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Some Simplicial Complexes of Universal Osborn Loops

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Abstract. A loop is shown to be a universal Osborn loop if and only if it has a particular simplicial complex. A loop is shown to be a universal Osborn loop and obeys two new identities if and only if it has another particular simplicial complex. A universal Osborn loop and four of its isotopes are shown to form a rectangular pyramid in a 3-dimensional space.

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1 Introduction and Preliminaries

A loop is called an Osborn loop if it obeys any of the two identities below.

$$\mathrm{OS}_3 \ : \ (x\cdot yz)x = xy\cdot[(x^\lambda\cdot xz)\cdot x] \tag{1.1}$$

$$\mathrm{OS}_5 \ : \ (x\cdot yz)x = xy\cdot[(x\cdot x^\rho z)\cdot x] \tag{1.2}$$

For a comprehensive introduction to Osborn loops and its universality, and a detailed literature review on it, readers should check Jaiyéọlá, Adéníran and Sòlárìn [3] and Jaiyéọlá [4]. In this present paper, we shall follow the style and notations used in Jaiyéọlá, Adéníran and Sòlárìn [3] and Jaiyéọlá [4]. The only concepts and notions which will be introduced here are those that were not defined in Jaiyéọlá, Adéníran and Sòlárìn [3] and Jaiyéọlá [4].

Definition 1.1. *Let $(L,\cdot)$ be a loop and $U,V,W\in SYM(L,\cdot)$.*

1. *If $(U,V,W)\in AUT(L,\cdot)$ for some V,W, then U is called autotopic.*
2. *If $(U,V,W)\in AUT(L,\cdot)$ such that $W=U, V=I$, then U is called λ-regular.*
3. *If $(U,V,W)\in AUT(L,\cdot)$ such that $U=I, W=V$, then V is called ρ-regular.*

Stein [5] and Drisko [2] while considering the action of isotopisms and autotopisms of loops, found it convenient to think of a loop $\mathcal{Q}=(Q,\cdot,\backslash,/)$ in terms of the set T_Q of all ordered triples (x,y,z) of elements of Q such that $x\cdot y=z$. An isotopism (α,β,γ) from G to H takes $(x,y,z)\in T_G$ to $(x\alpha,y\beta,z\gamma)\in T_H$. We shall adopt his conventions at some points in time. We shall denote by $[\alpha,\beta]$, the commutator of any $\alpha,\beta\in SYM(G,\cdot)$.
Let $(Q,\cdot,\backslash,/)$ be a loop, then we shall be making use of the following notations for principal isotopes of $(Q,\cdot)$.

- $(Q,*_0)$ represents $Q_{x,v}$;
- $(Q,\circ_0)$ represents $Q_{u,\phi_0(x,u,v)}$, $\phi_0(x,u,v)=(u\backslash([(uv)/(u\backslash(xv))]v))$;
- $(Q,\circ_1)$ represents $Q_{u,[u\backslash(xv)]}$;
- $(Q,*_1)$ represents $Q_{\phi_1(x,u,v),v}$, $\phi_1(x,u,v)=(u\backslash([(uv)/(u\backslash(xv))]v))$ for all $x,u,v\in Q$;
- $(Q,\circ_2)$ represents $Q_{x,\phi_2(x,u,v)}$, $\phi_2(x,u,v)=(u\backslash[(u/v)(u\backslash(xv))])$;
- $(Q,\circ_3)$ represents $Q_{[x\cdot u\backslash v]/v,[u\backslash(xv)]}$;
- $(Q,*_2)$ represents $Q_{u,e}$;
- $(Q,*_3)$ represents $Q_{e,v}$.

Let $(G,\cdot)$ be a loop and let

$$BS_2(G,\cdot)=\{\theta\in SYM(G)\ :\ G(a,b)\overset{\theta}{\cong}G(c,d)\text{ for some }a,b,c,d\in G\}.$$

As shown in Bryant and Schneider [1], $BS_2(G,\cdot)$ forms a group for a loop $(G,\cdot)$ and it shall be called the second Bryant-Schneider group (2nd BSG) of the loop.
Consider the following two notions in algebraic topology.

Definition 1.2. *Let V_Q be a set of isotopes of a loop $(Q,\cdot)$ and let $S_Q \subseteq 2^{V_Q}$ such that $\phi \in S_Q$. If S_Q is a topology on V_Q, then it is called the topology of isotopes of the loop Q and the pair (V_Q, S_Q) is called a topological space of isotopes of Q if (V_Q, S_Q) is a topological space.*

Based on the above notion of topological space of isotopes of a loop, the following facts are direct consequences.

Lemma 1.1. *Let $(Q,\cdot)$ be a loop and let V_Q be the set of isotopes of Q. Then, $\left(V_Q, 2^{V_Q}\right)$ is a topological space of isotopes of Q.*

Lemma 1.2. *Let $(Q,\cdot)$ be a G-loop and let V_Q be the set of isotopes of Q. Let $S_Q = \{X_i\}_{i\in\Omega} \subseteq 2^{V_Q}$ such that $\phi \in S_Q$ and $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a topological space of isotopes of Q.*

Corollary 1.3. *Let $(Q,\cdot)$ be a CC-loop or VD-loop or K-loop or Buchsteiner loop or extra loop or group. Let $S_Q = \{X_i\}_{i\in\Omega} \subseteq 2^{V_Q}$ such that $\phi \in S_Q$ and $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a topological space of isotopes of Q.*

Definition 1.3. *A simplicial complex is a pair (V, S) where V is a set of points called vertices and S is a given family of finite subsets, called simplexes, so that the following conditions are satisfied:*

1. *all points of V are simplexes;*
2. *any non-empty subset of a simplex is a simplex.*

A simplex consisting of $(n+1)$ points is called n-dimensional simplex.

