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SOME SIMILARITY SOLUTIONS OF THE NAVIER-STOKES EQUATIONS AND RELATED TOPICS

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Dedicated to Professor Fon-Che Liu on his sixtieth birthday

Abstract. We consider a semilinear equation arising from the Navier-Stokes equations – the governing equations of viscous fluid motion – and related model equations. The solutions of the semilinear equation represent a certain class of exact solutions of the Navier-Stokes equations. Both the equation and our models have nonlocal terms. We will show that the nonlocality will play an intriguing role for the blow-up and/or global existence of the solutions and that the convection term, which is often neglected in the study of the blow-up problems, plays a very decisive role. In addition to our new contributions, open problems and known facts are surveyed.

1. Introduction

The main purpose of the present paper is to mathematically analyze the following equation:

$$f_{txx} + f f_{xxx} - f_x f_{xx} = \nu f_{xxxx}, \tag{1}$$

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where ν is a constant called the viscosity, $t > 0$ is the time variable, and x is a real variable running in an interval $[\alpha, \beta]$. Here and hereafter the subscripts imply the differentiation, unless otherwise stated. The following generalized equation is also considered:

$$f_{txx} + f f_{xxx} - a f_x f_{xx} = \nu f_{xxxx}, \tag{2}$$

where $a \in R$ is a new parameter. The equation (1) derives, as we will explain in the next section, from the Navier-Stokes equations of incompressible viscous fluid motion. We are not sure who the first to discover (1) was but it seems to us that it first appeared in Riabouchinsky [27]. However, the main theme of the paper is not to analyze (1) but to consider its stationary solutions. Proudman and Johnson [26] consider the genuinely nonstationary solutions; hence the equation (1) may be called the Proudman-Johnson equation. It should, however, be noted that the stationary equation of (1) has a much longer history. In fact, it is Hiemenz [18] who first derived the equation (1) with f_{txx} omitted.

The equation (2) is an artificial one at this stage. However, we will show in the next section that it too has some physical meaning. The purpose of the generalization is as follows. The equation (1) has two nonlinear terms. We will show that both of them play a very important role and the balance between the two nonlinear terms is decisive. In order to study the balance, we come up with the generalized equation (2), where unbalance of the two nonlinear terms is measured by a. We show that some solutions blow up for some values of a and all the solutions decay to zero for other values.

Two boundary conditions are considered for (1) and (2). One is the following Dirichlet boundary condition:

$$f(t, \alpha) = f_x(t, \alpha) = f(t, \beta) = f_x(t, \beta) = 0. \tag{3}$$

The other is the periodic boundary condition.

It is also interesting to study the case where $a \to \infty$. This limiting process can be achieved by rescaling $f = a^{-1} F$ to obtain

$$F_{txx} + bFF_{xxx} - F_x F_{xx} = \nu F_{xxxx}, \tag{4}$$

where $b = 1/a$. This equation is the same one which was studied in Budd *et al.* [4]. Setting $b = 0$, integrating once, and defining $u = \frac{1}{2} f_x$, we obtain the following equation:

$$\begin{cases} u_t = \nu u_{xx} + u^2 - c(t) & (0 < t,\ \alpha < x < \beta), \\ \int_\alpha^\beta u(t, x) dx = 0 & (0 < t), \end{cases} \tag{5}$$

where $c(t)$ is an unknown quantity depending only on t. The second equation in (5) is a consequence of the boundary condition (3). If the term $c(t)$ is absent, the differential equation is a well-known and well-studied one, especially in the context of the blow-up problems. Addition of the unknown $c(t)$ requires one more condition, which is supplied as the second requirement in (5), i.e., $\int_\alpha^\beta u(t,x)dx = 0$.

The rest of the present paper is organized as follows. Section 2 is devoted to the derivation of the equations. Local existence of the solutions are demonstrated in Section 3. In Section 4, a blow-up theorem for (5) is proved when $\nu > 0$. Blow-up of solutions of (5) when $\nu = 0$ is treated in Section 5. We also consider (5) for the piecewise constant initial data. We then derive a blow-up result, which is sharper than the one in Section 4. We will also show how the result is different from those for the equation without $c(t)$. Section 6 is devoted to comparisons among the dynamics of three equations: the equation (1), our model (5), and the well-known equation without $c(t)$. Global existence results for (2) are established in Section 7. Blow-up of solutions of (2) when $\nu = 0$ is considered in Section 8. Steady-states of (2) are considered in Section 9. Section 10 is a remark about von Kármán's swirling flows.

2. Derivation of the Equations

We first explain how (1) derives from the two-dimensional Navier-Stokes equations, which are written as follows:

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} &= \nu\triangle\mathbf{u} - \frac{1}{\rho}\nabla p, \\ \operatorname{div}\mathbf{u} &= 0. \end{aligned}$$

Here the velocity vector $\mathbf{u}$ is given by $\mathbf{u} = (u,v) = (\psi_y, -\psi_x)$, where ψ is a scalar-valued function called the stream function. $\nu \geq 0$ is the viscosity and $\rho > 0$ is the mass density. We consider the Navier-Stokes equations in $\alpha < x < \beta, -\infty < y < +\infty$. We then employ the ansatz that $\psi(t,x,y) = yf(t,x)$ to obtain the equation (1). (Since the calculation necessary to derive (1) is elementary, we omit it here.) Accordingly, the solutions of (1) represent a certain class of exact solutions of the two-dimensional Navier-Stokes equations. If we assume the non-slip boundary condition on the boundary $x = \alpha$, $x = \beta$, then we may assume the boundary condition (3). We also consider (1) with the periodic boundary condition.

It is interesting to see that the equation (2) in the case where $a = 0$ is also derived from the Navier-Stokes equations. In fact, let us consider the Navier-Stokes equations in cylindrical coordinates in R^3 and assume that $(u_r, u_\theta, u_z) = (-\frac{r}{2}f_z(t,z), 0, f(t,z))$. (Here the subscripts in the left-hand side denote the direction, not the differentiation. On the other hand, f_z means the derivative of f with respect to z.) Then the Navier-Stokes equations are satisfied if f satisfies

$$f_{tzz} + f f_{zzz} = \nu f_{zzzz}. \tag{6}$$

If we consider the Navier-Stokes equations in m-dimensions with cylindrical coordinates and if we assume that $u_r = -\frac{r}{m-1} f(t,z), u_z = f(t,z)$, where $z = x_m$ and $r^2 = x_1^2 + \cdots + x_{m-1}^2$, then the Navier-Stokes equations are reduced to

$$f_{tzz} + f f_{zzz} + \frac{m-3}{m-1} f_z f_{zz} = \nu f_{zzzz}. \tag{7}$$

See, for instance, Weyl [32]. Thus, the artificial parameter a in (2) is physically substantiated for these discrete values. If $m = 2$, (7) is nothing but (1). If $-1 \le a < 3$, then the stationary equation of (2) is derived, without any approximation, from the two-dimensional Navier-Stokes equations; see Weyl [32].

Remark. Extensions in higher dimensions are possible in many ways. In fact, [17] and (independently) [35] consider the three-dimensional version, which consists of a coupled system of one-dimensional equations.

The equation (1) is equivalently written as follows, which is more convenient for the numerical computation: If we put $\omega = -f_{xx}$, then

$$\omega_t + f\omega_x - f_x\omega = \nu\omega_{xx}, \tag{8}$$

$$f = G(\omega), \tag{9}$$

$$f_x(t,\alpha) = f_x(t,\beta) = 0. \tag{10}$$

Here G is the Green operator for $-d^2/dx^2$ in $\alpha < x < \beta$ with the Dirichlet boundary condition $f|_{x=\alpha} = f|_\beta = 0$. Note that $f = G(\omega)$ is explicitly given as

$$\begin{aligned} f(x) = &\frac{1}{\beta-\alpha}\int_\alpha^x (\beta - x)(\xi - \alpha)\omega(t,\xi)d\xi \\ &+ \frac{1}{\beta-\alpha}\int_x^\beta (\beta-\xi)(x-\alpha)\omega(t,\xi)d\xi. \end{aligned} \tag{11}$$

Since it gives us

$$f_x(x) = \frac{-1}{\beta-\alpha}\int_\alpha^x (\xi-\alpha)\omega(t,\xi)d\xi + \frac{1}{\beta-\alpha}\int_x^\beta (\beta-\xi)\omega(t,\xi)d\xi,$$

the boundary condition (10) can be replaced equivalently by

$$\int_\alpha^\beta \omega(t,\xi)d\xi = \int_\alpha^\beta \xi\omega(t,\xi)d\xi = 0. \tag{12}$$

Thus we have reached the two, mutually equivalent sets of nonlinear and nonlocal equations: $(8, 9, 10)$ and $(8, 9, 12)$.

