PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

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ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.

A *symplectic* manifold is a manifold of dimension $2k$ with a closed 2-form α such that α^k is nonsingular. If M^{2k} is a closed symplectic manifold, then the cohomology class of α is nontrivial, and all its powers through k are nontrivial. M also has an almost complex structure associated with α, up to homotopy.

It has been asked whether every closed symplectic manifold has also a Kaehler structure (the converse is immediate). A Kaehler manifold has the property that its odd dimensional Betti numbers are even. H. Guggenheimer claimed [1], [2] that a symplectic manifold also has even odd Betti numbers. In the review [3] of [1], Liberman noted that the proof was incomplete. We produce elementary examples of symplectic manifolds which are not Kaehler by constructing counterexamples to Guggenheimer's assertion.

There is a representation ρ of $Z \oplus Z$ in the group of diffeomorphisms of T^2 defined by

$$(1,0) \xrightarrow{\rho} \mathrm{id}, \qquad (0,1) \xrightarrow{\rho} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

where "$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$" denotes the transformation of T^2 covered by the linear transformation of $\mathbf{R}^2$. This representation determines a bundle M^4 over T^2, with fiber T^2: $M^4 = \tilde{T}^2 \times_{Z \oplus Z} T^2$, where $Z \oplus Z$ acts on $\tilde{T}^2$ by covering transformations, and on T^2 by ρ (M^4 can also be seen as $\mathbf{R}^4$ modulo a group of affine transformations). Let Ω_1 be the standard volume form for T^2. Since ρ preserves Ω_1, this defines a closed 2-form Ω_1' on M^4 which is nonsingular on each fiber. Let ρ be projection to the base: then it can be checked that $\Omega_1' + p^* \Omega_1$ is a symplectic form. (It is, in general, true that $\Omega_1'' + K\rho^* \Omega_1$ is a symplectic form, for *any* closed Ω_1'' which is a volume form for each fiber, and K sufficiently large.) But $H_1(M^4) = Z \oplus Z \oplus Z$, so M^4 is not a Kaehler manifold.

Many more examples can be constructed. In the same vein, if M^{2k} is a closed symplectic manifold, and if N^{2k+2} fibers over M^{2k} with the fundamental class of the fiber not homologous to zero in N, then N is also a symplectic manifold. If, for instance, the Euler characteristic of the fiber is not zero, this

Received by the editors July 31, 1974.
AMS (MOS) subject classifications (1970). Primary 57D15, 58H05.

hypothesis is satisfied. To do this, one must see that if there is a closed 2-form α_1 whose integral on a fiber is nonzero, then α_1 is cohomologous to a 2-form α which is nonsingular on each fiber. To find α, first find a 2-form β, not necessarily closed, which is nonsingular on each fiber, and whose integral on each fiber agrees with that of α_1: this exists by convexity considerations. On each fiber, F, there is a form γ_F such that $\beta_F - (\alpha_1)_f = d(\gamma_F)$. This equation can also be solved differentiably in a small neighborhood of the base, so, by convexity considerations, there is a global 1-form γ such that on each fiber, $\beta_F - (\alpha_1)_F = d(\gamma_F)$. Let $\alpha = \alpha_1 + d(\gamma)$. If Ω_1 is a symplectic form for M^{2k}, then $\Omega = \alpha + K(\rho^* \Omega_1)$ is a symplectic form for N^{2k+2}, K is sufficiently large.

This construction, although it applies only to a narrow range of examples, nonetheless has a certain amount of flexibility. This leads me to make the

CONJECTURE. Every closed $2k$-manifold which has an almost complex structure τ and a real cohomology class α such that $\alpha^k \neq 0$ has a symplectic structure realizing τ and α.

I would like to thank Alan Weinstein for pointing out this question and for helpful discussions.

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