Definition 1.4. *Let V_Q be a set of isotopes of a loop $(Q,\cdot)$ and let $S_Q \subseteq 2^{V_Q}$. If $K_Q = (V_Q, S_Q)$ is a simplicial complex, then K_Q is called a trivial simplicial complex of isotopes of the loop Q.*

Definition 1.5. *Let V_Q be a set of isotopes of a loop $(Q,\cdot)$ and let $S_Q = \{X_i\}_{i\in\Omega} \subseteq 2^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. If $K_Q = (V_Q, S_Q)$ is a simplicial complex, then K_Q is called a non-trivial simplicial complex of isotopes or simplicial complex of isotopes of the loop Q.*

The facts below follow suite.

Lemma 1.4. *Let $(Q,\cdot)$ be a loop and let V_Q be the set of isotopes of Q. Then, $\left(V_Q, 2^{V_Q}\right)$ is a trivial simplicial complex of isotopes of Q.*

Lemma 1.5. *Let $(Q,\cdot)$ be a G-loop and let V_Q be the set of isotopes of Q. Let $S_Q = \{X_i\}_{i\in\Omega} \subseteq 2^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a simplicial complex of isotopes of Q.*

Corollary 1.6. *Let $(Q,\cdot)$ be a CC-loop or VD-loop or K-loop or Buchsteiner loop or extra loop or group. Let $S_Q = \{X_i\}_{i\in\Omega} \subseteq 2^{V_Q}$ such that $x_{i_j} \cong x_{i_k}$ for all $x_{i_j}, x_{i_k} \in X_i$. Then, (V_Q, S_Q) is a simplicial complex of isotopes of Q.*

Definition 1.6. *Let $K = (V,S)$ and $K' = (V',S')$ be two simplicial complexes. A simplicial map $f : K \to K'$ is a set map $f : V \to V'$ satisfying the property: for every simplex $x \in S$, the image $f(x) \in S'$.*

In this work, the notion of simplicial complex is used to characterize universal Osborn loops. The following results are important for the set objective.

Theorem 1.7. *(Jaiyéọlá , Adéníran and Sòlárìn* [3]*)*
Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a loop and $\gamma_0(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ for all $x,u,v \in Q$, then $\mathcal{Q}$ is a universal Osborn loop if and only if the commutative diagram

$$\begin{array}{ccc} & \xrightarrow{(R_{\phi_0(x,u,v)},L_u,I)} & (Q,\circ_0) \\ & & \Big\downarrow (\gamma_0,\gamma_0,\gamma_0)\ \textit{isomorphism} \\ (Q,\cdot) & \xrightarrow[\textit{principal isotopism}]{(R_v,L_x,I)} & (Q,*_0) \end{array} \tag{1.3}$$

holds.

Theorem 1.8. *(Jaíyéọlá* [4]*)*
Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a loop and $\gamma_1(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ for all $x,u,v \in Q$, then $\mathcal{Q}$ is a universal Osborn loop if and only if the commutative diagram

$$\begin{array}{ccc} & \xrightarrow{(R_v,L_{\phi_1(x,u,v)},I)} & (Q,*_1) \\ & & \Big\downarrow (\gamma_1,\gamma_1,\gamma_1)\ \textit{isomorphism} \\ (Q,\cdot) & \xrightarrow[\textit{principal isotopism}]{(R_{[u\backslash(xv)]},L_u,I)} & (Q,\circ_1) \end{array} \tag{1.4}$$

holds.

Theorem 1.9. *(Jaiyéọlá , Adéníran and Sòlárìn* [3]*)*
Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a loop and $\gamma_0(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ for all $x,u,v \in Q$, then $\mathcal{Q}$ is a universal Osborn loop implies the commutative diagram

$$\begin{array}{ccc} & \xrightarrow{(R_{\phi_2(x,u,v)},L_x,I)} & (Q,\circ_2) \\ & & \Big\uparrow (\gamma_0,\gamma_0,\gamma_0)\ \textit{isomorphism} \\ (Q,\cdot) & \xrightarrow[\textit{principal isotopism}]{(I,L_u,I)} & (Q,*_2) \end{array} \tag{1.5}$$

holds.

Theorem 1.10. *(Jaíyéọlá [4])*
Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop and $\gamma_1(x, u, v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ for all $x, u, v \in Q$, then $\mathcal{Q}$ is a universal Osborn loop implies the commutative diagram

$$\begin{array}{ccc} & & (Q,\circ_3) \\ & \nearrow^{(R_{[u\backslash(xv)]},L_{[x\cdot u\backslash v]/v},I)} & \Big\uparrow {\scriptstyle (\gamma_1,\gamma_1,\gamma_1)\ \textit{isomorphism}} \\ (Q,\cdot) & \xrightarrow[\textit{principal isotopism}]{(R_v,I,I)} & (Q,*_3) \end{array} \tag{1.6}$$

holds.

Lemma 1.11. *(Drisko [2])*
Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop. Then $Q_{f,g} \cong Q_{c,d}$ if and only if there exists $(\alpha, \beta, \gamma) \in AUT(\mathcal{Q})$ such that $(f, g, fg)(\alpha, \beta, \gamma) = (c, d, cd)$.