The equation (8) has three important terms. Namely, the dissipation term $\nu\omega_{xx}$, the convection term $f\omega_x$, and the stretching term $f_x\omega$. Strictly speaking, the stretching term helps stretching only if $f_x > 0$. However, we call $f_x\omega$ the stretching term, whatever the sign of f_x might be. This is an abuse of the terminology. By that abuse, we have thus arrived at an analogue of three-dimensional vortex dynamics of incompressible viscous fluid motion. Note however that this analogy comes from two-dimensional Navier-Stokes equations with infinite energy, and the results for (1) may well have little relevance with real vortex dynamics. (Notice that the physical dimension of ω in (1) is the same as that of vorticity gradient.) But we do not consider this issue of analogy any further and consider the equation (1) as a model equation of blow-up. In other words, (1) or (2) per se is interesting to us.

It is sometimes claimed that the convection term does not play an important role in the increase of the vorticity. We do not know to what extent this statement may be true. So we would like to study the role of the convection term in (1). To this end, we will consider what happens if we drop the convection term in (1). This amounts to considering the following equation

$$\omega_t - f_x\omega = \nu\omega_{xx}, \tag{13}$$

which is the same as the equation (4) with $b = 0$, whence is rewritten as (5). We would like to compare the results for (8) with those for (13) (or (5)). Roughly speaking, the results are: some solutions of (13) (hence of (5)) blow up in finite time, while all the solutions of (8) exist globally in time. Consequently we may say that the nonlinear convection term arrests the blow-up.

If the constant term $c(t)$ in (5) is absent, then the equation is rather famous. Notably Fujita [12] considers the Cauchy problem in which the nonlinear term u^2 is generalized to $u^{1+\alpha}$, and his results lead to further intensive studies of blow-up problems. See, for instance, Deng and Levine [10] or Levine [21].

Many blow-up models have been considered in relation to fluid mechanics. In particular, those with a nonlocal stretching are well-known [6, 7, 8, 28]. Models with nonlocal convection are proposed in [1]. We will show in this paper that the blow-up behavior arising in equation (5) is different from those in [1, 8, 28].

Notation is now defined. $L^2(\alpha, \beta)$ denotes the Hilbert space of all the square summable functions in (α, β) and $L^2(\alpha, \beta)/R$ denotes the orthogonal complement to the one-dimensional space of all the constant functions. $L^\infty(\alpha, \beta)$ denotes the Banach space of all the essentially bounded functions.

$L^\infty(\alpha,\beta)/R$ denotes

$$\left\{ f \in L^\infty(\alpha,\beta); \quad \int_\alpha^\beta f(x)dx = 0 \right\}.$$

$L^\infty(0,T;L^\infty(\alpha,\beta))$ and similar notations are defined straightforwardly. Sobolev spaces of L^2-type are used and denoted by H^m or $H^m(\alpha,\beta)$. All functions in this paper are real-valued.

3. Existence Local-in-time

We first prove the existence local-in-time of solutions of (2). Such a theorem would be an easy exercise if $\nu > 0$. As a matter of fact, we can prove the following theorem, whose proof is omitted.

Theorem 1. *Let $\nu > 0$ and assume that $\omega(0,\cdot) \in L^2$ satisfies* (12). *Then there exists $T > 0$ and a unique solution $\omega \in C^0([0,T];\, L^2) \cap C^1((0,T];L^2)$ of* (8), (9), *and* (12).

The existence theorem is less easy if $\nu = 0$. What we consider is

$$f_{txx} + f f_{xxx} - a f_x f_{xx} = 0 \quad (0 < t,\ \alpha < x < \beta), \tag{14}$$

$$f_{xx}(0,x) \in L^2 \quad \text{is given}, \tag{15}$$

$$f(t,\alpha) = f(t,\beta) = 0. \tag{16}$$

The boundary condition (16) may be replaced by the periodic boundary condition.

In order to prove a local existence for (14)–(16), we use a theorem by Kato and Lai [19]. We prove the existence in the case of periodic boundary condition, since the proof is almost the same in both cases. Accordingly, we assume that $\alpha = 0$ and $\beta = 2\pi$. We use the following theorem, which is a special case of a theorem by Kato and Lai [19] (here S^1 denotes the circle):

Theorem 2. *Let $V = H^1(S^1)/R$, $H = L^2(S^1)/R$, and $X = H^{-1}(S^1)/R$. Let A be a sequentially weakly continuous mapping from H into X such that*

$$\langle v, A(v)\rangle \geq -\beta(\|v\|^2) \quad \text{for } v \in V,$$

where $\beta \geq 0$ is an increasing function. Then for any $u(0) \in H$, there exists $T > 0$ and a solution of $u_t + A(u) = 0$ in the class

$$C_w([0,T];H) \cap C_w^1([0,T];X),$$

where C_w and C_w^1 indicate the weak continuity.

Proof of existence of (14). We define A as follows

$$A(v) = G(v)\frac{\partial v}{\partial x} - aG(v)_x v,$$

where $G(v)$ is the Green operator. Namely, it is defined by (11) for the Dirichlet boundary condition and by the following formula for the periodic boundary condition:

$$G(v) = \sum_{n\neq 0} \frac{a_n}{n^2} e^{inx} \quad \text{for} \quad v = \sum_{n\neq 0} a_n e^{inx}.$$

For $v \in H$, $G(v)$ belongs to C^1; hence it is easy to see that A maps H continuously into X. Since

$$\langle A(v), v\rangle = \int_0^{2\pi} G(v)v_x v dx - a\int_0^{2\pi} G(v)_x v^2 dx = -\frac{1+2a}{2}\int_0^{2\pi} G(v)_x v^2 dx,$$

and since

$$\|G(v)_x\|_{L^\infty} \leq \sqrt{2\pi}\|v\|,$$

where $\| \, \|$ is the L^2-norm, it holds that

$$\langle A(v), v\rangle \geq -\beta(\|v\|^2)$$

with

$$\beta(s) = |1+2a|\sqrt{\frac{\pi}{2}} s^{3/2}.$$

As a consequence, we are given a solution $v \in C_w([0,T];\ L^2/R) \cap C_w^1([0,T];\ H^{-1}/R)$. ■

The Kato-Lai theorem is not concerned with the uniqueness of the solution, which must be proved separately. We now prove that the solution is unique. To this end, let v and w be solutions and $v(0,\cdot) = w(0,\cdot)$. Set $f = G(v)$ and $g = G(w)$. Then, formally, they satisfy $f_{txx} + ff_{xxx} - af_x f_{xx} =$ and $g_{txx} + gg_{xxx} - ag_x g_{xx} = 0$, respectively. Then we obtain

$$(f_{xx} - g_{xx})_t + f(f_{xxx} - g_{xxx}) + (f-g)g_{xxx} - af_x(f_{xx} - g_{xx}) - a(f_x - g_x)g_{xx} = 0.$$

This gives us, by the integration by parts, the following equation:

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi}(f_x - g_x)^2 dx = \int_0^{2\pi}[(1+a)f_{xx} - (a+2)g_{xx}](f-g)(f_x - g_x)dx$$
$$+\frac{3+2a}{2}\int_0^{2\pi} f_x(f_x - g_x)^2 dx.$$

Note that f_{xx} and g_{xx} are uniformly bounded in $0 \le t \le T$ in L^2-norm. Therefore, with the aid of the Sobolev inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|f_x - g_x\|^2 \le M\|f_x - g_x\|^2 \quad (0 \le t \le T').$$

It is now easy to get the uniqueness.

Finally we remark that the solution actually belongs to $C([0,T]; L^2(S^1)/R)$. Namely, the continuity holds for strong topology. This is proved in Kato and Lai [19, page 23], by using the uniqueness of the solution.

Thus we have a unique, local, generalized solution for the case where $\nu = 0$.

4. Blow-up Theorems

The purpose of the present section is to show sufficient conditions for the blow-up of solutions to (5). Budd *et al.* [3] proved a theorem which guaranteed blow-ups of some solutions when $\nu > 0$. We prove a blow-up theorem which is different from theirs.

We begin with the case where $\nu > 0$. We consider

$$u_t - u_{xx} - u^2 + c(t) = 0 \qquad (0 < t,\ 0 < x < 1), \tag{17}$$

$$\int_0^1 u(t,x)dx = 0 \qquad (0 < t), \tag{18}$$

$$u(t,0) = u(t,1) = 0. \tag{19}$$

Both the coefficients of u_{xx} and u^2 are normalized to unity, since the normalization is enabled by the transformation $u(t,x) \to \eta u(\tau t, x)$ and $c(t) \to \gamma c(\tau t)$ with positive constants η, γ, and τ and a suitable linear transformation for x. When we consider the periodic boundary condition, we deal with

$$u_t - u_{xx} - u^2 + \int_{-\pi}^{\pi} u(t,x)^2 dx = 0 \qquad (0 < t, \ -\pi < x < \pi), \tag{20}$$

$$\int_{-\pi}^{\pi} u(t,x)dx = 0 \qquad (0 < t). \tag{21}$$

The point $x = \pi$ and $x = -\pi$ are identified and we regard u as a function defined for $0 < t$ and x on the circle.

