omega \to E''$ such that $m(A) = \int_A f(t) d\mu(t)$ for each $A \in \Sigma$.

The set $A \subset E$ is *dentable* if for every neighborhood V of 0 in E there exists an $x \in A$ such that x does not belong to the closed convex hull of $A \cap (x + V)^c$. The set $A \subset E$ is σ-*dentable* if for every neighborhood V of 0 in E there exists $x \in A$ such that $x \notin \sigma(A \cap (x + V)^c)$, where $\sigma(X) = \{\sum_{n \in \mathbb{N}} \lambda_n x_n : x_n \in X, \lambda_n > 0, \sum_{n \in \mathbb{N}} \lambda_n = 1\}$.

Theorem [96]. *Let E be a locally convex Hausdorff space and assume that its dual E' equipped with the strong topology $\beta(E', E)$ is quasi-complete (this is the case e.g. when E is barrelled, and it implies that E' is polar semi-reflexive). Let us furthermore assume that for every countable bounded subset B of E the closed subspace F generated by B is infrabarrelled. Then the following are equivalent:*

i) *E' has the Radon-Nikodým property.*

ii) *Every countable bounded non-empty subset of E' is dentable.*

iii) *Every bounded non-empty subset of E' is σ-dentable.*

iv) *Every equicontinuous non-empty subset B of E' has the property that for every neighborhood V of 0 in E' there is an $x' \in B$ such that x' does not belong to the convex hull of $B \cap (x' + V)^c$.*

v) *For each of the above mentioned subspaces F every bounded subset of the strong dual F' is ω-precompact.*

The same conclusion holds if E is supposed to be infrabarrelled [98] (without any conditions on E' or F).

Theorem [94]. *Let E be a quasi-complete locally convex Hausdorff space. The following are equivalent:*

i) *E has the Radon-Nikodým property.*

ii) *Every martingale (f_i, Σ_i) with values in the space E, such that the f_i are uniformly bounded, is a Cauchy net in the space $\mathcal{L}^1(\Omega, \Sigma, \mu; E)$ of μ-integrable functions.*

iii) *Every martingale $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ with values in E, such that the f_n are uniformly bounded, is a Cauchy sequence in the space $\overline{\mathcal{L}}^1(\Omega, \Sigma, \mu; E)$ of $\overline{\mu}$-integrable functions.*

iv) *Every countable bounded non-empty subset E is σ-dentable.*

v) *Every bounded non-empty subset E is σ-dentable.*

vi) *For every countable bounded non-empty subset B of E and every neighborhood V of 0 in E there exists $x \in B$ such that x does not belong to the convex hull of $B \cap (x + V)^c$.*

vii) *For every bounded non-empty subset B of E and every neighborhood V of 0 in E there exists $x \in B$ such that x does not belong to the convex hull of $B \cap (x + V)^c$.*

viii) *For every bounded non-empty subset B of E and every balanced convex closed neighborhood V of 0 in E the set $\pi_V(B)$ is dentable (or, equivalently, σ-dentable), where π_V denotes the canonical surjection from E onto the quotient space $E / \cap_{\lambda > 0} \lambda U$.*

Rodríguez-Salinas has written about twenty more works on the theory of integration. Ten of these papers were written in collaboration with José L. DeMaria. Some of their results are summarized in the expository lecture [119] which introduces such concepts as Spanish σ-algebras and Spanish sets. Besides the papers referred to above, don Baltasar wrote other papers on integration with Pedro Jiménez Guerra. One of the papers written jointly with Fernando Bombal Gordón [103] is related to the topics discussed above. Let me observe that Bombal developed a parallel theory of integration for functions with values in a vector space {3}, {4}, which, however, is not assumed to be topological but bornological.

Don Baltasar is still very active and his mathematical output has not diminished. I wish him many more years of healthy, happy mathematical activity.

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