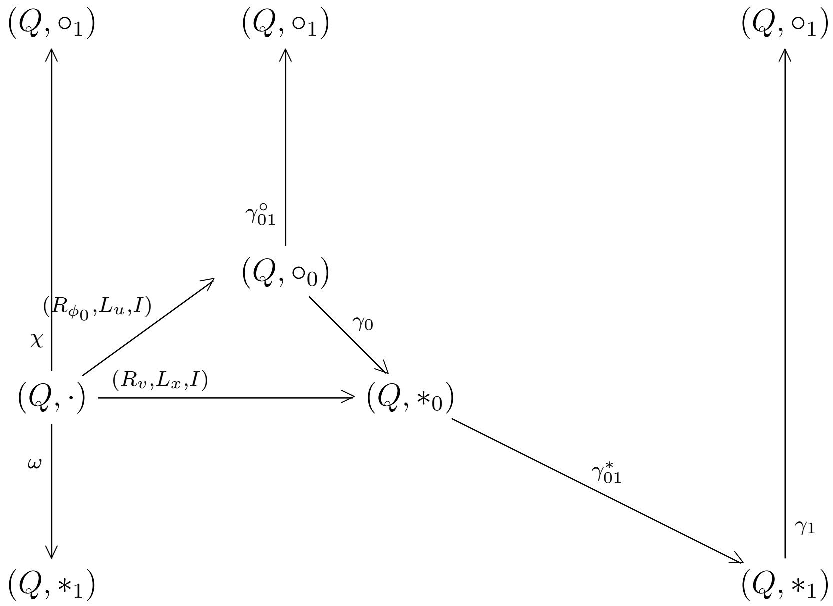
Theorem 1.12. *(Bryant and Schneider [1])*
Let $(Q, \cdot, \backslash, /)$ be a quasigroup. If $Q_{a,b} \overset{I}{\cong} Q_{c,d}$ if and only if $c\cdot b, a\cdot d \in N_\mu(Q_{a,b})$ and $a \cdot b = c \cdot d$.

2 Main Results

Theorem 2.1. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then, the following are necessary and sufficient for each other.*

1. $(Q, \circ_0) \overset{I}{\cong} (Q, \circ_1)$.
2. $(Q, *_0) \overset{I}{\cong} (Q, *_1)$.
3. *$\mathcal{Q}$ is a boolean group.*

Proof. By combining the commutative diagrams in Equation 1.3 and Equation 1.4, we have the commutative diagram below.



(2.1)

where $\chi = (R_{[u\backslash(xv)]}, L_u, I)$ and $\omega = (R_v, L_{\phi_1}, I)$. Let

$$(Q, \circ_0) \xrightarrow[\text{isotopism}]{(\delta_{01}^{\circ}, \varepsilon_{01}^{\circ}, \pi_{01}^{\circ})} (Q, \circ_1).$$

So, from Equation 2.1,

$$\begin{gathered}(R_{\phi_0(x,u,v)}, L_u, I)(\delta_{01}^{\circ}, \varepsilon_{01}^{\circ}, \pi_{01}^{\circ}) = (R_{[u\backslash(xv)]}, L_u, I) \Rightarrow \\ (R_{\phi_0(x,u,v)}\delta_{01}^{\circ}, L_u\varepsilon_{01}^{\circ}, \pi_{01}^{\circ}) = (R_{[u\backslash(xv)]}, L_u, I) \Leftrightarrow \\ R_{\phi_0(x,u,v)}\delta_{01}^{\circ} = R_{[u\backslash(xv)]},\ L_u\varepsilon_{01}^{\circ} = L_u \text{ and } \pi_{01}^{\circ} = I \Leftrightarrow \\ \delta_{01}^{\circ} = R_{\phi_0(x,u,v)}^{-1}R_{[u\backslash(xv)]},\ \varepsilon_{01}^{\circ} = L_u^{-1}L_u = I \text{ and } \pi_{01}^{\circ} = I.\end{gathered}$$

Thus, $(Q, \circ_0) \cong (Q, \circ_1)$ iff $\delta_{01}^{\circ} = \varepsilon_{01}^{\circ} = I$ iff

$$\begin{gathered}R_{\phi_0(x,u,v)}^{-1}R_{[u\backslash(xv)]} = I \Leftrightarrow \phi_0(x, u, v) = [u\backslash(xv)] \\ (u\backslash([(uv)/(u\backslash(xv))]v)) = [u\backslash(xv)] \Leftrightarrow x\backslash(uv) = u\backslash(xv).\end{gathered}$$

Similarly, by using the procedure above, it can be shown that $(Q, *_0) \cong (Q, *_1)$ iff $x\backslash(uv) = u\backslash(xv)$.
Keeping in mind that every Osborn loop of exponent 2 is an abelian group, hence, a Boolean group. This completes the proof. □

Remark 2.1. It can be observed that in a universal Osborn loop $\mathcal{Q} = (Q, \cdot, \backslash, /)$ and for $\gamma_0(x, u, v)$ and $\gamma_1(x, u, v)$ of Theorem 1.7 and Theorem 1.8,

$\gamma_0(x,u,v) = \gamma_1(x,u,v)$ if and only if $\Big[\mathbb{L}_u L_x, \mathbb{R}_v R_{[u\backslash(xv)]}\Big] = I$ for all $x,u,v \in Q$.

The proof of Theorem 2.1 can also be achieved by making use of Theorem 1.12. Take $a = u$, $b = \phi_0(x,u,v)$, $c = u$ and $d = u\backslash(xv)$. Then, $(Q,\circ_0) \overset{I}{\cong} (Q,\circ_1)$ iff

1. $u\phi_0(x,u,v) \in N_\mu\big((Q,\circ_0)\big)$,
2. $u[u\backslash(xv)] \in N_\mu\big((Q,\circ_0)\big)$,
3. $u\phi_0(x,u,v) = u[u\backslash(xv)] \Leftrightarrow$ $\mathcal{Q}$ is a Boolean group.

Theorem 2.2. *Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a universal Osborn loop. Then $(Q,\circ_0) \cong (Q,\circ_1)$ if and only if there exists $(I,\beta,\gamma) \in AUT(\mathcal{Q})$ such that*

$$uv = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u \tag{2.2}$$

for all $x,u,v \in Q$.