We now prove the following

Theorem 3. *If u_0 satisfies the following three conditions*

- $u_0(0) = u_0(1) = 0$ *or the periodic boundary condition,*

- $\int_0^1 u_0(x)dx = 0$,
- $\frac{1}{3}\int_0^1 u_0(x)^3 dx - \frac{1}{2}\int_0^1 u_{0,x}(x)^2 dx > 0$,

then the solution blows up in finite time.

Proof. Since u is of mixed sign, we cannot use Jensen's inequality, which is a powerful tool in the case of convex nonlinearity. Also, neither the comparison theorem nor an everywhere-positive eigenfunction, which play a crucial role in some nonlinear parabolic equations, can be used for our equation. Fortunately we can prove that the theorem above is a special case of the theorem obtained by Levine [20]. He considers an abstract evolution equation

$$u_t = -Au + F(u), \qquad u(0) = u_0, \tag{22}$$

in a Hilbert space H. Here A is a self-adjoint positive definite operator in H with the domain D, and F denotes a nonlinear term which is assumed to satisfy $F(0) = 0$. He actually considers in a more general setting but (22) is enough for our purpose. He assumes the following conditions:

1. The Fréchet derivative of $F : D \to H$ exists and is continuous. The Fréchet derivative is assumed to be a symmetric operator.
2. u_0 is assumed to belong to D.
3. There exists an $\alpha > 0$ such that
$$2(\alpha + 1)g(x) \leq (F(x), x) \qquad (\forall x \in D),$$
where g is defined as
$$g(v) = \int_0^1 (F(tv), v)dt.$$
4. The initial function u_0 satisfies
$$g(u_0) > \frac{1}{2}(u_0, Au_0).$$

Then he proves that the solution cannot exist globally in time. In particular, he proves the existence of $t_0 < \infty$ such that

$$\lim_{t \to t_0} \int_0^t \|u(s)\|^2 ds = +\infty,$$

where $\| \ \|$ denotes the norm of H.

In application to the present theorem, it is sufficient to note that $H = L^2(0,1)/R$, $A = -Pd^2/dx^2$, $F(w) = P(w^2)$, and $g(v) = (1/3)\int_0^1 v(x)^3 dx$, where P denotes the orthogonal projection from $L^2(0,1)$ onto $L^2(0,1)/R$.

One small question about the smoothness of u_0 remains here. Levine's theorem requires $u_0 \in D \subset H^2(0,1)$. However, it is not difficult to see that a solution exists for $u_0 \in L^2(0,1)/R$ and $u(t,\cdot) \in D$ for any $t > 0$. Keeping this fact in mind, we get to our conclusion. ∎

4.1. The inviscid case

Let us now consider the equation when the viscosity is zero, which was not considered in [3]. The boundary condition $u = 0$ on $x = 0, 1$ is discarded since it stems from the non-slip condition, which is not assumed in the inviscid fluid motion. Accordingly, let u be a real-valued function of $t > 0$ and $x \in [0,1]$ satisfying

$$u_t = u^2 - \int_0^1 u(t,x)^2 dx \tag{23}$$

and

$$\int_0^1 u(t,x) dx = 0. \tag{24}$$

We assume that the initial function $u(0,x) = u_0(x)$ satisfies

$$\int_0^1 u_0(x) dx = 0.$$

The existence of solutions which is local in time is guaranteed by the following theorem:

Theorem 4. *If u_0 belongs to $L^\infty(0,1)/R$, then there exists a $T > 0$ such that a solution u of* (23) *and* (24) *with $u(0,x) = u_0(x)$ exists and is unique in $L^\infty(0,T; L^\infty(0,1)/R)$.*

Since the right-hand side of (23) defines a smooth mapping from $L^\infty(0,1)$ into itself, the proof is elementary.

We now prove the following theorem:

Theorem 5. *If $u_0 \in L^\infty(0,1)/R$ satisfies*

$$\int_0^1 u_0(x)^3 dx > 0,$$

then the solution of (23) *and* (24) *blows up in finite time.*

Proof. It is easy to derive the following two equations:

$$\frac{d}{dt}\frac{1}{2}\int_0^1 u(t,x)^2dx = \int_0^1 u(t,x)^3dx, \tag{25}$$

$$\frac{d}{dt}\frac{1}{3}\int_0^1 u(t,x)^3dx = \int_0^1 u(t,x)^4dx - \left(\int_0^1 u(t,x)^2dx\right)^2. \tag{26}$$

The equation (25) is obtained by multiplying (23) by u and integrating it on $(0,1)$. The equation (26) is obtained similarly by (23) $\times\, u^2$.

We now derive the following inequality. Suppose that v is not identically zero and that $\int_0^1 v(x)dx = 0$. Then

$$\left(\int_0^1 v(x)^2dx\right)^2 \le \int_0^1 v(x)^4dx - \frac{\left(\int_0^1 v(x)^3dx\right)^2}{\int_0^1 v(x)^2dx}. \tag{27}$$

In order to derive this inequality, we take an arbitrary $\xi \in R$. Then,

$$\int_0^1 v^2dx = \int_0^1 (v^2-\xi v)dx \le \left(\int_0^1 (v^2-\xi v)^2dx\right)^{1/2}.$$

Namely, we obtain

$$\left(\int_0^1 v^2dx\right)^2 \le \int_0^1 (v^2-\xi v)^2dx = \int_0^1 v^4 - 2\xi\int_0^1 v^3 + \xi^2\int_0^1 v^2$$

for any $\xi \in R$. If we choose a ξ such that the right-hand side becomes the smallest, then we obtain the inequality (27).

By (26) and (27), we have

$$\frac{d}{dt}\frac{1}{3}\int_0^1 u(t,x)^3dx \ge \frac{\left(\int_0^1 u(t,x)^3dx\right)^2}{\int_0^1 u(t,x)^2dx}.$$

Let us define $\phi(t) = \int_0^1 u(t,x)^2dx$. Then the inequality is rewritten as

$$\frac{1}{6}\phi''(t) \ge \frac{\frac{1}{4}(\phi'(t))^2}{\phi(t)}, \qquad \text{or} \qquad \frac{\phi''}{\phi'} \ge \frac{3}{2}\frac{\phi'}{\phi}, \tag{28}$$

where the prime implies the differentiation. On the other hand, we have $\phi(0) > 0$ and $\phi'(0) > 0$. Thus we may integrate the inequality (28) so that

$$\frac{2}{3}\log\phi'(t) \ge \log\phi(t) + c_0,$$

where c_0 is the integration constant. This inequality holds true as far as both ϕ and ϕ' are positive. The inequality, however, leads us to

$$\phi'(t) \geq A_0 \phi(t)^{3/2},$$

where $A_0 = \exp(3c_0/2)$ is a positive constant. The conclusion of the theorem follows easily now. ■

Remark. In view of this theorem, one may well wonder if the opposite condition $\int_0^1 u_0(x)^3 dx < 0$ forces the solution to exist globally in time. However this is not true. There exist those solutions which satisfy $\int_0^1 u_0(x)^3 dx < 0$ but blow up in finite time. Examples are given at the end of the next section.

Remark. The proof above seems to be included in Levine's theorem, which we have used for the case of $\nu > 0$. It seems to the authors that Levine's theorem holds true even if the dissipation term, u_{xx} or $-Au$, is deleted. If so, Theorem 5 too is a special case of Levine's theorem. The local existence would be difficult if the term Au is missing. However, if local existence is guaranteed by other methods and if only the blow-up is concerned, Levine's argument is equally applicable and seems to prove the blow-up for (23) and (24). Since we are not completely sure about this and since the proof above, though it is essentially the same as the one in [20], is short enough, we put it in the present section. Another reason why we give a proof here is the question on the smoothness of u_0. In the case where Au is dropped, the equation is no longer a parabolic one and it is not so clear to what a function space the initial data supposed to belong.

5. Influence of the Projection in Inviscid Case

We now briefly compare $(23, 24)$, which is equivalently written as $u_t = P(u^2)$, with the following one:

$$u_t = u^2. \tag{29}$$

We will show that the existence of the projection makes a subtle change concerning the blow-up of the solutions. We first note that (29) is essentially a family of ordinary differential equations. The blow-up occurs for (29) if and only if there exists an x_0 such that $u_0(x_0) > 0$. On the other hand, the behavior of the solutions to $(23, 24)$ is somewhat more complex. We will show this by piecewise constant initial data.