Proof. Following Lemma 1.11, $(Q,\circ_0) \cong (Q,\circ_1)$ if and only if there exists $(\alpha,\beta,\gamma) \in AUT(\mathcal{Q})$ such that

$$\begin{gathered}(u,\phi_0(x,u,v),u\phi_0(x,u,v))(\alpha,\beta,\gamma) = (u,[u\backslash(xv)],xv) \Leftrightarrow\\ (u\alpha,\phi_0(x,u,v)\beta,(u\phi_0(x,u,v))\gamma) = (u,[u\backslash(xv)],xv) \Leftrightarrow\\ u\alpha = u,\ \phi_0(x,u,v)\beta = [u\backslash(xv)] \text{ and } (u\phi_0(x,u,v))\gamma = xv \Leftrightarrow\\ \alpha = I,\ \{u\backslash([(uv)/(u\backslash(xv))]v)\}\beta = u\backslash(xv) \text{ and } \{[(uv)/(u\backslash(xv))]v\}\gamma = xv \Leftrightarrow\\ \alpha = I,\ [(uv)/(u\backslash(xv))]R_v\mathbb{L}_u\beta = xR_v\mathbb{L}_u \text{ and } [(uv)/(u\backslash(xv))]R_v\gamma = xR_v \Leftrightarrow\\ \alpha = I,\ (uv)/(u\backslash(xv)) = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \text{ and } [(uv)/(u\backslash(xv))] = xR_v\gamma^{-1}\mathbb{R}_v\\ \Leftrightarrow \alpha = I,\ uv = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u \text{ and } uv = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u \Leftrightarrow\end{gathered}$$

there exists $(I,\beta,\gamma) \in AUT(\mathcal{Q})$ such that

$$uv = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u.$$

□

Remark 2.2. If the autotopism (α,β,γ) in Theorem 2.2 is the identity autotopism, then we shall have the equivalence of 1. and 3. of Theorem 2.1.

Corollary 2.3. *Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be a universal Osborn loop. Then $(Q,\circ_0) \cong (Q,\circ_1)$ implies that there exists $(I,\beta,\gamma) \in AUT(\mathcal{Q})$ such that $\gamma = \mathbb{L}_u\beta L_u$ for all $u \in Q$. Hence,*

1. *$\gamma = \beta$ iff $[\beta, L_u] = I$ or $[\gamma, L_u] = I$. Thence, β is a ρ-regular permutation.*

2. $\gamma = L_u$ *iff* $\beta = L_u$*. Thence,* $\mathcal{Q}$ *is an abelian group.*

Proof. The proof of these follows from the fact in Theorem 2.2 that

$$xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u \Rightarrow$$

$\mathbb{L}_u\beta L_u = \gamma$ for all $u \in Q$. □

Theorem 2.4. *Let* $\mathcal{Q} = (Q, \cdot, \backslash, /)$ *be a universal Osborn loop. Then* $(Q, *_0) \cong (Q, *_1)$ *if and only if there exists* $(\delta, I, \pi) \in AUT(\mathcal{Q})$ *such that*

$$uv = x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \tag{2.3}$$

for all $x, u, v \in Q$*.*

Proof. Following Lemma 1.11, $(Q, *_0) \cong (Q, *_1)$ if and only if there exists $(\delta, \varepsilon, \pi) \in AUT(\mathcal{Q})$ such that $(x, v, xv)(\delta, \varepsilon, \pi) = (\phi_1(x, u, v), v, \phi_1(x, u, v)v)$. The procedure of the proof of the remaining part is similar to that of Theorem 2.2. □

Remark 2.3. If the autotopism $(\delta, \varepsilon, \pi)$ in Theorem 2.4 is the identity autotopism, then we shall have the equivalence of 2. and 3. of Theorem 2.1.

Corollary 2.5. *Let* $\mathcal{Q} = (Q, \cdot, \backslash, /)$ *be a universal Osborn loop. Then* $(Q, *_0) \cong (Q, *_1)$ *implies that there exists* $(\delta, I, \pi) \in AUT(\mathcal{Q})$ *such that* $\pi = \mathbb{R}_v\delta R_v$ *for all* $v \in Q$*. Hence,*

1. $\pi = \delta$ *iff* $[\delta, R_v] = I$ *or* $[\pi, R_v] = I$*. Thence,* δ *is a* λ*-regular permutation.*

2. $\delta = R_v$ *iff* $\pi = R_v$*. Thence,* $\mathcal{Q}$ *is an abelian group.*

Proof. The proof of these follows from the fact in Theorem 2.4 that

$$x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \Rightarrow$$

$\pi = \mathbb{R}_v\delta R_v$ for all $v \in Q$. □

Theorem 2.6. *Let* $\mathcal{Q} = (Q, \cdot, \backslash, /)$ *be a universal Osborn loop. Then* $(Q, \circ_0) \cong (Q, \circ_1)$ *and* $(Q, *_0) \cong (Q, *_1)$ *if and only if there exists* $(I, \beta, \gamma), (\delta, I, \pi) \in AUT(\mathcal{Q})$ *such that*

$$\begin{gathered} uv = xR_v\mathbb{L}_u\beta^{-1}L_u\mathbb{R}_v \cdot xR_v\mathbb{L}_u = \\ xR_v\gamma^{-1}\mathbb{R}_v \cdot xR_v\mathbb{L}_u = x \cdot x\delta R_v\mathbb{L}_u = x \cdot xR_v\pi\mathbb{L}_u \end{gathered} \tag{2.4}$$

for all $x, u, v \in Q$

Proof. This is achieved by simply combining Theorem 2.2 and Theorem 2.4. $\square$

Theorem 2.7. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. If $(Q, \circ_0) \overset{\gamma_{01}^{\circ}}{\cong} (Q, \circ_1)$ and $(Q, *_0) \overset{\gamma_{01}^{*}}{\cong} (Q, *_1)$, then $\gamma_0\gamma_{01}^{*}\gamma_1 = \gamma_{01}^{\circ}$.*

Proof. The commutative diagram in Equation 2.1 proves this. $\square$

Corollary 2.8. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. If $(Q, \circ_0) \cong (Q, \circ_1)$ and $(Q, *_0) \cong (Q, *_1)$, then the following are necessary and sufficient for each other.*

1. $\beta = I$.
2. $\gamma = I$.
3. $\delta = I$.
4. $\pi = I$.
5. $(Q, \circ_0) \overset{I}{\cong} (Q, \circ_1)$.
6. $(Q, *_0) \overset{I}{\cong} (Q, *_1)$.
7. $\mathcal{Q}$ *is a boolean group.*

Proof. To prove the equivalence of 1. to 4. and 7., use Equation 2.4 of Theorem 2.6. The proof of the equivalence of 5. to 7. follows from Theorem 2.1. $\square$

Remark 2.4. Corollary 2.8 is a very important result in this study. It gives us the main distinctions between Theorem 2.1 and Theorem 2.6. That is, the necessary and sufficient condition(s) under which the isomorphisms $(Q, \circ_0) \cong (Q, \circ_1)$ and $(Q, *_0) \cong (Q, *_1)$ will be trivial. And the condition(s) is when any of the autotopic permutations of β, γ, δ and π of Theorem 2.2 and Theorem 2.4 is equal to the identity mapping.

Next, it is important to deduce the actual definitions of the autotopic mappings β, γ, δ, π and the isomorphisms γ_{01}^{*} and γ_{01}°. Recall that by the necessary part of Lemma 1.11, if $\mathcal{Q} = (Q, \cdot, \backslash, /)$ is a loop and $Q_{f,g} \overset{\theta}{\cong} Q_{c,d}$, then there exists $(A, B, C) \in AUT(\mathcal{Q})$ such that $(f, g, fg)(A, B, C) = (c, d, cd)$. According to the proof of this,

$$\begin{gathered}(A, B, C) = (R_g\theta R_d^{-1}, L_f\theta L_c^{-1}, \theta) \Leftrightarrow \\ A = R_g\theta R_d^{-1},\ B = L_f\theta L_c^{-1} \text{ and } C = \theta.\end{gathered} \tag{2.5}$$

Thus,

$$I = \alpha = R_{\phi_o(x,u,v)}\gamma_{01}^{\circ}R_{[u\backslash(xv)]}^{-1},\ \beta = L_u\gamma_{01}^{\circ}L_u^{-1} \text{ and } \gamma = \gamma_{01}^{\circ}$$
$$\gamma_{01}^{\circ} = \mathbb{R}_{\phi_o(x,u,v)}R_{[u\backslash(xv)]},\ \beta = L_u\mathbb{R}_{\phi_o(x,u,v)}R_{[u\backslash(xv)]}\mathbb{L}_u^{-1} \text{ and}$$
$$\gamma = \mathbb{R}_{\phi_o(x,u,v)}R_{[u\backslash(xv)]}$$

and

$$\delta = R_v\gamma_{01}^* R_v^{-1},\ I = \varepsilon = L_x\gamma_{01}^* L_{\phi_1(x,u,v)}^{-1} \text{ and } \pi = \gamma_{01}^*$$
$$\delta = R_v\gamma_{01}^* \mathbb{R}_v^{-1},\ \gamma_{01}^* = \mathbb{L}_x L_{\phi_1(x,u,v)} \text{ and } \pi = \gamma_{01}^*$$
$$\delta = R_v\mathbb{L}_x L_{\phi_1(x,u,v)}\mathbb{R}_v^{-1},\ \gamma_{01}^* = \mathbb{L}_x L_{\phi_1(x,u,v)} \text{ and } \pi = \mathbb{L}_x L_{\phi_1(x,u,v)}.$$

Therefore, Theorem 2.2 and Theorem 2.4 can now be restated as follows.

Theorem 2.9. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, \circ_0) \overset{\gamma_{01}^\circ}{\cong} (Q, \circ_1)$ if and only if*

$$y \cdot u\backslash[(uz)\psi_0] = (yz)\psi_0 \text{ and } uv = xR_v(R_v\psi_0)^{-1} \cdot xR_v\mathbb{L}_u \qquad (2.6)$$

where $\psi_0 = \mathbb{R}_{\phi_o(x,u,v)}R_{[u\backslash(xv)]}$ for all $x, y, z, u, v \in Q$

Proof. Simply substitute

$$\beta = L_u\mathbb{R}_{\phi_o(x,u,v)}R_{[u\backslash(xv)]}\mathbb{L}_u^{-1} \text{ and } \gamma = \mathbb{R}_{\phi_o(x,u,v)}R_{[u\backslash(xv)]}$$

into Equation 2.2 of Theorem 2.2. □

Theorem 2.10. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. Then $(Q, *_0) \overset{\gamma_{01}^*}{\cong} (Q, *_1)$ if and only if*

$$[(yv)\psi_1]/v \cdot z = (yz)\psi_1 \text{ and } uv = x \cdot u\backslash[(xv)\psi_1] \qquad (2.7)$$

where $\psi_1 = \mathbb{L}_x L_{\phi_1(x,u,v)}$ for all $x, y, z, u, v \in Q$

Proof. Simply substitute

$$\delta = R_v\mathbb{L}_x L_{\phi_1(x,u,v)}\mathbb{R}_v^{-1} \text{ and } \pi = \mathbb{L}_x L_{\phi_1(x,u,v)}$$

into Equation 2.3 of Theorem 2.4. □

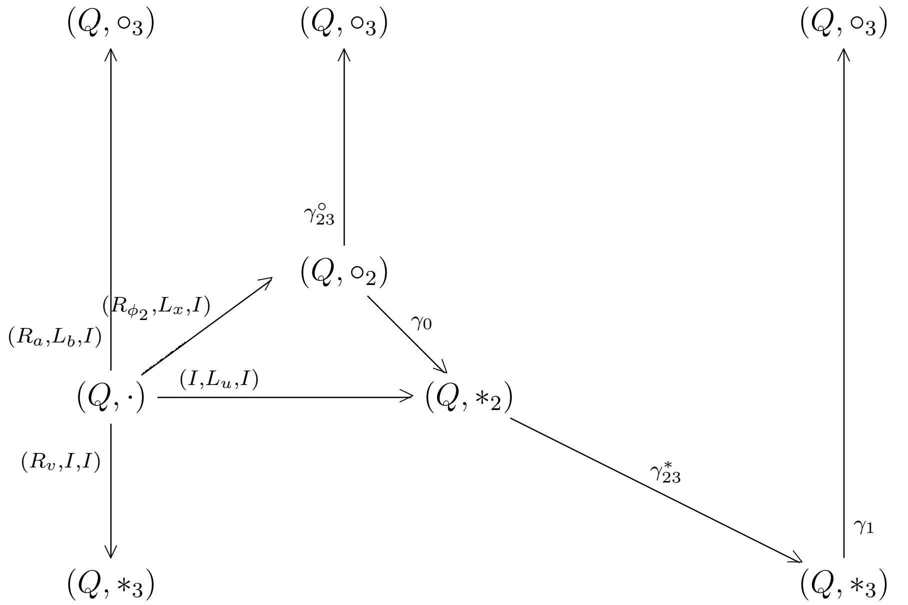
Lemma 2.11. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop.*