Suppose that the initial function is of the following form:

$$u_0(x) = w_0 \chi_A + z_0 \chi_B,$$

where w_0 and z_0 are constants, χ_A is the characteristic function of a measurable set $A \subset [0,1]$, and B is its complement: $B = [0,1] \setminus A$. In order to ensure $\int_0^1 u_0 = 0$, we assume that $w_0\alpha + z_0(1-\alpha) = 0$, where α denotes the Lebesgue measure of A. Then it is easily seen that the solution u of (23) remains to be piecewise constant for $t > 0$ and that

$$u(t,x) = w(t)\chi_A + z(t)\chi_B,$$

where $w(t)$ and $z(t)$ satisfy

$$\begin{aligned} \dot{w}(t) &= (1-\alpha)\left(w^2 - z^2\right), \\ \dot{z}(t) &= \alpha\left(z^2 - w^2\right). \end{aligned}$$

Here and hereafter, the dot denotes the differentiation. Since $\alpha w(t) + (1-\alpha)z(t) \equiv 0$ is easily seen, the set of the ordinary differential equations above is reduced to the following single equation:

$$\dot{w}(t) = \frac{1-2\alpha}{1-\alpha} w^2.$$

Therefore we get to the following

Theorem 6. *If $(\alpha - 1/2)w_0 < 0$, then the solutions blow up in finite time. If $\alpha = 1/2$, then the solution represents a stationary solution. If $(\alpha-1/2)w_0 > 0$, then the solution exists for all the time and decays to zero.*

Corollary 1. *The evolution equation (23), (24) has a continuum of steady-states. Every neighborhood (in the L^∞-norm) of the steady-states obtained in the above way contains those functions such that u blows up in finite time if u starts from them. Also every neighborhood has those functions such that u decays to zero if u starts from them.*

It is easy to see that if the initial function is piecewise constant, then u remains to be so for $t > 0$. More specifically, if

$$u_0(x) = \sum_{k=1}^{N} w_k(0)\chi_{A_k},$$

where A_k $(k = 1, 2, \cdots, N)$ are mutually disjoint measurable sets in $[0,1]$ such that $\cup_{k=1}^N A_k = [0,1]$, then u is of the following form:

$$u(t,x) = \sum_{k=1}^{N} w_k(t)\chi_{A_k}.$$

$\{w_k(t)\}$ satisfies the following ordinary differential equations:

$$\dot{w}_n(t) = w_n^2 - \sum_{k=1}^{N} \alpha_k w_k^2 \qquad (n = 1, 2, \cdots, N), \tag{30}$$

where α_k denotes the Lebesgue measure of A_k, and hence satisfies $\sum_{k=1}^N \alpha_k = 1$. We easily see that

$$\sum_{k=1}^{N} \alpha_k w_k(t) \equiv 0 \qquad (0 \le t). \tag{31}$$

It is possible to classify the asymptotic behavior of all the solutions of (30). First of all, we note that $\{w_j(0)\}$ is a stationary solution of (30) if and only if $|w_j(0)|$ is independent of j. Next, we assume that $N \ge 3$, that all α_j are positive, and that $w_j(0)$ is different from each other. This is allowed since otherwise the system is reduced to the case where N is smaller. In particular, this assumption implies that $\{w_j(0)\}$ is not a stationary solution. The remaining part of the present section is devoted to proving the following two theorems.

Theorem 7. *Let J be an index such that $w_J(0) = \max_{1 \le j \le N}\{w_j(0)\}$ and assume that $\alpha_J < 1/2$. Then $w_J(t)$ blows up in finite time.*

Theorem 8. *Let J be chosen as above. Suppose that $\alpha_J > 1/2$. Then the solution exists globally and we have*

$$\lim_{t\to\infty} w_i(t) = 0$$

for all i. If $\alpha_J = 1/2$, then

$$w_J(t) \to U_\infty, \qquad w_i(t) \to -U_\infty \quad (i \ne J),$$

where U_∞ is a positive constant.

The proof is completed with the aid of some propositions. We begin with the following

Lemma 1. *$w_i(t) \ne w_j(t)$ for any $i \ne j$ and any t. If $w_j(t_0) \le 0$ for some t_0, then $w_j(t) < 0$, for all $t > t_0$.*

Proof. If $w_i(t_0) = w_j(t_0)$ for some t_0, then $w_i(t) = w_j(t)$ for all t. Thus, by the assumption we have made, we see that $w_i(t) \ne w_j(t)$ for any $i \ne j$ and any t. The second statement is proved similarly. ∎

Note that $w_J(t)$ remains to be the largest among $\{w_k(t)\}$ for all t. Note also a trivial relation $w_J(t) > 0$ for all t.

Lemma 2. *Suppose that J is taken as above. If there exists an index $I \neq J$ such that $w_I(t)$ is positive for all t, then the solution blows up in finite time.*

Proof. Note that $w_J(0) - w_I(0) > 0$. By equation (30) and the positivity of w_I, we have

$$\frac{d(w_J - w_I)}{dt} = w_J^2 - w_I^2 \geq (w_J - w_I)^2. \tag{32}$$

It is obvious that $w_J - w_I$ blows up in finite time because $w_J(0) - w_I(0) > 0$. Thus w_J too blows up in finite time. ∎

Proposition 1. *Suppose that $\sum_{j=1}^N \alpha_j w_j(0)^3 > 0$. Let J be the index as above. Then $w_J(t)$ blows up in finite time.*

This is a special case of Theorem 5.

Proposition 2. *Let J be as above and assume that $\alpha_J < 1/2$. If $\dot{w}_J(0) = w_J(0)^2 - \sum_{j=1}^N \alpha_j w_j(0)^2 > 0$, then $w_J(t)$ blows up in finite time.*

Proof. We first define

$$I_2(t) = \sum_{j=1}^N \alpha_j w_j^2(t), \qquad I_3(t) = \sum_{j=1}^N \alpha_j w_j^3(t).$$

Equation (30) gives us the following relation:

$$\frac{dI_2(t)}{dt} = 2I_3(t).$$

In view of Proposition 1, it suffices to prove that there exists a $t_0 > 0$ such that $I_3(t_0) > 0$. Suppose this is not true, then $I_3(t) \leq 0$ for all t. Then we obtain

$$\frac{dI_2(t)}{dt} = 2I_3(t) \leq 0, \qquad \text{and hence} \qquad I_2(t) \leq I_2(0).$$

Thus

$$\frac{dw_J}{dt} = w_J^2 - I_2(t) \geq w_J^2 - I_2(0).$$

Let $c = \sqrt{I_2(0)}$. Then we have $w_J(0) > c$ by the assumption. Also, $\dot{w}_J(t) > 0$ as long as $w_J(t) > c$. Consequently, $w_J(t) > c$ for all t. Thus it must hold that

$$\frac{1}{w_J^2 - c^2} \frac{dw_J}{dt} \geq 1.$$

Integrating this inequality, we obtain

$$1 \geq \frac{w_J(t) - c}{w_J(t) + c} \geq \frac{w_J(0) - c}{w_J(0) + c} \exp(2ct).$$

This is a contradiction for sufficiently large t, since c is positive. ∎

Corollary 2. *If $\alpha_J < 1/2$, then any globally existing solution, if it exists, satisfies $\dot{w}_J(t) \leq 0$ for all t.*

Proof of Theorem 7. Assume that there exists a global solution. Then by the last corollary, the solution satisfies $\dot{w}_J(t) \leq 0$. Also we may assume, without losing generality, that $w_j(t) < 0$ for all t and $j \neq J$. Now let us take I such that $w_I(0) < 0$ is the largest among $\{w_j(0)\}_{j \neq J}$. Then it holds that $-w_I(0) < w_J(0)$. In fact, if otherwise, then, for all $k \neq J$, it holds that $w_k(0) \leq w_I(0) \leq -w_J(0)$, whence

$$\alpha_J w_J(0) = \sum_{k \neq J} \alpha_k(-w_k(0)) \geq \sum_{k \neq J} \alpha_k w_J(0) = (1 - \alpha_J) w_J(0).$$

Since $w_J(0) > 0$ and $\alpha_J < 1/2$, this is a contradiction. Now it is obvious that

$$\dot{w}_I(0) < \dot{w}_J(0) \leq 0.$$

By the same reasoning, we can conclude that

$$\dot{w}_I(t) < \dot{w}_J(t) \leq 0$$

for all t. Therefore $w_j(t) \leq w_I(0) < 0$ for all $j \neq J$. Consequently,

$$\alpha_J w_J(t) = \sum_{k \neq J} \alpha_k(-w_k(t))$$

is bounded from below by a positive constant.