1. *$\mathcal{Q}$ is a universal Osborn loop and obeys Equation 2.6 if and only if $\gamma_0, \gamma_{01}^\circ \in BS_2(\mathcal{Q})$.*
2. *$\mathcal{Q}$ is a universal Osborn loop and obeys Equation 2.7 if and only if $\gamma_1, \gamma_{01}^* \in BS_2(\mathcal{Q})$.*

Proof. This follows by combining Theorem 1.7, Theorem 1.8, Theorem 2.2 and Theorem 2.4 □

Remark 2.5. It is a self exercise to confirm if $(Q, \circ_0) \overset{\gamma_{01}^\circ}{\cong} (Q, \circ_1)$ and $(Q, *_0) \overset{\gamma_{01}^*}{\cong} (Q, *_1)$ in some universal Osborn loops like Moufang loops and extra loops by simply verifying Equation 2.6 and Equation 2.7. Furthermore, the relation $\gamma_0\gamma_{01}^*\gamma_1 = \gamma_{01}^\circ$ of Theorem 2.7 is justifiable as well. It must be noted also, that in any universal Osborn loop $\mathcal{Q}$, Equation 2.6 and Equation 2.7 are necessary and sufficient conditions for $\gamma_{01}^*, \gamma_{01}^\circ \in BS_2(\mathcal{Q})$.

By combining the commutative diagrams in Equation 1.5 and Equation 1.6, we have the commutative diagram below.



(2.8)

where $a = [u\backslash(xv)]$ and $b = \{[x \cdot u\backslash v]/v\}$.

Theorem 2.12. *Let* $\mathfrak{Q} = (Q, \cdot, \backslash, /)$ *be a universal Osborn loop. Then* $(Q, \circ_2) \cong (Q, \circ_3)$ *if and only if there exists* $(\lambda, \mu, \nu) \in AUT(\mathfrak{Q})$ *such that*

$$\lambda = R_{u\backslash v}\mathbb{R}_v,\ \mu = L_u\mathbb{L}_{u\backslash v} \text{ and } [x \cdot xR_v\mathbb{L}_u\mu^{-1}]\nu = x\lambda \cdot xR_v\mathbb{L}_u \tag{2.9}$$

for all $x, u, v \in Q$.

Proof. Following Lemma 1.11, $(Q, \circ_2) \cong (Q, \circ_3)$ if and only if there exists $(\lambda, \mu, \nu) \in AUT(\mathfrak{Q})$ such that $(x, \phi_2(x, u, v), x\phi_2(x, u, v))(\lambda, \mu, \nu) = ([x \cdot u\backslash v]/v, [u\backslash(xv)], \{[x \cdot u\backslash v]/v\}[u\backslash(xv)])$. The procedure of the proof of the remaining part is similar to that of Theorem 2.2. □

Lemma 2.13. *Let* $\mathfrak{Q} = (Q, \cdot, \backslash, /)$ *be a universal Osborn loop. Then* $(Q, \circ_2) \overset{\gamma^\circ_{23}}{\cong} (Q, \circ_3)$ *if and only if there exists* $(\lambda, \mu, \gamma^\circ_{23}) \in AUT(\mathfrak{Q})$ *such that*

$$\begin{gathered}\gamma^\circ_{23} = \mathbb{R}_{\phi_2(x,u,v)}R_{u\backslash v}\mathbb{R}_v R_{[u\backslash(xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u\backslash v} L_{\{[x\cdot u\backslash v]/v\}} \text{ and} \\ [x \cdot xR_v\mathbb{L}_u\mu^{-1}]\gamma^\circ_{23} = x\lambda \cdot xR_v\mathbb{L}_u\end{gathered} \tag{2.10}$$

for all $x, u, v \in Q$.

Proof. Considering the commutative diagram in Equation 2.8 and using Equation 2.5,

$$\lambda = R_{\phi_2(x,u,v)}\gamma^\circ_{23}R^{-1}_{[u\backslash(xv)]},\ \mu = L_x\gamma^\circ_{23}L^{-1}_{\{[x\cdot u\backslash v]/v\}} \text{ and } \nu = \gamma^\circ_{23}.$$

The final conclusion follows from Theorem 2.12. □

Corollary 2.14. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a universal Osborn loop. $\gamma_{23}^{\circ} \in BS_2(\mathcal{Q})$ if and only if there exists $(\lambda, \mu, \gamma_{23}^{\circ}) \in AUT(\mathcal{Q})$ such that*

$$\begin{array}{c}\gamma_{23}^{\circ} = \mathbb{R}_{\phi_2(x,u,v)} R_{u\backslash v} \mathbb{R}_v R_{[u\backslash(xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u\backslash v} L_{\{[x\cdot u\backslash v]/v\}} \text{ and} \\ [x \cdot x R_v \mathbb{L}_u \mu^{-1}]\gamma_{23}^{\circ} = x\lambda \cdot x R_v \mathbb{L}_u \end{array} \tag{2.11}$$

for all $x, u, v \in Q$.

Proof. This follows from Lemma2.13. □

Corollary 2.15. *Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be a loop. $\mathcal{Q}$ is a universal Osborn loop and $\gamma_{23}^{\circ} \in BS_2(\mathcal{Q})$ implies $\gamma_0 \in BS_2(\mathcal{Q})$ and there exists $(\lambda, \mu, \gamma_{23}^{\circ}) \in AUT(\mathcal{Q})$ such that*

$$\begin{array}{c}\gamma_{23}^{\circ} = \mathbb{R}_{\phi_2(x,u,v)} R_{u\backslash v} \mathbb{R}_v R_{[u\backslash(xv)]} = \mathbb{L}_x L_u \mathbb{L}_{u\backslash v} L_{\{[x\cdot u\backslash v]/v\}} \text{ and} \\ [x \cdot x R_v \mathbb{L}_u \mu^{-1}]\gamma_{23}^{\circ} = x\lambda \cdot x R_v \mathbb{L}_u \end{array} \tag{2.12}$$

for all $x, u, v \in Q$.

Proof. This follows from Theorem 1.9 and Lemma2.13. □

Simplicial Complex of Isotopes of a Universal Osborn Loop

Theorem 2.16. *Let $(Q, \cdot)$ be a loop. Let $V_0(Q) = \{(Q, \cdot), (Q, \circ_0), (Q, *_0)\}$ and $S_0(Q) = \Big\{\{(Q, \cdot)\}, \{(Q, \circ_0)\}, \{(Q, *_0)\}, \{(Q, \circ_0), (Q, *_0)\}\Big\}$. Then, $(Q, \cdot)$ is a universal Osborn loop if and only if $K_0(Q) = \Big(V_0(Q), S_0(Q)\Big)$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This is proved with the help of Theorem 1.7. □

Theorem 2.17. *Let $(Q, \cdot)$ be a loop. Let $V_1(Q) = \{(Q, \cdot), (Q, \circ_1), (Q, *_1)\}$ and $S_1(Q) = \Big\{\{(Q, \cdot)\}, \{(Q, \circ_1)\}, \{(Q, *_1)\}, \{(Q, \circ_1), (Q, *_1)\}\Big\}$. Then, $(Q, \cdot)$ is a universal Osborn loop if and only if $K_1(Q) = \Big(V_1(Q), S_1(Q)\Big)$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This is proved with the help of Theorem 1.8. □

Theorem 2.18. *Let $(Q, \cdot)$ be a loop. Let $V_2(Q) = \{(Q, \cdot), (Q, \circ_2), (Q, *_2)\}$ and $S_2(Q) = \Big\{\{(Q, \cdot)\}, \{(Q, \circ_2)\}, \{(Q, *_2)\}, \{(Q, \circ_2), (Q, *_2)\}\Big\}$. If $(Q, \cdot)$ is a universal Osborn loop, then $K_2(Q) = \Big(V_2(Q), S_2(Q)\Big)$ is a simplicial complex of isotopes of $(Q, \cdot)$.*

Proof. This is proved with Theorem 1.9. □

Theorem 2.19. *Let* $(Q,\cdot)$ *be a loop. Let* $V_3(Q)=\big\{(Q,\cdot),\ (Q,\circ_3),\ (Q,*_3)\big\}$ *and* $S_3(Q)=\Big\{\{(Q,\cdot)\},\ \{(Q,\circ_3)\},\ \{(Q,*_3)\},\ \big\{(Q,\circ_3),\ (Q,*_3)\big\}\Big\}$. *If* $(Q,\cdot)$ *is a universal Osborn loop, then* $K_3(Q)=\Big(V_3(Q),S_3(Q)\Big)$ *is a simplicial complex of isotopes of* $(Q,\cdot)$.

Proof. This is proved with the aid of Theorem 1.10. □

Corollary 2.20. *Let* $(Q,\cdot)$ *be a loop. Let* $V_i(Q)=\big\{(Q,\cdot),(Q,\circ_i),(Q,*_i)\big\}$ *and* $S_i(Q)=\Big\{\{(Q,\cdot)\},\{(Q,\circ_i)\},\{(Q,*_i)\},\big\{(Q,\circ_i),(Q,*_i)\big\}\Big\}$ *for* $i=0,1$. *Then,* $(Q,\cdot)$ *is a universal Osborn loop if and only if* $K_{01}(Q)=K_0(Q)\cup K_1(Q)=\Bigg(V_0(Q)\cup V_1(Q),S_0(Q)\cup S_1(Q)\Bigg)$ *is a simplicial complex of isotopes of* $(Q,\cdot)$.

Proof. This follows from Theorem 2.16 and Theorem 2.17. □

Corollary 2.21. *Let* $(Q,\cdot)$ *be a loop. Let* $V_i(Q)=\big\{(Q,\cdot),(Q,\circ_i),(Q,*_i)\big\}$ *and* $S_i(Q)=\Big\{\{(Q,\cdot)\},\{(Q,\circ_i)\},\{(Q,*_i)\},\big\{(Q,\circ_i),(Q,*_i)\big\}\Big\}$ *for* $i=2,3$. *If* $(Q,\cdot)$ *is a universal Osborn loop, then* $K_{23}(Q)=K_2(Q)\cup K_3(Q)=\Bigg(V_2(Q)\cup V_3(Q),S_2(Q)\cup S_3(Q)\Bigg)$ *is a simplicial complex of isotopes of* $(Q,\cdot)$.

Proof. This follows from Theorem 2.18 and Theorem 2.19. □

Corollary 2.22. *Let* $(Q,\cdot)$ *be a loop. Let* $V_i(Q)=\big\{(Q,\cdot),(Q,\circ_i),(Q,*_i)\big\}$ *and* $S_i(Q)=\Big\{\{(Q,\cdot)\},\{(Q,\circ_i)\},\{(Q,*_i)\},\big\{(Q,\circ_i),(Q,*_i)\big\}\Big\}$ *for* $i=0,1,2,3$. *If* $(Q,\cdot)$ *is a universal Osborn loop, then*

$$K_{0123}(Q)=\bigcup_{i=0}^{3}K_i(Q)=\Bigg(\bigcup_{i=0}^{3}V_i(Q),\bigcup_{i=0}^{3}S_i(Q)\Bigg)$$ *is a simplicial complex of isotopes of* $(Q,\cdot)$.