On the other hand, take any $i \neq J$. Then we have

$$\frac{d(w_I - w_i)}{dt} = w_I^2 - w_i^2 \leq -(w_I - w_i)^2.$$

Integrating both sides, we obtain

$$0 \leq w_I(t) - w_i(t) \leq \frac{w_I(0) - w_i(0)}{(w_I(0) - w_i(0))t + 1} \qquad (0 \leq t).$$

Thus we may write

$$w_i = w_I + b_i \qquad (i \neq J),$$

where $b_i(t) = O(t^{-1})$ as $t \to \infty$. By this observation, we have

$$\begin{aligned}\frac{d}{dt}w_J &= (1-\alpha_J)w_J^2 - \sum_{i\neq J}\alpha_i w_i^2 \\ &= (1-\alpha_J)w_J^2 - \sum_{i\neq J}\alpha_i (w_I + b_i)^2 \\ &= (1-\alpha_J)w_J^2 - (1-\alpha_J)w_I^2 - 2d_1(t)w_I + d_2(t),\end{aligned}$$

where $d_1(t) = \sum_{k\neq J}\alpha_k b_k(t) = O(t^{-1})$ and $d_2 = O(t^{-2})$. Since

$$\alpha_J w_J = -\sum_{k\neq J}\alpha_k(w_I + b_k) = -(1-\alpha_J)w_I - d_1,$$

with the same d_1 as above, we obtain

$$\frac{d}{dt}w_J = \frac{1-2\alpha_J}{1-\alpha}w_J^2 - r_2(t),$$

where $r_2(t) = d_2 + d_1^2/(1-\alpha_J) = O(t^{-2})$. Since $w_J(t)$ is bounded from below by a positive constant, there exists a $t_0 > 0$ such that $\dot{w}(t_0) > 0$. This is against our assumption. ■

Proof of Theorem 8. Note that

$$\alpha_J w_J = -\sum_{k\neq J}\alpha_k w_k \leq \left(\sum_{k\neq J}\alpha_k\right)^{1/2}\left(\sum_{k\neq J}\alpha_k w_k^2\right)^{1/2}.$$

This gives us

$$-\sum_{k\neq J}\alpha_k w_k^2 \leq -\frac{\alpha_J^2}{1-\alpha_J}w_J^2.$$

Consequently,

$$\begin{aligned}\frac{d}{dt}w_J(t) &= (1-\alpha_J)w_J^2 - \sum_{k\neq J}\alpha_k w_k^2 \\ &\leq (1-\alpha_J)w_J^2 - \frac{\alpha_J^2}{1-\alpha_J}w_J^2 = \frac{1-2\alpha_J}{1-\alpha_J}w_J^2.\end{aligned}$$

This inequality gives us the conclusion if $\alpha_J > 1/2$.

If $\alpha_J = 1/2$, w_J is still nonincreasing and $w_J > 0$. Consequently, there exists $U_\infty \in [0,\infty)$ such that

$$\lim_{t\to\infty} w_J(t) = U_\infty.$$

By the boundedness of $w_J(t)$ and Lemma 2, it must hold that $w_k(t) < 0$ for all sufficiently large t and all $k \neq J$. In order to prove that $U_\infty > 0$, we note that there exists a $j \neq J$ such that $0 < -w_j(0) < w_J(0)$. For this j we have

$$\dot{w}_j(t) \leq \dot{w}_J(t) \leq 0.$$

Consequently, $w_j(t) \leq w_j(0) < 0$. If $U_\infty = 0$, then $\lim_{t\to\infty} w_k(t) = 0$ for all k. Thus U_∞ must be a positive constant. ■

It is interesting to note that any decaying solution has in its any L^∞-neighborhood many solutions which blow up. In fact, the situation is best understood from Figure 1. The function in the left-hand side decays to zero, while the function in the right-hand side blows up by Theorem 7.

Remark: In our numerical experiments with various initial data, we find that, for almost all initial data, the solution blows up in such a way that the maximum tends to infinity while all the other w_j's become negative eventually. Thus it is natural to ask whether the situation in Lemma 2 actually occurs or not. Our experiments suggest that it does not occur but we cannot prove it.

FIG. 1. Blow-up solution in a neighborhood of decaying solution. In the left, the interval length α_J is greater than 1/2. In the right, the interval length α_J is less that 1/2.

5.1. Converse of Theorem 5 is not true

We consider piecewise constant initial data with three components. Using the same notation in Section 4, we have

$$\int_0^1 u_0(x)^3 dx = \alpha_1 w_1^3 + \alpha_2 w_2^3 + \alpha_3 w_3^3, \tag{33}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1,$$

and

$$\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 = 0.$$

The proof is completed by Theorem 7 if we have shown that, by choosing $w_3 < 0 < w_2 < w_1$ and α_j appropriately, we can accomplish both $\int_0^1 u_0(x)^3 dx < 0$ and $\alpha_1 < 1/2$. To this end, we put $\alpha_1 = 1/2 - \epsilon$, $\alpha_2 = 1/2 - 2\epsilon$, and $\alpha_3 = 3\epsilon$ with $\epsilon > 0$. Given $0 < w_2 < w_1$, we define w_3 by $w_3 = -(\alpha_1 w_1 + \alpha_2 w_2)/\alpha_3$ and substitute it into (33). We obtain

$$\begin{aligned}\int_0^1 u_0(x)^3 dx = \left(\frac{1}{2} - \epsilon\right) w_1^3 + \left(\frac{1}{2} - 2\epsilon\right) w_2^3 \\ -\frac{1}{9\epsilon^2}\left[\frac{(w_1+w_2)^3}{8} - \frac{3\epsilon}{4}(w_1+w_2)^2(w_1+2w_2)\right. \\ \left. +\frac{3\epsilon^2}{2}(w_1+w_2)(w_1+2w_2)^2 - \epsilon^3(w_1+2w_2)^3\right].\end{aligned}$$

The right-hand side is negative if ϵ is sufficiently small. ■

6. Comparison of (5) with (1)

The purpose of the present section is to explain how different (5) is from (1). As for (1), there are some papers which study blow-up. Childress *et al.* [5] reports that there is a solution which blows up in finite time. The first author of the present paper and M. Shōji have made several numerical experiments on (1) and (3) with different numerical methods and different initial data [24, 25]. We could not find any blow-up solution, though the solution $\omega = -f_{xx}$ can be very large especially near the boundary. Due to this boundary layer, the numerical computation by finite difference method experiences a serious difficulty if the number of mesh points is insufficient. This may well be regarded erroneously as a blow-up, but the solution continues to exist if more mesh points are used and if the time-step is adaptively chosen. The second author of the present paper also gave careful numerical experiments in [35] and confirmed the conclusion in [24, 25]. So we believe that the solution does not blow up for (1) under the boundary condition (3).

On the other hand, Grundy and McLaughlin [16] show numerically that there is a blow-up if the boundary condition is the following inhomogeneous one:

(34) $\quad f(t,\alpha) = \rho_1, \quad f_{xx}(t,\alpha) = -\sigma_1, \quad f(t,\beta) = \rho_2, \quad f_{xx}(t,\beta) = -\sigma_2,$

where ρ_j and σ_j are constants. Further, the paper shows that the blow-up rate is $O((t_0 - t)^{-2})$, if the solution blows up at $t = t_0$. But the occurrence of the blow-up for (1) under (34) does not seem to be proved rigorously by anybody. Also, Cox [9] gives numerical evidences for the blow-up of (1) and (34).

Another important discovery of [16] is that the blow-up occurs everywhere in the interval. Namely,

$$\lim_{t \to t_0} |u(t,x)| = \infty$$

for all $x \in (\alpha, \beta)$. Our numerical experiments also confirm their results, which are not reported here.

So, as numerical experiments show, blow-ups are present if we replace the boundary condition (3) with some inhomogeneous one. We, however, strongly believe that any solution to (1) and (3) exists globally in time and that any solution tends to zero as $t \to \infty$. In particular, the global attractor would consist of $u \equiv 0$ only. Except for numerical evidences, we do not have a proof for this statement. The uniqueness of the steady-state which is proved in Section 9 seems to support this conjecture, too. Anyway, there seems to be no mathematical proof of blow-up or global existence of the solutions to (1) with any boundary condition, except for the global existence in the case where the initial value is small (see [5]). The reason that we cannot use Levine's theorem for (1) which was used in Section 2 for (5) is that Levine's theorem requires the symmetry of the Fréchet derivative of the nonlinear term F. The nonlinear term of (1) does not seem to meet the requirement.

We may summarize as follows. The equation (1) does not admit blow-up for homogeneous boundary data but blow-ups may happen for inhomogeneous boundary data. We also remark that if we generalize (1) to coupled equations for 3D flows, then blow-up can occur for homogeneous boundary data [17].

The equation (5) is obtained by dropping the convection term from the equation (1). At first glance, the difference appears to be insignificant, since the function of the convection term is to move things sideward: the nonlinear convection term may yield a singularity of shock-wave type as in the Burgers equation but it seems to have nothing to do with the mechanism by which the solution itself becomes large. However, we have seen that solutions of equation (5) with homogeneous boundary condition can blow up. This fact is an indirect evidence that the convection term may play an important role once the solution grows large enough. We may say that the convection term can suppress the blow-up. Other numerical evidences for the existence of solutions to (1) and related equations are reported in a paper by the second author [35].

Remark. The present paper is by no means the first to recognize the importance of nonlocal nonlinear convection term. In fact, P. Constantin shows in [6] that 3D Euler equations for incompressible fluid, after a certain modification, have a solution which blows up in finite time. His result indicates the importance of the convection term.

It is also important to note how different the blow-ups are from those appearing in the equation without the projection. Let us consider the following equation:

$$u_t - u_{xx} - u^2 = 0 \qquad (0 < t,\ 0 < x < 1). \tag{35}$$

Both (5) and (35) are considered with the homogeneous Dirichlet boundary condition. The blow-up behavior of (35) is well-known; see Giga and Kohn [14, 15]. We compute a typical solution numerically and the graph is given in Figure 2.