Proof. This is proved by combining Corollary 2.20 and Corollary 2.21. □

Theorem 2.23. *Let* $(Q,\cdot)$ *be a loop. Let* $V_{01}(Q)=\big\{(Q,\cdot),(Q,\circ_0),(Q,*_0),(Q,\circ_1),(Q,*_1)\big\}$ *and* $S_{10}(Q)=\Big\{\{(Q,\cdot)\},\ \{(Q,\circ_0)\},\ \{(Q,*_0)\},\ \{(Q,\circ_1)\},\ \{(Q,*_1)\},\ \big\{(Q,\circ_0),(Q,*_0)\big\},\ \big\{(Q,\circ_1),\ (Q,*_1)\big\},\ \big\{(Q,\circ_0),\ (Q,\circ_1)\big\},\ \big\{(Q,*_0),\ (Q,*_1)\big\},\ \big\{(Q,\circ_0),\ (Q,*_1)\big\},\ \big\{(Q,\circ_1),\ (Q,*_0)\big\},\ \big\{(Q,\circ_0),\ (Q,\circ_1),\ (Q,*_0)\big\},$

$\big\{(Q,\circ_0),\ (Q,\circ_1),\ (Q,*_1)\big\},\ \big\{(Q,*_0),\ (Q,*_1),\ (Q,\circ_0)\big\},$ $\big\{(Q,*_0),\ (Q,*_1),\ (Q,\circ_1)\big\},\ \big\{(Q,\circ_0),\ (Q,\circ_1),\ (Q,*_0),\ (Q,*_1)\big\}\Big\}$*. Then,* $(Q,\cdot)$ *is a universal Osborn loop and obey Equation 2.6 and Equation 2.7 if and only if* $K_{10}(Q) = \Big(V_{01}(Q), S_{10}(Q)\Big)$ *is a simplicial complex of isotopes of* $(Q,\cdot)$*.*

Proof. This is proved with the aid of Theorem 2.16, Theorem 2.17, Theorem 2.9 and Theorem 2.10. □

Theorem 2.24. *Let* $(Q,\cdot)$ *be a universal Osborn loop.*
Let $V_i(Q) = \big\{(Q,\cdot),(Q,\circ_i),(Q,*_i)\big\}$,
$S_i(Q) = \Big\{\{(Q,\cdot)\},\{(Q,\circ_i)\},\{(Q,*_i)\},\{(Q,\circ_i),(Q,*_i)\}\Big\}$ *and*
$K_i = (V_i(Q), S_i(Q))$ *for* $i = 0,1,2,3$*. Define* $f_{ij}\ :\ K_i \to K_j$ *as*

$$f_{ij}\ :\ \begin{cases}(Q,\cdot) & \longmapsto (Q,\cdot)\\ (Q,\circ_i) & \longmapsto (Q,\circ_j)\quad i,j = 0,1,2,3 \text{ such that } i\neq j.\\ (Q,*_i) & \longmapsto (Q,*_j)\end{cases}$$

Then, f_{ij} *is a simplicial map.*

Proof. This is proved by Theorem 2.16, Theorem 2.17, Theorem 2.18 and Theorem 2.19. □

Theorem 2.25. *Let* $(G,\cdot)$ *and* $(H,\star)$ *be two loop isotopes under the triple* (A,B,C)*. For* $D\in\{A,B,C\}$*, if* $D = E_1E_2\cdots E_i\cdots E_n$*,* $E_i\ :\ G\to H$*,* $i = 1,\cdots n$ *been bijections such that there does not exist* $r\geq n$ *for which* $D = E_1E_2\cdots E_i\cdots E_r$*, then the length of* D*,* $|D| = n$ *units. If* $D = I$*, the identity mapping, then* $|D| = 0$*. The length of the isotopism* $(G,\cdot)\xrightarrow[\text{Isotopism}]{(A,B,C)}(H,\star)$ *is giving by* $|(A,B,C)| = |A|+|B|+|C|$ *units. For an isotopism* $(G,\cdot)\xrightarrow[\text{Isotopism}]{(A,B,C)}(H,\star)$*, let the two loops* $(G,\cdot)$ *and* $(H,\star)$ *represent points in a* 3*-dimensional space and let an isotopism from* $(G,\cdot)$ *to* $(H,\star)$ *be a line with* $(G,\cdot)$ *and* $(H,\star)$ *as end-points. The set of loops* $V_{01}(Q) = \big\{(Q,\cdot),(Q,\circ_0),(Q,*_0),(Q,\circ_1),(Q,*_1)\big\}$ *where* $(Q,\cdot)$ *is a universal Osborn loop, form a rectangular pyramid with apex* $(Q,\cdot)$*.*

Proof. We shall make use of the combined commutative diagram (2.1) as shown in the proof of Theorem 2.1. There are four isotopes of $(Q,\cdot)$ as shown in the combined commutative diagram (2.1), namely $(Q,\circ_i),(Q,*_i)$ for $i = 0,1$. The length of each of the isotopisms $(R_{[u\backslash(xv)]},L_u,I)$, (R_{ϕ_0},L_u,I), (R_v,L_{ϕ_1},I), (R_v,L_x,I) is 2 units. The length of each of the isomorphisms

$\gamma_0(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ and $\gamma_1(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ is 12 units. The length of each of the isomorphisms $\gamma_{01}^{\circ} = \mathbb{R}_{\phi_o(x,u,v)} R_{[u\backslash(xv)]}$ and $\gamma_{01}^{*} = \mathbb{L}_x L_{\phi_1(x,u,v)}$ is 6 units. Hence, the four loop isotopes $(Q,\circ_i),(Q,*_i)$ for $i = 0,1$ of $(Q,\cdot)$ form a rectangle. Thus, taking $(Q,\cdot)$ as an apex and the four isotopism as lines drawn from the apex to the four vertices of the rectangle, we have a rectangular pyramid. □

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