On the other hand, a graph of blow-up in (5) is shown in Figure 3. Solutions in Figures 2 and 3 are computed by a finite difference scheme using the idea of Nakagawa [22] and Tabata [31]. There are visible differences between Figure 2 and Figure 3. A big difference is that u becomes positive in less and less part in the x-interval, while the central part of Figure 2 does not shrink. In equation (35), the blow-up generically occurs at a single point. On the other hand, the solution of (5) satisfies the following two asymptotics:

$$\lim_{t\to t_0} u(t,x) = \begin{cases} +\infty & x = a, \\ -\infty & x \neq a, \pm 1, \end{cases} \tag{36}$$

where t_0 is a blow-up time, a is the point at which u blows up to $+\infty$. In addition, it holds that

$$\lim_{t\to t_0} \frac{u(t,x)}{u(t,a)} = 0.$$

FIG. 2. The graph of the solution u to (35) (left), the maximum of u and t (right). The initial value is $aU(x)$, where U is the steady-state and a is equal to 1.01.

FIG. 3. The graph of the solution u to (17)–(19). The initial value is $300 \sin (2\pi x)$.

FIG. 4. The graph of the solution u. Initial data are $u_0(x) = -50x^3 + 30x$.

Thus, u blows up everywhere but the $+\infty$ blow-up occurs at a single point. In this sense, the blow-up is localized. This remarkable fact is proved in [3].

The asymptotic form of the function $u(t, x)$ near the blow-up time is known for (35); see [13, 14, 15]. The asymptotic form of the function $u(t, x)$ near the blow-up for (5) is given in [3].

FIG. 5. The graph of the solution u. The same initial data as in Figure 4.

7. Existence for Generalized Equations

We consider (2). Only the case where $\nu > 0$ is considered in this section and the case for $\nu = 0$ is considered in the next section.

Theorem 9. *If $a = -1/(2n)$ with a positive integer n, $a = 0$, or $a = -2$, then all the solutions exist globally in time and tend to zero as $t \to \infty$.*

Proof. Multiplying (2) by f, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} f_x(t,x)^2 dx - \left(1 + \frac{a}{2}\right)\int_{-1}^{1} f_x(t,x)^3 dx = -\nu \int_{-1}^{1} f_{xx}(t,x)^2 dx.$$

From this we can prove the global existence for all initial data if $a = -2$. This conclusion is valid for both the homogeneous Dirichlet boundary condition and the periodic boundary condition.

When we assume the periodic boundary condition, we multiply (2) by f_{xx} to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} f_{xx}^2 dx - \left(a + \frac{1}{2}\right)\int_{-1}^{1} f_x f_{xx}^2 dx = -\nu \int_{-1}^{1} f_{xxx}^2 dx.$$

From this we can prove the global existence for all initial data if $a = -1/2$. Therefore solutions of (7) exist globally in time if $m = 5$.

Similarly we can prove the global existence if $a = -1/(2n)$ for $n = 1, 2, \cdots$ and if the periodic boundary condition is assumed. In fact, we multiply (2) by f_{xx}^{2n-1} and integrate by parts to obtain

$$\frac{1}{2n}\frac{d}{dt}\int_{-1}^{1} f_{xx}^{2n} dx - \left(a + \frac{1}{2n}\right)\int_{-1}^{1} f_x f_{xx}^{2n} dx = -\nu \int_{-1}^{1} f_{xx}^{2n-2} f_{xxx}^2 dx.$$

Global existence is also guaranteed for the case of $a = 0$. This is because the equation becomes

$$\omega_t + G(\omega)\omega_x = \nu\omega_{xx}$$

and the maximum principle holds true in this case. ■

We have unsuccessfully tried to prove the existence or blow-up for other values of a. Figures 4 and 5 seem to indicate that any solutions of (2) do not blow up if $0 \le a \le 1$ and that some of them can blow up in finite time if $1 < a < \infty$.

For $a < 0$, it is not easy to see where the boundary between the global existence and blow-up is. See Figure 6.

FIG. 6. The graph of the solution u. $\nu = 0.001$. The same initial data as in Figure 4.

Remark. In their analysis of nonstationary boundary-later equation, E and Engquist [11] were lead to the following initial-boundary value problem:

$$v_t = v_{xx} + v^2 - \left(\int_0^x v(t,y)dy\right) v_x \qquad (0 < x < \infty), \tag{37}$$

$$v(t,0) = 0, \qquad v(t,\infty) = 0.$$

They proved that some solutions blow up in finite time.

Using their argument, we can prove the following theorem:

Theorem 10. *Let w satisfy*

$$w_t = w_{xx} + w^2 - \left(\int_0^x w(t,y)dy\right) w_x \qquad (0 \le x < 2\pi)$$

and the periodic boundary condition. Then w blows up in finite time if

$$\frac{1}{2}\int_0^{2\pi} w_x(0,x)^2 dx - \frac{1}{4}\int_0^{2\pi} w(0,x)^3 dx < 0.$$

An equivalent form of (1) with $\nu = 1$ is given by:

$$\begin{aligned} u_t &= u_{xx} + u^2 - \left(\int_0^x u(t,y)dy\right) u_x + \gamma(t) \qquad (0 < x < 2\pi), \\ 0 &= \int_0^{2\pi} u(t,x). \end{aligned}$$

Therefore we may say that the presence of the constant term, $\gamma(t)$, can arrest the blow-up of solutions.

The results of the present section are certainly fragmental and the authors welcome the readers who strengthen them.

8. The Case Where $\nu = 0$

Throughout this section, we consider

$$f_{txx} + f f_{xxx} - a f_x f_{xx} = 0 \quad (0 \le x < 2\pi) \tag{38}$$

under the periodic boundary condition. Integrating once, we obtain

$$f_{tx} + f f_{xx} - \frac{a+1}{2} f_x^2 = \gamma(t),$$

where

$$\gamma(t) = -\frac{a+3}{4\pi}\int_0^{2\pi} f_x^2.$$

Then we easily obtain the following two equalities:

$$\frac{d}{dt}\int_0^{2\pi} f_x^2 = (2+a)\int_0^{2\pi} f_x^3, \tag{39}$$

$$\frac{d}{dt}\int_0^{2\pi} f_x^3 = \frac{5+3a}{2}\int_0^{2\pi} f_x^4 - \frac{3(3+a)}{4\pi}\left(\int_0^{2\pi} f_x^2\right)^2. \tag{40}$$

We now prove the following lemma.

Lemma 3. *Suppose that a function w defined in $[0, 2\pi]$ has zero mean :*

$$\int_0^{2\pi} w(x)dx = 0.$$

Then, unless w is identically zero, it holds that

$$\left(\int_0^{2\pi} w^2\right)^2 \le 4\pi\left[\int_0^{2\pi} w^4 - \frac{\left(\int_0^{2\pi}|w|^3\right)^2}{2\int_0^{2\pi} w^2}\right]. \tag{41}$$

Proof. Let $A = A(x)$ denote the characteristic function of the set of point x such that $w(x) > 0$ and let B denote $1 - A$. Then for any $\xi \in R$, we have

$$\int_0^{2\pi} w^2 A = \int_0^{2\pi}(w^2 A + \xi w) \le \sqrt{2\pi}\left(\int_0^{2\pi}(w^4 A + 2\xi w^3 A + \xi^2 w^2)\right)^{1/2}. \tag{42}$$

Similarly, we get

$$\int_0^{2\pi} w^2 B = \int_0^{2\pi}(w^2 B + \xi w) \le \sqrt{2\pi}\left(\int_0^{2\pi} w^4 B - 2\xi w^3 B + \xi^2 w^2)\right)^{1/2}. \tag{43}$$

By these inequalities, we obtain

$$\begin{aligned}\int_0^{2\pi} w^2 &= \int w^2 A + \int w^2 B\\ &\le \sqrt{2\pi}\left(2\int w^4 A + 2\int w^4 B + 4\xi^2\int_0^{2\pi} w^2 + 4\xi\int w^3 A - 4\xi\int w^3 B\right)^{1/2}\\ &\le \sqrt{4\pi}\left(\int_0^{2\pi} w^4 + 2\xi^2\int_0^{2\pi} w^2 + 2\xi\int_0^{2\pi}|w|^3\right)^{1/2}.\end{aligned}$$

If we choose ξ which minimizes the right-hand side, then we obtain (41). ■

Applying this lemma to $w = f_x$, we obtain by (39) and (40)

$$\begin{aligned}\ddot{\phi}(t) &= \frac{(2+a)(5+3a)}{2}\int_0^{2\pi} f_x^4 - \frac{3(3+a)(2+a)}{4\pi}\phi(t)^2\\ &\ge \frac{(2+a)(5+3a)}{2}\left[\frac{1}{4\pi}\phi(t)^2 + \frac{\left(\int_0^{2\pi}|f_x|^3\right)^2}{2\phi(t)}\right]\\ &\quad - \frac{3(3+a)(2+a)}{4\pi}\phi(t)^2\\ &= -\frac{(2+a)(13+3a)}{8\pi}\phi(t)^2 + \frac{(2+a)(5+3a)}{4}\frac{\left(\int_0^{2\pi}|f_x|^3\right)^2}{\phi},\end{aligned}$$

where we have put $\phi(t) = \int_0^{2\pi} f_x(t,x)^2 dx$ and we have assumed that $(2+a)(5+3a) > 0$. On the other hand, Hölder's inequality gives us

$$\frac{1}{2\pi}\left(\int_0^{2\pi} f_x^2\right)^3 \le \left(\int_0^{2\pi} |f_x|^3\right)^2.$$

Combining these inequalities, we obtain

$$\begin{aligned}\ddot{\phi}(t) &\ge -\frac{(2+a)(13+3a)}{8\pi}\phi(t)^2 + \frac{(2+a)(5+3a)}{8\pi}\phi(t)^2 \\ &= -\frac{a+2}{\pi}\phi(t)^2.\end{aligned}$$

Therefore we have reached the following proposition: if $a < -2$, then $\phi(t) = \int_0^{2\pi} f_x^2$ satisfies

$$\ddot{\phi}(t) \ge -\frac{a+2}{\pi}\phi(t)^2.$$

We now prove the following theorem.

Theorem 11. *Assume that $a < -2$ and that*

$$\int_0^{2\pi} f_x(0,x)^3 dx > 0.$$

Then the solution of (38) *blows up in finite time.*

Proof. The function $\phi(t)$ satisfies

$$b\phi^2 \le \ddot{\phi} \tag{44}$$

with a constant $b > 0$. By the assumption, $\dot{\phi}(0) > 0$. Consequently, $\dot{\phi}(t) > 0$ for $0 \le t < t_0$ for some $t_0 > 0$. As far as $\dot{\phi}(t) > 0$, it holds that

$$b\phi^2\dot{\phi} \le \ddot{\phi}\dot{\phi}.$$

Integrating it, we obtain

$$c_0 \le \frac{1}{2}\left(\dot{\phi}\right)^2 - \frac{b}{3}\phi^3,$$

where

$$c_0 = \frac{1}{2}\left(\dot{\phi}(0)\right)^2 - \frac{b}{3}\phi(0)^3$$

is a constant. If $c_0 \ge 0$, then

$$\sqrt{\frac{2b}{3}}\,\phi(t)^{3/2} \le \dot{\phi}(t).$$

It would be easy to show by this inequality that ϕ blows up in finite time. This inequality is valid as far as $\dot{\phi}(t) \geq 0$. Since it guarantees monotonicity of ϕ, ϕ never turns negative once $\dot{\phi} \geq 0$. Thus, we have proved that ϕ blows up in finite time if $c_0 \geq 0$.

If there exists a t_1 such that $\frac{1}{2}\dot{\phi}(t_1)^2 - \frac{b}{3}\phi(t_1)^3 > 0$, then we can prove the blow-up in the same way. If such a t_1 does not exist, then we may assume that

$$\left(\dot{\phi}(t)\right)^2 \leq \frac{2b}{3}\phi(t)^3.$$

By this inequality and (44), we have

$$\left(\dot{\phi}(t)\right)^{4/3} \leq \left(\frac{2b}{3}\right)^{2/3} \phi(t)^2 \leq \left(\frac{2b}{3}\right)^{2/3} \frac{1}{b}\ddot{\phi}(t).$$

If we put $\psi = \dot{\phi}$, then we obtain

$$b\left(\frac{3}{2b}\right)^{2/3} \psi^{4/3} \leq \dot{\psi}.$$

In a way similar to the one above, we can prove the blow-up. ■

8.1. 3D axisymmetric Euler

We consider (38) in the case where $a = 0$. Note that this equation gives us exact solutions for axisymmetric Euler flow. The equation is

$$f_{txx} + f f_{xxx} = 0 \quad (0 \leq x < 2\pi) \tag{45}$$

and the periodic boundary condition is imposed. As we have seen in Section 3, there exists a unique local solution for $f_{xx}(0, \cdot) \in L^2/R$. It is not difficult to see that f is smooth for short time if $f_{xx}(0, \cdot)$ is smooth.

Figure 7 shows the graph of a numerical solution. This graph shows a very sharp internal layer which seems to indicate a singularity of shock-wave type. However, we can prove that the smooth solution does not lose its smoothness for any t.

We rewrite (45) as follows:

$$v_t + G(v)v_x = 0.$$

We prove that, if $v(0, \cdot)$ belongs to $L^2(S^1)/R \cap C^0(S^1)$, then v belongs to the same function space for all $t > 0$. This can be seen as follows. Note first that

FIG. 7. The graph of the solution of (45). Initial data are $f_{xx}(0,x) = 20\sin(\pi x)$.

$v \in C^0([0,T];L^2)$ implies f_x is bounded in $[0,T] \times S^1$, where $f = G(v)$. This means that the characteristic curves

$$\dot{x} = f(t, x(t)), \quad x(0) = \xi$$

are well-defined. Let $\phi(t,\xi)$ denote the solution of this ordinary differential equation. Then

$$v(t, \phi(t,\xi)) = a(\xi),$$

where a denotes $v(0,\cdot)$. This shows that the solution satisfies the following a priori estimate

$$\|v(t,\cdot)\|_{L^\infty} \leq \|v(0,\cdot)\|_{L^\infty}. \tag{46}$$

Since the L^∞-norm of f_x is bounded by a constant which depends only on the L^∞-norm of v, the characteristic curves exist in a time interval whose length depends only on $\|v(0,\cdot)\|_{L^\infty}$. This, together with a repeated use of (46), guarantees the continuity of v for all t.

This fact shows that the shock-like phenomenon in Figure 7 actually does not contain singularity at all. The figure shows only that the L^∞-norm of f_{xxx} grows very rapidly.

9. Remarks on Stationary Viscous Flow

We proved theorems showing that a solution blows up if its initial value is sufficiently large. On the other hand, it can be proved easily that the solution exists globally in time if the L^2-norm of initial function is sufficiently small (see Childress *et al.* [5] for (1)). The precise boundary between blow-up and global existence seems to be difficult to locate, as is often the case in many blow-up problems.

Concerning this question, it may be helpful to study the stationary problem, since, in many evolution equations, steady-states stand on the border of blow-up and global existence. Throughout this section, ν is assumed to be positive.

9.1. The case where $a = \infty$

The case of the periodic boundary condition, the easiest case, is studied first. The problem is to find a 2π-periodic function $U = U(x)$ such that

$$U_{xx} + U^2 - \frac{1}{2\pi}\int_{-\pi}^{\pi} U(x)^2 dx = 0, \tag{47}$$

$$\int_{-\pi}^{\pi} U(x) dx = 0. \tag{48}$$

The solution can be expressed by elliptic functions in an explicit but tedious form. Instead, we compute the solution as follows. Following the method in [23], we consider the following generalized problem:

$$U_{xx} + AU + U^2 - \frac{1}{2\pi}\int_{-\pi}^{\pi} U(x)^2 dx = 0, \tag{49}$$

$$\int_{-\pi}^{\pi} U(x) dx = 0, \tag{50}$$

where A is a parameter, which is artificial in our context. This problem is the same as the one in [23], where $-U$ instead of U is taken as the unknown. Reference [23] shows that any solution of (49) and (50) with real A including $A = 0$ gives an exact solution of the Navier-Stokes equations called Oseen's spiral flow. It is interesting to see that the same equations lead to stationary solutions in a different situation.

We now outline how we can solve (49) and (50). We first note that if $U(x)$ is a solution to (49) and (50) then $U(x - a)$ with a constant a is also a solution. In order to eliminate this indeterminacy, we henceforth consider only those solutions which are even functions of x. By linearizing (49) at $U = 0$, we obtain

$$U_{xx} + AU = 0, \tag{51}$$

FIG. 8. Numerical solutions of (47) and (48). The solution in the right-hand side is obtained from the one in the left-hand side by shifting the former by π. Namely, $U(x)$ and $U(x-\pi)$.

$$\int_{-\pi}^{\pi} U(x)dx = 0. \tag{52}$$

This is an eigenvalue problem with A being the eigenvalue. It has a nontrivial solution if and only if

$$A = n^2, \qquad U = \text{constant} \times \cos nx,$$

where n is a positive integer. Therefore, nontrivial solutions may bifurcate from $A = n^2$. It is shown numerically in [23] that the bifurcations are subcritical pitchforks and the branches of the nontrivial solutions exist for all $A \in (-\infty, n^2)$. In particular, there exists a solution with $A = 0$. This is true for all $n = 1, 2, \cdots$ and we obtain infinitely many solutions of (47) and (48). Two solutions obtained by the continuation from $(A, U) = (1, 0)$ are shown in Figure 8.

It is easy to see that, if $U(x)$ is a solution to (47) and (48), then $\tilde{U}(x) = n^2 U(nx)$ with n a positive integer is a solution to the same boundary value problem, too. Accordingly, the stationary solutions which are obtained by the continuation from $(A, U) = (n^2, 0)$ are nothing but $n^2 U(nx)$ with U the solution in Figure 8.

The stationary solutions are not stable. In fact, our numerical experiments show that u blows up if $u(0, x) = aU(x)$ with $a > 1$ and that u decays to zero if $u(0, x) = aU(x)$ with $0 < a < 1$.

It should be noted that the Dirichlet boundary value problem:

$$u_{xx} + u^2 - c_0 = 0, \tag{53}$$

$$\int_{-\pi}^{\pi} u(x)dx = 0, \tag{54}$$

$$u(\pm\pi) = 0, \tag{55}$$

has nontrivial solutions. In fact, if we take a constant a appropriately, then $U(x-a)$, where U is the solution in Figure 8, satisfies all the requirements.

The situation is quite different if we consider the equation without the projection. In fact, let us consider the following boundary value problem:

$$u_{xx} + u^2 = 0 \tag{56}$$

with the periodic boundary condition. It is easy to prove the following proposition.

Proposition 3. *If* (56) *is fulfilled in* $a < x < b$ *and* u *satisfies* $u(a) = u(b)$, $u_x(a) = u_x(b)$, *then* u *is identically zero.*

Proof. From (56), we obtain

$$\frac{1}{2}u_x^2 + \frac{1}{3}u^3 = c \tag{57}$$

with a constant c. Suppose that u is not identically zero. Then it assumes its maximum and minimum at, say, x_1 and x_2, respectively. Since equation (57) implies that $u(x_1) = u(x_2)$, u must be a constant function. Then (56) forces u to be identically zero. ∎

A quite similar reasoning leads us to the same conclusion under the homogeneous Neumann boundary condition. On the other hand, it is well-known that there exists a *unique* solution which satisfies

$$u_{xx} + u^2 = 0, \tag{58}$$

$$u(a) = u(b) = 0. \tag{59}$$

Figure 9 shows the stationary solution of (58) and (59) in $[0, 1]$.

9.2. General Case

We consider

$$ff_{xxx} - af_xf_{xx} = \nu f_{xxxx} \qquad (0 \le x < 2\pi) \tag{60}$$

FIG. 9. Numerical solution of (58) and (59).

under the periodic boundary condition. Here f is a function of x only and ν is assumed to be positive. For this problem, we have the following propositions which guarantee the nonexistence of nontrivial solutions.

Proposition 4. *If* $a = 0, 1,$ *or* $-1/(2n)$ *with a positive integer* n, *then* $f \equiv$ constant *is the only solution to* (60).

Proof. If $a = 0$, then (60) can be rewritten as

$$\left(\nu f_{xxx} e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi}\right)_x = 0.$$

Consequently, we have

$$\nu f_{xxx} = c_0 e^{\frac{1}{\nu}\int_0^x f(\xi)d\xi},$$

where c_0 is a constant. By integrating on $[0, 2\pi)$, we obtain $c_0 = 0$. Therefore $f_{xxx} = 0$.

Assume that $a = 1$ and differentiate (60). We then obtain

$$\nu f^{(V)} - f f_{xxxx} = -f_{xx}^2.$$

This equation can be rewritten as

$$\left(\nu f_{xxxx} e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi}\right)_x = -f_{xx}^2 e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi}.$$

By integration, we obtain

$$0 = -\int_0^{2\pi} f_{xx}^2 e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi} dx.$$

Consequently, $f_{xx} \equiv 0$ if $a = 1$.

If we multiply (60) by f_{xx}^{2n-1} with a positive integer n and integrate it on $[0, 2\pi)$, then we have

$$-(2n-1)\nu \int_0^{2\pi} f_{xxx}^2 f_{xx}^{2n-2} = -\left(a + \frac{1}{2n}\right) \int_0^{2\pi} f_x f_{xx}^{2n}.$$

Therefore, $f_{xx} \equiv 0$ if $a = -1/(2n)$.

If we multiply (60) by f_{xxxx} and integrate it on $[0, 2\pi)$, then we have

$$\nu \int_0^{2\pi} f_{xxxx}^2 = \left(a - \frac{1}{2}\right) \int_0^{2\pi} f_x f_{xxx}^3.$$

Therefore, $f_{xxxx} \equiv 0$ if $a = 1/2$. ∎

Proposition 5. *If* $-3 \le a \le -1$, *then* $f \equiv$ constant *is the only solution to* (60).

Proof. Integrating (60), we obtain

$$f f_{xx} - \frac{1+a}{2} f_x^2 = \nu f_{xxx} + \gamma,$$

where

$$\gamma = -\frac{3+a}{4\pi} \int_0^{2\pi} f_x^2.$$

We therefore have

$$-\frac{1+a}{2} f_x^2 e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi} - \gamma e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi} = \left(\nu f_{xx} e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi}\right)_x.$$

By integrating this equation, we obtain

$$0 = -\frac{1+a}{2} \int_0^{2\pi} f_x^2 e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi} dx + \frac{3+a}{4\pi} \int_0^{2\pi} f_x^2 dx \int_0^{2\pi} e^{-\frac{1}{\nu}\int_0^x f(\xi)d\xi} dx,$$

from which the conclusion follows. ∎

We now define $g = \frac{a}{2\nu} f$. Then the equation (60) is written as

$$\frac{1}{a} g g_{xxx} - g_x g_{xx} = \frac{1}{2} g_{xxxx} \qquad (0 \le x < 2\pi). \tag{61}$$

Putting $b = 1/a$, we get to

$$b g g_{xxx} - g_x g_{xx} = \frac{1}{2} g_{xxxx} \qquad (0 \le x < 2\pi). \tag{62}$$

FIG. 10. Plot of solution curves, where (b, S) are plotted for each solution. Here $b = 1/a$ and S is the sum of all the Fourier coefficients.

If $b = 0$, then the equation is reduced to

$$U_{xx} = U^2 - \frac{1}{2\pi} \int_0^{2\pi} U(x)^2 dx,$$

where $U(x) = -g_x(x)$. This equation possesses nontrivial solutions as we have seen in the previous subsection.

For nonzero b, we can compute the solutions of (62) by the path-continuation method as was done in [4]. Figure 10 shows our result of numerical computation, which is carried out by the Fourier-spectral method. It suggests that nontrivial solutions exist for $-1/3 < b < 1$, i.e., for $-\infty < a < -3$ and $1 < a < +\infty$. Therefore the nonexistence results for $a = -3$ and $a = 1$ would be a sharp one. We, however, do not know what happens for $-1 < a < 1$, except for the discrete values discussed in Proposition 4. It may be that the trivial solution is the only solution for all $a \in [-3, 1]$.

The asymptotic behavior as $b \to 1$ and $b \to -1/3$ is another interesting problem. It is argued in Budd *et al.* [4] that, as $b \to 1$, the steady-state tends to constant $\times(1-b)^{-1} \sin x$ in the case where the boundary condition is $f = f_{xx} = 0$ at the boundary. They determine the constant by an asymptotic expansion. Their boundary condition is essentially the same as ours and, in our case, we have

$$g(x) = \frac{1}{2(1-b)} \sin x + O(1)$$

as $b \to 1$. Here $O(1)$ is uniformly bounded as $b \to 1$.

FIG. 11. Plot of two solutions. $b = 0.99$ in the right and $b = -0.33$ in the left.

Budd *et al.* [4] does not consider the case where $b < 0$. In our computation we obtain

$$g(x) \sim \frac{\text{constant}}{1-b} x \qquad (-\pi < x < \pi)$$

as $b \to -1/3$. See Figure 11.

Remark. If we consider (60) with $a = 1$ under nonhomogeneous boundary conditions, then multiple solutions can exist. See [2, 9, 16, 29, 30, 34] and the references therein.

10. Von Kármán's Swirling Flow

The three-dimensional axisymmetric flow (6) is generalized so as to include nonzero azimuthal component. In fact, the ansatz

$$(u_r, u_\theta, u_z, p) = \left(-\frac{r}{2} f_z(t,z), rg(t,z), f(t,z), \rho\nu f_z - \frac{\rho}{2} f^2 + \rho \int_0^z f_t \right),$$

which was invented by von Kármán, leads us to the following coupled equations:

$$f_{txx} + f f_{xxx} + 4 g g_x = \nu f_{xxxx}, \tag{63}$$

$$g_t + f g_x - f_x g = \nu g_{xx}. \tag{64}$$

Let us consider those equations in the interval $-\alpha < z < \alpha$ with the boundary condition

$$f(t, \pm\alpha) = f_z(t, \pm\alpha) = 0, \quad g(t, -\alpha) = \gamma_1, \quad g(t, \alpha) = \gamma_2. \tag{65}$$

Physical meaning of this boundary condition is evident: the fluid is contained between two planes $z = -\alpha$ and $z = \alpha$ and the two bounding planes are rotated about z-axis with constant angular velocities γ_1 and γ_2. Let the initial values be $f_{xx}(0, x)$ and $g(0, x)$. As is already noted, the global existence of the solution is guaranteed by the maximum principle if $\gamma_1 = \gamma_2 = 0$ and if $g(0, x) \equiv 0$. We do not know of existence or blow-up if general initial data are given.

It is worthwhile to note that there are many stationary solutions to (63), (64), and (65). See Zandbergen and Dijkstra [33].

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